Hadron Structure Theory III

Alexei Prokudin
The plan:

- **Lecture I**: Transverse spin structure of the nucleon

- **Lecture II**: Transverse Momentum Dependent distributions (TMDs)  
  Semi Inclusive Deep Inelastic Scattering (SIDIS)

- **Tutorial**: Calculations of SIDIS structure functions using Mathematica

- **Lecture III**: Advanced topics. Evolution of TMDs
If you have any comment on the mathematica package, or the tutorial, send me a message prokudin@jlab.org.
Factorization theorems help us to relate functions that describe the hadron structure and the experimental observables.

Factorization is a *controllable approximation* and the goal of theorists and phenomenologists is to test and improve the region of applicability of factorization and/or construct new factorization theorems.

Hadron structure is the ultimate goal of measurements and phenomenology.

The polarized proton in momentum space as “seen” by the virtual photon.
Transverse Momentum Dependent distributions

Individual TMDs can be projected out of the correlator

Unpolarized quarks

\[
\frac{1}{2} \text{Tr} \left[ \gamma^+ \Phi(x, k_{\perp}) \right] = f_1 - \frac{\varepsilon^{jk} k_{\perp}^j S_T^k}{M_N} f_{1T}^\perp
\]

Longitudinally polarized quarks

\[
\frac{1}{2} \text{Tr} \left[ \gamma^+ \gamma_5 \Phi(x, k_{\perp}) \right] = S_L g_1 + \frac{k_{\perp} \cdot S_T}{M_N} g_{1T}^\perp
\]

Transversely polarized quarks

\[
\frac{1}{2} \text{Tr} \left[ i\sigma^j + \gamma^+ \Phi(x, k_{\perp}) \right] = S_T^j h_1 + S_L \frac{k_{\perp}^j}{M_N} h_{1L}^\perp + \frac{\kappa^{jk} S_T^k}{M_N^2} h_{1T}^\perp + \frac{\varepsilon^{jk} k_{\perp}^k}{M_N} h_{1}^\perp
\]

\[
\kappa^{jk} \equiv (k_{\perp}^j k_{\perp}^k - \frac{1}{2} k_{\perp}^2 \delta^{jk})
\]
Quark TMDs

<table>
<thead>
<tr>
<th></th>
<th>U</th>
<th>L</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>U</td>
<td></td>
<td></td>
</tr>
<tr>
<td>U</td>
<td>$f_1$</td>
<td></td>
<td>$h_1$</td>
</tr>
<tr>
<td>L</td>
<td></td>
<td>$g_1$</td>
<td>$h_{1L}$</td>
</tr>
<tr>
<td>T</td>
<td>$f_{1T}$</td>
<td>$g_{1T}$</td>
<td>$h_1$</td>
</tr>
</tbody>
</table>

8 functions in total (at leading twist)

Each represents different aspects of partonic structure

Each depends on Bjorken-$x$, transverse momentum, the scale

Each function is to be studied

### Quark TMD Fragmentation Functions

<table>
<thead>
<tr>
<th></th>
<th>U</th>
<th>L</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>U</td>
<td></td>
<td></td>
</tr>
<tr>
<td>U</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Each represents different aspects of partonic structure.

Each depends on Bjorken-$z$, transverse momentum, the scale.

Each function is to be studied.

---

TMD distributions

More at higher twist!

\[
\frac{1}{2} \text{Tr} \left[ 1 \Phi(x, k_{\perp}) \right] = \frac{M_N}{P^+} \left[ e - \frac{\varepsilon^{jk} k^j \cdot S_T^k}{M_N} e_T^\perp \right],
\]

\[
\frac{1}{2} \text{Tr} \left[ i \gamma_5 \Phi(x, k_{\perp}) \right] = \frac{M_N}{P^+} \left[ S_L e_L + \frac{k_{\perp} \cdot S_T}{M_N} e_T \right],
\]

\[
\frac{1}{2} \text{Tr} \left[ \gamma^j \Phi(x, k_{\perp}) \right] = \frac{M_N}{P^+} \left[ \frac{k^j}{M_N} f^\perp + \varepsilon^{jk} S_T^k f_T + S_L \frac{\varepsilon^{jk} k_{\perp}^k}{M_N} f_L^\perp - \frac{\varepsilon^{jk} \varepsilon^{kl} S_T^l}{M_N^2} f_T^\perp \right],
\]

\[
\frac{1}{2} \text{Tr} \left[ \gamma^j \gamma_5 \Phi(x, k_{\perp}) \right] = \frac{M_N}{P^+} \left[ \frac{S_T^j g_T}{M_N} + S_L \frac{k_{\perp}^j}{M_N} g_L^\perp + \frac{\kappa_{jk} k_{\perp}^k}{M_N^2} g_T^\perp + \frac{\varepsilon^{jk} k_{\perp}^k}{M_N} g^\perp \right],
\]

\[
\frac{1}{2} \text{Tr} \left[ i \sigma^{jk} \gamma_5 \Phi(x, k_{\perp}) \right] = \frac{M_N}{P^+} \left[ \frac{S_T^j k_{\perp}^k - S_T^k k_{\perp}^j}{M_N} h_T^\perp - \varepsilon^{jk} h \right],
\]

\[
\frac{1}{2} \text{Tr} \left[ i \sigma^{+-} \gamma_5 \Phi(x, k_{\perp}) \right] = \frac{M_N}{P^+} \left[ S_L h_L + \frac{k_{\perp} \cdot S_T}{M_N} h_T \right].
\]
More at higher twist!

\[
\frac{1}{2} \text{Tr} \left[ 1 \Phi(x, k_\perp) \right] = \frac{M_N}{P^+} \left[ e - \frac{\varepsilon^{jk} k^j}{k \cdot S_T} \right],
\]

\[
\frac{1}{2} \text{Tr} \left[ i \gamma_5 \Phi(x, k_\perp) \right] = \frac{M_N}{P^+} \left[ S_T \frac{k^j}{M_N} g_T + S_L \frac{\varepsilon^{jk} k^j}{M_N} f_L - \frac{\kappa^{jk} \varepsilon^{kl} S_L^l}{M_N^2} f_T^\perp \right],
\]

\[
\frac{1}{2} \text{Tr} \left[ \gamma^j \Phi(x, k_\perp) \right] = \frac{M_N}{P^+} \left[ \frac{S_T^j g_T^j}{M_N} + \frac{S_L^j g_L^j}{M_N} + \frac{\kappa^{jk} S_T^k}{M_N} g_T^\perp + \frac{\varepsilon^{jk} k^j}{M_N} g^\perp \right],
\]

\[
\frac{1}{2} \text{Tr} \left[ i \sigma^{jk} \Phi(x, k_\perp) \right] = \frac{M_N}{P^+} \left[ \frac{S_T^j k^j}{M_N} - \frac{S_T^j k^j}{M_N} h_T^\perp - \varepsilon^{jk} h \right],
\]

\[
\frac{1}{2} \text{Tr} \left[ x \Phi(x, k_\perp) \right] = \frac{M_N}{P^+} \left[ S_L h_L + \frac{k_\perp \cdot S_T}{M_N} h_T \right].
\]
We are in a good company
Semi Inclusive Deep Inelastic Scattering (SIDIS)
One can rewrite the cross-section in terms of 18 structure functions.

Each structure function encodes parton dynamics via convolutions of TMDs when factorization is applicable.

Mulders, Tangerman (1995),
Boer, Mulders (1998)
Bacchetta et al (2007)
Semi Inclusive Deep Inelastic scattering

\[ \ell P \rightarrow \ell' \pi X \]

One can rewrite the cross-section in terms of 18 structure functions.

Each structure function encodes parton dynamics via convolutions of TMDs when factorization is applicable.


\[
x = \frac{Q^2}{2 \, P \cdot q}, \quad y = \frac{P \cdot q}{P \cdot l}, \quad z = \frac{P \cdot P_h}{P \cdot q}, \quad Q^2 = -q^2
\]
Semi Inclusive Deep Inelastic scattering

\[ \ell P \rightarrow \ell' \pi X \]

One can rewrite the cross-section in terms of 18 structure functions

Each structure function encodes parton dynamics via convolutions of TMDs when factorization is applicable


The TMD factorization is valid in the region

\[ P_{hT}/z \ll Q \]

Interesting QCD regime, when recoil is happening from a low transverse momentum – important for studies of non perturbative physics.
Semi Inclusive Deep Inelastic scattering

\[ \ell P \to \ell' \pi X \]

One can rewrite the cross-section in terms of 18 structure functions.

Each structure function encodes parton dynamics via convolutions of TMDs when factorization is applicable.


The TMD factorization is valid in the region

\[ F \sim \int d^2 k_\perp d^2 p_\perp \delta^{(2)}(z k_\perp + p_\perp - \vec{P}_{hT}) \omega f(x, k_\perp) D(z, p_\perp) \]

Final transverse momentum is related to transverse momenta of parent and fragmenting partons.
What do we know about structure functions in SIDIS?
Sivers function
Non universal

Collins function
Universal
Definitions

Sivers function: unpolarized quark distribution inside a transversely polarized nucleon

\[
f_{q/h^\uparrow}(x, k_\perp, \vec{S}) = f_{q/h}(x, k_\perp^2) - \frac{1}{M} f_{1T}^{q}(x, k_\perp^2) \vec{S} \cdot (\vec{P} \times \vec{k}_\perp)
\]

Spin independent

Sivers 1989

Collins function: unpolarized hadron from a transversely polarized quark

\[
D_{q/h}(z, p_\perp, \vec{s}_q) = D_{q/h}(z, p_\perp^2) + \frac{1}{z M_h} H_{1}^{q}(z, p_\perp^2) \vec{s}_q \cdot (\vec{k} \times \vec{p}_\perp)
\]

Spin dependent

Collins 1992
Suppose the spin is along $Y$ direction:

Deformation in momentum space is:

This is the “dipole” deformation.
Suppose the spin is along Y direction: 
\[ S_T = (0, 1) \]
Deflection in momentum space is: 
\[ x \cdot f(x^2 + y^2) \]
This is the “dipole” deformation.

No correlation:

Correlation:
Tomographic scan of the nucleon

$\chi f_1(x, k_T, S_T)$

$u$ quark

$k_x$ (GeV) $k_y$

$d$ quark

$k_x$ (GeV) $k_y$

Anselmino et al 2009
Tomographic scan of the nucleon

Internal motion of quarks is correlated with the spin of the proton!
Definitions

Sivers function: $f_{1T}^{\perp q}$ describes strength of correlation

$$\vec{S} \cdot (\hat{P} \times \vec{k}_\perp)$$

Sivers 1989

Collins function: $H_1^{\perp q}$ describes strength of correlation

$$\vec{s}_q \cdot (\hat{k} \times \vec{p}_\perp)$$

Collins 1992

Both functions extensively studied experimentally, phenomenologically, theoretically

$$\ell P \to \ell' \pi X$$

Sivers function and Collins function can give rise to Single Spin Asymmetries in scattering processes. For instance in Semi Inclusive Deep Inelastic process

$$d\sigma(S') \sim \sin(\phi_h + \phi_S) h_1 \otimes H_1^{\perp} + \sin(\phi_h - \phi_S) f_{1T}^{\perp} \otimes D_1 + ...$$

Sivers function

Large – $N_c$ result

\[ f_{1T}^u = - f_{1T}^d \]  

\[ \Rightarrow \text{Confirmed by phenomenological extractions} \]

\[ \Rightarrow \text{Confirmed by experimental measurements} \]

Relation to GPDs (E) and anomalous magnetic moment

\[ f_{1T}^q \sim \kappa q \]

\[ \Rightarrow \text{Predicted correct sign of Sivers asymmetry in SIDIS} \]

\[ \Rightarrow \text{Shown to be model-dependent} \]

\[ \Rightarrow \text{Used in phenomenological extractions} \]
Sivers function

Sum rule

→ Conservation of transverse momentum

→ Average transverse momentum shift of a quark inside a transversely polarized nucleon

\[ \langle k^i_T, q \rangle = \varepsilon^{ij}_T S^j_T f_{1T}^{(1)}q(x) \]

\[ f_{1T}^{(1)}q(x) = \int d^2 k_\perp \frac{k_\perp^2}{2M^2} f_{1T}^{\perp}q(x, k_\perp^2) \]

→ Sum rule

\[ \sum_{a=q,g} \int_{0}^{1} dx \langle k^i_T, a \rangle = 0 \]
\[ \sum_{a=q,g} \int_{0}^{1} dx f_{1T}^{\perp(1)a}(x) = 0 \]
Colored objects are surrounded by gluons, profound consequence of gauge invariance: Sivers function has opposite sign when gluon couple after quark scatters (SIDIS) or before quark annihilates (Drell-Yan)

\[ f_{1T}^{\perp \text{SIDIS}} = - f_{1T}^{\perp \text{DY}} \]

Crucial test of TMD factorization and collinear twist-3 factorization
Several labs worldwide aim at measurement of Sivers effect in Drell-Yan
BNL, CERN, GSI, IHEP, JINR, FERMILAB etc
Barone et al., Anselmino et al., Yuan,Vogelsang, Schlegel et al., Kang,Qiu, Metz,Zhou etc
The verification of the sign change is an NSAC (DOE and NSF) milestone
First experimental hint on the sign change: $A_N$ in $W$ and $Z$ production

$\rightarrow$ Sign change: $\chi^2/d.o.f. \sim 1.2$

$\rightarrow$ No sign change: $\chi^2/d.o.f. \sim 3.2$

Large uncertainties of predictions

No antiquark Sivers functions
Future SIDIS and DY data related to target-spin-independent Boer-Mulders asymmetries. TMD PDFs between SIDIS and DY as predicted by QCD related to the pion Boer-Mulders PDFs, the obtained results may be used to study this function further and zero with a significance of about one standard deviation. Since both TSA calculations of Ref. #9 and can be used to study the universality of the nucleon transversity function.

The TSA of Ref. #37 is compared with recent theoretical predictions from Refs. #19. The average value for the TSA is measured to be above zero at about one standard deviation of the total uncertainty. In Fig. 6, the measured mean Sivers asymmetry and the theoretical predictions for different schemes from Refs. #20, which is related to the nucleon pretzelosity TMD PDFs, is measured to be above zero. First measurement of transverse-spin-dependent azimuthal asymmetries.

→ First experimental hint on the sign change in Drell-Yan

→ Sign change

→ No sign change

→ COMPASS results hint on sign change
Collins function

Schafer-Teryaev sum rule

→ Conservation of transverse momentum

\[ \langle P_T^i(z) \rangle \sim H_1^{\perp (1)}(z) \quad H_1^{\perp (1)}(z) = \int d^2p_\perp \frac{p_\perp^2}{2z^2M_h^2} H_1^{\perp}(z, p_\perp^2) \]

→ Sum rule

\[ \sum_h \int_0^1 dz \langle P_T^i(z) \rangle = 0 \]

→ If only pions are considered

Universality of TMD fragmentation functions

\[ H_1^{\perp f_{\text{av}}}(z) \sim -H_1^{\perp \text{unf}}(z) \]

Gamberg, Mukherjee, Mulders 2011
Boer, Kang, Vogelsang, Yuan 2010

→ Very non trivial results

→ Agrees with phenomenology, allows global fits
SIDIS and $e^+e^-$: combined global analysis

\[
\frac{d\sigma(S_{\perp})}{dx_B dy d z_h d^2 P_{h\perp}} = \sigma_0(x_B, y, Q^2) \left[ F_{UU} + \sin(\phi_h + \phi_s) \frac{2(1-y)}{1+(1-y)^2} F_{UT}^{\sin(\phi_h+\phi_s)} + ... \right]
\]

\[
F_{UT}^{\sin(\phi_h+\phi_s)} \sim h_1(x_B, k_{\perp}) H_1^\perp(z_h, p_{\perp})
\]
thransversity
 Collins function

\[
Z_{\text{collins}}^{h_1 h_2} \sim H_1^\perp(z_1, p_{1\perp}) H_1^\perp(z_2, p_{2\perp})
\]
Collins function
Collins function

\[
\frac{d\sigma^{e^+e^- \rightarrow h_1 h_2 + X}}{dz_{h_1} dz_{h_2} d^2 P_{h\perp} d \cos \theta} = \frac{N_c \pi \alpha_{em}^2}{2Q^2} \left[ (1 + \cos^2 \theta) Z_{uu}^{h_1 h_2} + \sin^2 \theta \cos(2\phi_0) Z_{\text{collins}}^{h_1 h_2} \right]
\]
Fitted quark transversity and Collins function: $x (z)$-dependence

Collins function: $p_T$-dependence

Compatible with LO extraction
Various processes allow study and test of evolution, universality and extractions of distribution and fragmentation functions. We need information from all of them

\[ f(x) \otimes D(z) \]  
Semi Inclusive DIS – convolution of distribution functions and fragmentation functions

\[ \ell + P \rightarrow \ell' + h + X \]

\[ f(x_1) \otimes f(x_2) \]
Drell-Yan – convolution of distribution functions

\[ P_1 + P_2 \rightarrow \ell \ell + X \]

\[ D(z_1) \otimes D(z_2) \]
e+ e- annihilation – convolution of fragmentation functions

\[ \bar{\ell} + \ell \rightarrow h_1 + h_2 + X \]

\[ f(x_1) \otimes f(x_2) \otimes D(z) \]
Hadron-hadron – convolutions of PDF and fragmentation functions

\[ h_1 + h_2 \rightarrow h_3(\gamma, jet, W, \ldots) + X \]

Combining measurements from all above is important
Why TMDs, factorization, and evolution
Why QCD evolution is interesting?

Study of evolution gives us insight on different aspects and origin of confined motion of partons, gluon radiation, parton fragmentation.

Evolution allows to connect measurements at very different scales.

TMD evolution has also a universal non-perturbative part. The result of evolution cannot be uniquely predicted using evolution equations until the non-perturbative part is reliably extracted from the data.
TMD factorization

Collins, Soper (1983)
Collins, Soper, Sterman (1985)
Collins (2011)

Collins (2011)

Collins, Soper (1983)
Collins, Soper, Sterman (1985)
Collins (2011)

Collins (2011)

Meng, Olnes, Soper (1992)
Ji, Ma, Yuan (2004)
Collins (2011)
TMD factorization

**Collins, Soper (1983)**
**Collins, Soper, Sterman (1985)**
**Collins (2011)**

**TMD evolution equations**

- e^+e^-
  - electron
  - positron
  - pion

- lepton
  - proton
  - lepton
  - pion

- PROTON
  - proton
  - positron

- SIDIS
  - electron
  - proton

- PP
  - proton
  - pion

Meng, Olness, Soper (1992)
Ji, Ma, Yuan (2005)
Collins (2011)

Qiu, Sterman (1990)

**Only one scale is measured in PP**
TMD factorization is not applicable?
TMD factorization

Collins, Soper (1983)
Collins (2011)

Collins, Soper, Sterman (1985)
Collins (2011)

Collins, Soper (1983)
Collins, Soper, Sterman (1985)
Collins (2011)

Collins, Soper (1983)
Collins, Soper, Sterman (1985)
Collins (2011)

Twist-3 factorization
DGLAP equations

Collins, Soper (1983)
Collins, Soper, Sterman (1985)
Collins (2011)

Meng, Olness, Soper (1992)
Ji, Ma, Yuan (2005)

Collins (2011)

• Twist-3 functions are related to TMD via OPE
• TMD and twist-3 factorizations are related in high QT region
• Global analysis of TMDs and twist-3 is possible:
  All four processes can be used.
• Data are from HERMES, COMPASS, JLab,
  BaBar, Belle, RHIC, LHC, Fermilab

Qiu, Sterman (1990)

Global fit is needed.
Work in progress
Why TMD Evolution?
Factorization of regions: (1) $k//P_1$, (2) $k//P_2$, (3) $k$ soft, (4) $k$ hard

Drell-Yan:

\[ p + p \rightarrow [\gamma^* \rightarrow \ell^+ \ell^-] + X \]

Factorized form and mimicking “parton model”

\[
\frac{d\sigma}{dQ^2 dy d^2 q_{\perp}} \propto \int d^2 k_{1\perp} d^2 k_{2\perp} d^2 \lambda_{\perp} H(Q) f(x_1, k_{1\perp}) f(x_2, k_{2\perp}) S(\lambda_{\perp}) \delta^2(k_{1\perp} + k_{2\perp} + \lambda_{\perp} - q_{\perp}) 
\]

\[
= \int \frac{d^2 b}{(2\pi)^2} e^{i q_{\perp} \cdot b} H(Q) f(x_1, b) f(x_2, b) S(b)
\]

\[
= \int \frac{d^2 b}{(2\pi)^2} e^{i q_{\perp} \cdot b} H(Q) F(x_1, b) F(x_2, b) 
\]

mimic “parton model”
TMDs evolve

Just like collinear PDFs, TMDs also depend on the scale of the probe = evolution

Collinear PDFs

\[ F(x, Q) \]

✓ DGLAP evolution
✓ Resum \[ \left[ \alpha_s \ln \left( \frac{Q^2}{\mu^2} \right) \right]^n \]
✓ Kernel: purely perturbative

TMDs

\[ F(x, k_\perp; Q) \]

✓ Collins-Soper/rapidity evolution equation
✓ Resum \[ \left[ \alpha_s \ln^2 \left( \frac{Q^2}{k_\perp^2} \right) \right]^n \]
✓ Kernel: can be non-perturbative when \( k_\perp \sim \Lambda_{QCD} \)

\[ F(x, Q_i) \]
\[ R^{coll}(x, Q_i, Q_f) \]
\[ F(x, Q_f) \]
\[ F(x, k_\perp, Q_i) \]
\[ R^{TMD}(x, k_\perp, Q_i, Q_f) \]
\[ F(x, k_\perp, Q_f) \]
TMD evolution and non-perturbative component

Fourier transform back to the momentum space, one needs the whole b region (large b): need some non-perturbative extrapolation

Many different methods/proposals to model this non-perturbative part

\[ F(x, k_\perp; Q) = \frac{1}{(2\pi)^2} \int d^2 b e^{ik_\perp \cdot b} F(x, b; Q) = \frac{1}{2\pi} \int_0^\infty db b J_0(k_\perp b) F(x, b; Q) \]

Collins, Soper, Sterman 85, ResBos, Qiu, Zhang 99, Echevarria, Idilbi, Kang, Vitev, 14, Aidala, Field, Gamberg, Rogers, 14, Sun, Yuan 14, D’Alesio, Echevarria, Melis, Scimemi, 14, Rogers, Collins, 15, Vladimirov, Scimemi 17...

Eventually evolved TMDs in b-space

Since the polarized scattering data is still limited kinematics, we can use unpolarized data to constrain/extract key ingredients for the non-perturbative part

✓ Non-perturbative: fitted from data
✓ The key ingredient – \( \ln(Q) \) piece is spin-independent
TMD factorization has a validity region \( P_{hT}/z \ll Q \) (two scale problem)

In order to describe cross section in a wide region of transverse momentum one needs to add a \( Y \) term

\[
\frac{d\sigma}{dP_{hT}} = \left( \frac{d\sigma}{dP_{hT}} \right)_{TMD} + Y
\]

Improved approach that aims to describe low-\( Q \) region:

\( P_{hT}/z \ll Q \)

It seems too easy...
In fact it is not easy...

If we consider NLO corrections this diagram diverges

\[ \ell = (1 - \alpha)p \]

\[ \propto \int_0^1 d\alpha \frac{\alpha}{1 - \alpha} \]
In fact it is not easy...

If we consider NLO corrections this diagram diverges

\[ \ell = (1 - \alpha) p \]

\[ \propto \int_0^1 d\alpha \frac{\alpha}{1 - \alpha} \]
In fact it is not easy...

If we consider NLO corrections this diagram diverges

\[ \ell = (1 - \alpha)p \]

\[ \alpha \int_0^1 d\alpha \frac{\alpha}{1 - \alpha} \]

**Physics:** The gluon becomes collinear to the Wilson line (struck quark) and its rapidity goes to \(-\infty\)

“Rapidity divergence”
In fact it is not easy...

We know how to deal with it:

\[ \propto \int_0^1 d\alpha \frac{\alpha}{(1 - \alpha)_+} T(\alpha) = \]

\[ = \int_0^1 d\alpha \frac{\alpha T(\alpha) - T(1)}{(1 - \alpha)} \]

“+ prescription”

\[ T(\alpha = 1) - T(1) = 0 \]
In fact it is not easy...

Not working for TMDs:

\[ \alpha \int_{0}^{1} d\alpha \frac{\alpha}{(1 - \alpha)_+} T(\alpha, k_{\perp}) = \]

\[ = \int_{0}^{1} d\alpha \frac{\alpha T(\alpha, k_{\perp}) - T(1, 0_{\perp})}{(1 - \alpha)} \]

“+ prescription”

\[ T(\alpha = 1, k_{\perp}) - T(1, 0_{\perp}) \neq 0 \]

TMD related studies have been extremely active in the past few years, lots of progress have been made

We look forward to the future experimental results from COMPASS, RHIC, Jefferson Lab, LHC, Fermilab, future Electron Ion Collider

Many TMD related groups are created throughout the world:
Italy, Netherlands, Belgium, Germany, Japan, China, Russia, and the USA