Nearly Perfect Fluidity:
From Cold Atoms to Hot Quarks

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RHIC serves the perfect fluid
Heavy Ion Collisions are very complicated (time-dependent, strongly correlated quantum many body physics), but at RHIC & LHC a very simple theory appears to work.

\textit{παντα ρει (everything flows)}

Heraclitus of Ephesus, 535 - 475 BC

In this talk I will try to address the questions “Why?” and “How general is this phenomenon?”.
Hydrodynamics (undergraduate version): Newton’s law for continuous, deformable media.
**Fluids: Gases, liquids, plasmas, ...**

Hydrodynamics (postmodern): Effective theory of non-equilibrium long-wavelength, low-frequency dynamics of any many-body system.

\[ \tau \sim \tau_{\text{micro}} \]

\[ \tau \sim \lambda \]

\[ \tau \gg \tau_{\text{micro}}: \text{ Dynamics of conserved charges.} \]

Water: \((\rho, \epsilon, \vec{\pi})\)
Simple non-relativistic fluid

Simple fluid: Conservation laws for mass, energy, momentum

\[
\frac{\partial \rho}{\partial t} = -\vec{\nabla}(\rho \vec{v}) \quad \frac{\partial \epsilon}{\partial t} = -\vec{\nabla} j^\epsilon
\]

\[
\frac{\partial}{\partial t}(\rho v_i) = -\nabla_j \Pi_{ij}
\]

mass \times \text{acceleration} = \text{force}

Constitutive relations: Stress tensor

\[
\Pi_{ij} = P \delta_{ij} + \rho v_i v_j + \eta \left( \nabla_i v_j + \nabla_j v_i - \frac{2}{3} \delta_{ij} \nabla_k v_k \right) + O(\nabla^2)
\]

reactive \quad \text{dissipative} \quad 2nd \text{ order}

Expansion \( \Pi_{ij}^0 \gg \delta \Pi_{ij}^1 \gg \delta \Pi_{ij}^2 \)
Regime of applicability

Expansion parameter $Re^{-1} = \frac{\eta(\partial v)}{\rho v^2} = \frac{\eta}{\rho L v} \ll 1$

$$\frac{1}{Re} = \frac{\eta}{\hbar n} \times \frac{\hbar}{mvL}$$

Bath tub: $mvL \gg \hbar$ hydro reliable

Heavy ions: $mvL \sim \hbar$ need $\eta < \hbar n$

Note: Bacteria swim in the regime $Re^{-1} \gg 1$ but $Ma^2 \cdot Re^{-1} \ll 1$. 
Breakdown of fluid dynamics

Fluid dynamics is a universal theory but the breakdown of hydro, the emergence of non-hydrodynamic modes, is not.

Two extreme cases: Non-interacting particles or strongly collective, but non-hydrodynamic ($\omega_m(q \to 0) \neq 0$) modes.

Ballistic motion

Quasi-normal modes
Shear viscosity and friction

Momentum conservation at $O(\nabla v)$

$$
\rho \left( \frac{\partial}{\partial t} \vec{v} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla P + \eta \nabla^2 \vec{v}
$$

Navier-Stokes equation

Viscosity determines shear stress ("friction") in fluid flow

$$
F = A \eta \frac{\partial v_x}{\partial y}
$$
Kinetic theory

Kinetic theory: conserved quantities carried by quasi-particles. Quasi-particles described by distribution functions $f(x,p,t)$.

\[
\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p + \vec{F} \cdot \vec{\nabla}_p f_p = -C[f_p]
\]

\[
C'[f_p] = \begin{array}{c}
\begin{bmatrix}
\end{array}
\end{array} - \begin{array}{c}
\begin{bmatrix}
\end{array}
\end{array}
\]

Shear viscosity corresponds to momentum diffusion

\[
\eta \sim \frac{1}{3} n \bar{p} l_{mfp}
\]
Shear viscosity: Low density limit

Weakly interacting gas, \( l_{mfp} \sim \frac{1}{n\sigma} \)

\[ \eta \sim \frac{1}{3} \frac{\bar{p}}{\sigma} \]

shear viscosity independent of density

Maxwell (1860): ”Such a consequence of the mathematical theory is very startling and the only experiment I have met with on the subject does not seem to confirm it.”
Shear viscosity: Additional properties

Non-interacting gas \((\sigma \to 0)\): \[ \eta \to \infty \]

non-interacting and hydro limit \((T \to \infty)\) limit do not commute

Strongly interacting gas:

\[ \frac{\eta}{n} \sim \bar{p}l_{mfp} \geq \hbar \]

Quantum bound. But: Kinetic theory mat not be reliable!
And now for something completely different . . .

This is an irreversible process, $\Delta S > 0$. 
And now for something completely different . . .

Ringdown can be described in terms of stretched horizon that behaves as a sheared fluid

\[ \eta = \frac{s}{4\pi} \]

Note: Unusual thermodynamics, e.g. \( \zeta, C' < 0 \).
Idea can be made precise using the “AdS/CFT correspondence”

Strongly coupled thermal field theory on $R^4$ $\Leftrightarrow$ Weakly coupled string theory on $AdS_5$ black hole

CFT temperature $\Leftrightarrow$ Hawking temperature of black hole

CFT entropy $\Leftrightarrow$ Hawking-Bekenstein entropy $\sim$ area of event horizon
Holographic duals: Transport properties

Thermal (conformal) field theory $\equiv AdS_5$ black hole

CFT entropy $\iff$ Hawking-Bekenstein entropy

$\sim$ area of event horizon

Graviton absorption cross section $\sim$ area of event horizon

Shear viscosity $\iff$

Strong coupling limit

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}$$

Son and Starinets (2001)

Strong coupling limit universal? Provides lower bound for all theories?

Answer appears to be no; e.g. theories with higher derivative gravity duals.
Perfect Fluids: The contenders

- QGP ($T=180$ MeV)
- Trapped Atoms ($T=0.1$ neV)
- Liquid Helium ($T=0.1$ meV)
Perfect Fluids: The contenders

QGP $\eta = 5 \cdot 10^{11} \text{Pa} \cdot \text{s}$

Trapped Atoms
$\eta = 1.7 \cdot 10^{-15} \text{Pa} \cdot \text{s}$

Liquid Helium
$\eta = 1.7 \cdot 10^{-6} \text{Pa} \cdot \text{s}$

Consider ratios $\eta/s$
Perfect Fluids: Not a contender

Queensland pitch-drop experiment

1927-2011 (8 drops)

\[ \eta = (2.3 \pm 0.5) \cdot 10^8 \text{ Pa s} \]
I. QCD and the Quark Gluon Plasma

\[ \mathcal{L} = \bar{q}_f (i \slashed{D} - m_f) q_f - \frac{1}{4 g^2} G^a_{\mu \nu} G^{a \mu \nu} \]
Hydrodynamic expansion converts coordinate space anisotropy to momentum space anisotropy.

\[ p_0 \left. \frac{dN}{d^3p} \right|_{p_z=0} = v_0(p_\perp) \left( 1 + 2v_2(p_\perp) \cos(2\phi) + \ldots \right) \]
Viscosity and Elliptic Flow

Viscous correction to $v_2$ (blast wave model)

$$\frac{\delta v_2}{v_2} = -\frac{1}{3} \frac{1}{\tau_f T_f} \left( \frac{\eta}{s} \right) \left( \frac{p_\perp}{T_f} \right)^2$$

Grows with $p_\perp$, decreases with system size

Many details: Dependence on initial conditions, freeze out, etc.

**conservative bound**

$$\frac{\eta}{s} < 0.25$$

Glauber predicts flat initial spectrum $(n \geq 3)$. Observed flow spectrum consistent with sound attenuation

$$\delta T^{\mu\nu}(t) = \exp \left( -\frac{2}{3} \frac{\eta}{s} \frac{k^2 t}{T} \right) \delta T^{\mu\nu}(0)$$
Everything flows (including p+Pb, and maybe even p+p)

Signatures of collective expansion (radial and elliptic flow) in high multiplicity p+Pb collisions.

Mass ordering of mean $p_T$

CMS (2013)

Mass ordering of $v_2(p_T)$

Alice (2013)

Further evidence for short mean free path? Or suppression of non-hydrodynamic modes?
II. Dilute Fermi gas: BCS-BEC crossover

\[ \mathcal{L}_{\text{eff}} = \psi^\dagger \left( i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 \]
Consider simple square well potential

\[ a < 0 \quad a = \infty, \, \epsilon_B = 0 \quad a > 0, \, \epsilon_B > 0 \]
Unitarity limit

Now take the range to zero, keeping $\epsilon_B \sim 0$

Universal scattering amplitude $\mathcal{T} = \frac{1}{ik}$
Feshbach resonances

Atomic gas with two spin states: \( \uparrow \) and \( \downarrow \)

Feshbach resonance

\[
a(B) = a_0 \left( 1 + \frac{\Delta}{B - B_0} \right)
\]
Fermi gas at unitarity: Field Theory

Non-relativistic fermions at low momentum

\[ \mathcal{L}_{\text{eff}} = \psi^\dagger \left( i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 \]

Unitary limit: \( a \to \infty \) (DR: \( C_0 \to \infty \))

This limit is smooth (HS-trafo, \( \Psi = (\psi^\uparrow, \psi^\dagger_\downarrow) \))

\[ \mathcal{L} = \Psi^\dagger \left[ i\partial_0 + \sigma_3 \frac{\vec{\nabla}^2}{2m} \right] \Psi + (\Psi^\dagger \sigma_+ \Psi \phi + h.c.) - \frac{1}{C_0} \phi^* \phi , \]
Universal fluid dynamics

Many body system: Effective cross section $\sigma_{tr} \sim n^{-2/3}$ (or $\sigma_{tr} \sim \lambda^2$)

Systems remains hydrodynamic despite expansion
Almost ideal fluid dynamics

Hydrodynamic expansion converts coordinate space anisotropy to momentum space anisotropy

O'Hara et al. (2002)
Determination of $\eta(n, T)$

Measurement of $A_R(t, E_0)$ determines $\eta(n, T)$. But:

The corona is not a fluid. Can we ignore this issue?

No. Hubble flow & low density viscosity $\eta \sim T^{3/2}$ lead to paradoxical fluid dynamics.

$$\dot{Q} = \int \sigma \cdot \delta \Pi = \infty$$
**Possible Solutions**

Combine hydrodynamics & Boltzmann equation. Not straightforward.

Hydrodynamics + non-hydro degrees of freedom ($E_a; a = x, y, z$)

\[
\frac{\partial E_a}{\partial t} + \vec{\nabla} \cdot j^e_a = -\frac{\Delta P_a}{2\tau}
\]

\[
\Delta P_a = P_a - P
\]

\[
\frac{\partial E}{\partial t} + \vec{\nabla} \cdot \vec{j}^e = 0
\]

\[
E = \sum_a E_a
\]

$\tau$ small: Fast relaxation to Navier-Stokes with $\tau = \eta/P$

$\tau$ large: Additional conservation laws. Ballistic expansion.
$A_R = \sigma_x / \sigma_y$ as function of total energy. Data: Joseph et al (2016). $E / (NE_F) \sim 0.6$ is the superfluid transition.

Grey, Blue, Green: LO, NLO, NNLO fit.

$\eta = \eta_0 (mT)^{3/2} \left\{ 1 + \eta_2 n\lambda^3 + \eta_3 (n\lambda^3)^2 + \ldots \right\}$
Reconstructed $\eta/n$

Red band: This work. Right figure zooms in on $T_c/s \sim 0.17T_F$.

Black points: Same data, simplified theory. Dashed line: T-matrix theory (Enss et al.). Green band: QMC (Bulgac et al.)

$$\eta(T \gg T_c) = (0.28 \pm 0.02)(mT)^{3/2}$$

$$\frac{\eta}{n}\bigg|_{T_c} = 0.41 \pm 0.15$$

$$\eta_0(th) = \frac{15}{32\sqrt{\pi}} = 0.269$$

$$\eta/s\bigg|_{T_c} = 0.56 \pm 0.20$$
Remarkably, the best fluids that have been observed are the coldest and the hottest fluid ever created in the laboratory, cold atomic gases ($10^{-6}$K) and the quark gluon plasma ($10^{12}$K) at RHIC.

Both of these fluids come close to a bound on the shear viscosity that was first proposed based on calculations in string theory, involving non-equilibrium evolution of back holes in 5 (and more) dimensions.

Quantum limited viscosity and relaxation time explain applicability of fluid dynamics in very small, very short lived systems. Nature of non-hydrodynamic modes remains to be explored.
Outlook: Critical Fluctuations

[Graphical representation of critical fluctuations with data points and axes labeled.]
Historical digressions
Historical digression: Mott’s minimal conductivity

(Sir) Nevill Mott predicted that the metal-insulator transition cannot be continuous; there is a minimal conductivity.

\[ \frac{\sigma}{n^{1/3}} \geq \frac{1}{(3\pi^2)^{2/3}} \frac{e^2}{\hbar} \]

This idea is not correct, the metal-insulator transition can be continuous.
Historical digression: Minimal shear viscosity

Danielewicz & Gyulassy argue that the shear viscosity cannot be zero.

than \( \langle p \rangle^{-1} \). Requiring \( \lambda_i \gtrsim \langle p \rangle_i^{-1} \) leads to the lower bound

\[
\eta \gtrsim \frac{1}{3} n ,
\]  

(3.3)

where \( n = \sum n_i \) is the total density of quanta. What seems amazing about (3.3) is that it is independent of dynamical details. There is a finite viscosity regardless of

Is this idea correct?