GROUP THEORY APPROACH

TO SCATTERING

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ABSTRACT

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For certain physical systems, there exists a dynamical group which contains the operators connecting states with the same energy but belonging to potentials with different strengths. This group is called the potential group of that system. The SO(2,1) potential group structure is introduced to describe physical systems with mixed spectra, such as Morse and Pöschl-Teller potentials. The discrete spectrum describes bound states and the continuous spectrum describes scattering states. A solvable class of one-dimensional potentials given by Natanzon belongs to this structure with an SO(2,2) potential group. The potential group structure provides us with an algebraic procedure generating the recursion relations for the scattering matrix, which can be formulated in a purely algebraic fashion, divorced from any differential realization. This procedure, when applied to the three-dimensional scattering problem with SO(3,1) symmetry, generates the scattering matrix of the Coulomb problem. Preliminary phenomenological models for elastic scattering in a heavy-ion collision are constructed on this basis. The results obtained here can be regarded as an important extension of the group theory techniques to scattering problems similar to that developed for bound state problems.
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Chapter 1.

Introduction

It has been a century since Marius Sophus Lie began his research on what has evolved into one of the most beautiful and fruitful branches of modern mathematics—the theory of Lie groups. At first, it was unclear whether Lie groups could be of possible use to natural scientists. However, in this century, Lie groups have come to play an increasingly important role in modern physical theories.

Lie groups found their way into physics primarily through the development of modern quantum theory. In this theory, physical observables appear in the form of hermitian operators (matrices), while processes are described by unitary transformations generated by hermitian operators. Operators that close under commutation form a finite dimensional Lie algebra, while transformations described by a finite number of continuous parameters that close under multiplication belong to a Lie group.

Applications of Lie groups were first introduced through symmetry groups. Symmetry means degeneracy. The greater the symmetry, the greater the degeneracy. The symmetry group $G$ (with a Lie algebra $g$) of a system with a Hamiltonian $H$ is defined by the invariance of the Hamiltonian $H$ under transformations of the group $G$, i.e.

$$G H G^{-1} = H,$$

or equivalently,

$$[H, g] = 0.$$

Then, by the Wigner theorem, the state vectors spanning a fixed energy eigenspace carry a representation of the group $G$. No operator in the Lie algebra $g$ can connect state vectors belonging to different representations of the Lie group $G$. Therefore, symmetry groups are very useful in explaining level degeneracy and selection rules. Symmetry methods have been successfully applied in various fields of physics.

Gradually, physicists came to realize that dynamical groups, which contain additional operators that do not commute with the Hamiltonian $H$ can be even more useful than
symmetry groups from a computational viewpoint. Using dynamical groups we can reach any eigenstate from any other state by applying a sequence of elements in the Lie algebra. Thus, all excited states can be computed from the ground state, which, in turn, can be computed by variational methods. Therefore, dynamical groups are useful in generating energy spectra and calculating transition matrix elements. Purely algebraic models of physical systems have been used in the description of collective states in nuclei and molecules.

However, most applications have dealt with the bound state problem and the groups used are compact, so that their unitary representations could reproduce the observed discrete and finite dimensional spectra. In a recent series of papers, both bound and scattering states of certain one-dimensional potentials have been shown to be related to unitary representations of certain compact groups and their analytical continuations to non-compact groups. These papers provide the first steps towards the extension of group techniques to the description of systems which have both discrete and continuous spectra.

In this series of papers, the authors point out, that, besides symmetry groups and dynamical groups, there exist other kinds of groups, called "potential groups". The potential group of a physical system contains operators which connect states with the same energy but belonging to potentials with different strengths. In this thesis we are primarily concerned with the potential group approach to scattering problems. Since the scattering states are associated with a continuous spectrum, the Hamiltonian can not be a compact operator. The use of non-compact groups is a novel feature in the algebraic description of scattering states. Therefore, the algebraic techniques needed in this approach are quite different from those used in bound state problems.

We have found that several one-dimensional problems can be described by the potential group $SO(2, 1)$. The $SO(2, 1)$ algebra consists of three generators $J_1, J_2$ and $J_3$ satisfying the commutation relations

$$
\begin{align*}
[J_1, J_2] &= -iJ_3, \\
[J_2, J_3] &= iJ_1, \\
[J_3, J_1] &= iJ_2,
\end{align*}
$$

(1.1)
where $J_3$ is a compact generator. The $SO(2,1)$ algebra can be regarded as an analytic continuation of the $SO(3)$ algebra. The difference between these two algebra lies in the sign in the first commutation relation in Eq.(1.1), which has a major effect on the topological properties of these two groups. The $SO(3)$ group leaves the sphere in three-space $x_1^2 + x_2^2 + x_3^2 = 1$ invariant, while the $SO(2,1)$ group leaves the hyperboloid $x_1^2 + x_2^2 - x_3^2 = 1$ invariant. Fig.(1.1) shows the manifolds of the sphere and the hyperboloid. The unboundedness of the hyperboloid reflects the non-compactness of the group.

The $SO(2,1)$ Casimir invariant is

$$C_2 = -J_1^2 - J_2^2 + J_3^2.$$  \hspace{1cm} (1.2)

For a physical system with a dynamical symmetry $SO(2,1) \supset SO(2)$ the Hamiltonian $H$ can be diagonalized in a basis $|j, m\rangle$ defined by

$$\begin{align*}
C_2 |j, m\rangle &= j(j + 1)|j, m\rangle, \\
J_3 |j, m\rangle &= m|j, m\rangle. \\
\end{align*}$$  \hspace{1cm} (1.3)

The system with the structure of $SO(2,1)$ potential group has the specific feature that the Hamiltonian involves only the Casimir operator. For example, consider the system whose Hamiltonian can be written as

$$H = -(C_2 + \frac{1}{4}).$$  \hspace{1cm} (1.4)

The non-compact group $SO(2,1)$ has both a discrete principal series and a continuous principal series of unitary representations. For the continuous principal series representations

$$j = -\frac{1}{2} + ik, (0 < k < \infty),$$

the energy spectrum is

$$\langle H \rangle = -(j + \frac{1}{2})^2 = k^2 > 0.$$

The basis for this continuous representation can be shown to correspond to the scattering states in the physical problem. For the discrete principal series representations

$$j = -1/2 - s, s = 0, \frac{1}{2}, 1, \ldots,$$
the energy spectrum is
\[ \langle H \rangle = -(j + \frac{1}{2})^2 = -s^2 < 0, \]
and it corresponds to the bound states of the Schrödinger problem. Therefore, both bound and scattering states can be described by unitary representations of the same group, and they form a complete set of state vectors. As we shall see, the condition for the representation being in the discrete series can generate the bound state spectrum; the continuous series of representations lead to recursion relations derived in a purely algebraic fashion from which the scattering matrix can be obtained\[^{11}\].

In chapters 2 and 3, we will realize a lot of solvable potentials with potential groups $SO(2,1) \simeq SU(1,1)$ and $SO(2,2)$. These potentials have many applications in modern physics. For example, Eckart potential is used in the electron penetration of a potential barrier\[^{20}\]; Morse potential has been extensively applied to the molecule-molecule interaction\[^{17}\]; Ginocchio potential finds applications in describing the mean field of the nuclear shell model\[^{14}\]; etc.. Some of them have been realized with dynamical groups. However, the potential group approach is the only way to provide us with a mixed spectrum. Therefore, we expect that the study of potential group can be of great help to the understanding of physical systems with mixed spectra.

While in a bound state problem we are mostly interested in the spectrum and transition matrix elements, in a scattering problem the quantity of interest is the scattering matrix. In the usual potential approach, the scattering matrix is defined through the asymptotic behavior of the scattering states. But in the framework in which a state is described as an abstract vector, it was not known before how to incorporate the notion of an asymptotic limit. In an earlier treatment\[^{10}\] of the Coulomb problem, the only known example in which the scattering matrix was calculated algebraically, the difficulty was avoided, but in a way that could not be generalized to other problems. More recently\[^{8}\], a dynamical $SU(1,1)$ potential group was used to derive the scattering matrix for the one-dimensional Pöschl-Teller potential. However, the method employed was not completely algebraic because in describing the asymptotic behavior of the scattering states the algebraic language was abandoned and use was made of an explicit realization of $SU(1,1)$. 

\[^{11}\]
In this thesis we construct an algebraic framework to characterize the asymptotic behavior of the scattering state vectors. Since scattering states behave asymptotically like free waves and the Euclidean group contains the symmetry operations (translations and rotations) that leave the free-particle energy invariant, it is quite natural to use the representations of the Euclidean group to describe the asymptotic limit of the scattering states. The interplay of the dynamical potential group and the Euclidean group provides us with the recursion relations from which the scattering matrix can be calculated in a completely algebraic way.

This procedure can be generalized to models in which only a dynamical group Hamiltonian is specified and no differential Schrödinger equation is available. For the one-dimensional problem a general form (up to quadratic terms) of the $SU(1,1)$ generators in terms of the generators of the Euclidean group $E(2)$ is derived, which bridges the representations of the dynamical potential group in the asymptotic limit and the Euclidean group and leads to a general form of the reflection amplitude for the one-dimensional problem with an $SU(1,1) \supset O(2)$ dynamical symmetry.

It is obvious that our formulation of the asymptotic limit using the Euclidean group holds true for scattering problems with symmetry groups, since symmetry groups also leave the energy invariant. The extension of this method to three-dimensional Coulomb scattering problem where the relevant groups are $O(3,1)$ and $E(3)$ provides us with an example that generalizes this algebraic technique to higher dimensions. The general form (up to quadratic terms) for the problem with an $O(3,1)$ symmetry is also derived to relate the symmetry group and the Euclidean group, such that three-dimensional elastic scattering models can be constructed separate from any differential realization.

Reports in this paper are only the second steps towards the extension of group techniques to scattering problems. The research in this approach is still progressing rapidly. It is expected that, once this approach becomes better understood theoretically, it will be possible to construct phenomenological models of nuclear and atomic scattering, that will be the counterpart of the algebraic models used in the study of bound state spectra. At the
present stage only some preliminary scattering models are discussed. The relevant potential
groups are $SU(3,1)$ and $SO(3,2)$.

The structure of this thesis is as follows:

Following the introduction is a chapter devoted to one-dimensional problems with the
potential group $SO(2,1) \simeq SU(1,1)$. The differential realizations of the group $SO(2,1)$
appropriate to some one-dimensional potentials, such as Morse potential, Pöschl-Teller po-
tential and Ginocchio potential [112] are first introduced. Then we discuss the algebraic
methods to generate the bound state spectra and the scattering matrices. Finally, the
algebraic procedure for obtaining the scattering matrix is divorced from any differential
realization after the general form (up to quadratic terms) of the $SO(2,1)$ structure is de-
\riv ed. The ideas developed in Chapter 2 form the basis of all the discussion in the following
chapters.

In Chapter 3 the algebraic method is generalized to the potential group $SO(2,2)$. We first introduce the $SO(2,2)$ structure of the hypergeometric equation. A solvable class
of one-dimensional potential problems associated with hypergeometric functions, such as
Natanzon and Ginocchio potentials [13],[14], are, therefore, realized with the same structure.
Since Ginocchio interpreted his three-parameter potential as a three-dimensional potential,
we improve the realization with the $SO(3,2)$ group, where the three dimensional angular
momentum is explicit, and reexamine some of the discussion of the Ginocchio potential
from the viewpoint of the representation theory of the $SO(3,2)$ group.

Chapter 4 is devoted to the generalization of the group technique to higher dimensions.
Specific attention is paid to the Coulomb scattering problem and the general form of the
$SO(3,1)$ symmetry. A few scattering models with dynamical potential groups $SU(3,1)$ and
$SO(3,2)$ are discussed in Chapter 5. Open problems in the group theory approach to scat-
tering are pointed out in the conclusion. The representation theory for some non-compact
groups [16],[23],[24],[27],[29] referred to in the discussion is summarized in the appendices.
Fig. (1.1). The compact group $SO(3)$ leaves the sphere invariant (left), while the non-compact group $SO(2,1)$ leaves the hyperboloid invariant (right).
Chapter 2.

The Potential Group SO(2,1)

The success of the algebraic approach to bound state problems (especially the interacting boson models) raises the question of whether a similar approach can be useful in scattering problems. However, before such practical algebraic scattering models are found, one has to learn how to extend the algebraic approach to the continuum. In order to do that, it is very useful to study simple potential scattering problems for which the exact connection between the Schrödinger equation and a group representation can be found. By studying the algebraic description of both bound and scattering states of such problems we are able to formulate a group theory approach to scattering. Therefore, in this chapter, we shall first introduce some differential realizations of the potential group SO(2,1) (≈ SU(1,1)), which are associated with the one-dimensional Schrödinger equation with certain solvable potentials, such as the Morse potential, the Pöschl-Teller potential and the first class of Ginocchio potentials. Then, we shall see how both bound and scattering states appear as a representation basis of this group. Using these examples, we shall develop a technique to calculate the scattering matrix in an abstract algebraic fashion, divorced from any specific differential realization. Thus, the scattering matrix for a general SO(2,1) dynamical symmetry problem can be obtained.

It turns out that the potential group approach can be related to the factorization method introduced by Infeld and Hull[15] in the context of solving differential equation (although they did not use it for scattering problems). We shall also discuss this connection.

2.1. Differential Realizations of the SO(2,1) Potential Group

In the paper "The Factorization Method"[15], it is shown that certain classes of second order differential operators can be "factorized" as products of two first order differential operators. This method can be applied to a great variety of equations in mathematical physics and proves to be a powerful tool for determining the eigenfunctions of these equations.
Infeld and Hull classify six different types of factorizations, A–F. It turns out\[16\] that the six factorization types are obtainable from a study of four Lie algebras:

1. $O(3)$—the Lie algebra of the rotation group;
2. $E(2)$—the Lie algebra of the Euclidean group in the two-space;
3. $H(4)$—a four-dimensional solvable Lie algebra;
4. $E(3)$—the Lie algebra of the Euclidean group in the three-space.

We will show that a further extension of this method to the $SO(2,1)$ algebra—the Lie algebra of the (2,1) space leads to the differential realizations of one-dimensional Schrödinger equations with Morse and Pöschl-Teller potentials.

In Reference [16], a four-dimensional Lie algebra $G(a,b)$ is introduced, which contains four generators $J_+, J_-, J_3$ and $I$. They satisfy the following commutation relations:

\[
\begin{align*}
[J_3, J_\pm] &= \pm J_\pm, \\
[J_+, J_-] &= 2a^2 J_3 - 2b I, \\
[J_\pm, I] &= 0, \\
[J_3, I] &= 0.
\end{align*}
\]

(2.1.1)

The Casimir operator of the Lie algebra $G(a,b)$, which commutes with every element in $G(a,b)$, is

\[
C_{a,b} = J_+ J_- + a^2 J^2_3 - (2b + a^2) J_3.
\]

(2.1.2)

It is easy to see that

\[
G(0,0) \cong E(2) \oplus (I);
\]

\[
G(1,0) \cong O(3) \oplus (I);
\]

\[
G(0,1/2) \cong H_4;
\]

where $(I)$ denotes the one-dimensional Lie algebra generated by $I$. A well known example of $H_4$ is the algebra formed by boson operators $b^\dagger, b, b^\dagger b$ and $I$.

If we further continue "$a$" to complex values, then we have

\[
G(i,0) \cong SO(2,1) \oplus (I).
\]

Since $a^2 = -1$ and $b = 0$, the corresponding algebra formed by $J_\pm$ and $J_3$ is $SO(2,1)$, i.e.

\[
\begin{align*}
[J_+, J_-] &= -2J_3, \\
[J_3, J_\pm] &= \pm J_\pm.
\end{align*}
\]

(2.1.3)
Consider the differential realization:
\[
\begin{align*}
J_\pm &= e^{\pm i\phi} \left[ \pm \frac{\partial}{\partial x} + k_1(x)(i \frac{\partial}{\partial \phi} \mp \frac{1}{2}) + k_0(x) \right], \\
J_3 &= -i \frac{\partial}{\partial \phi},
\end{align*}
\]
(2.1.4)
where \(k_1(x)\) and \(k_0(x)\) are to be determined and the numbers "\(\mp \frac{1}{2}\)" are introduced to make \(J_+\) and \(J_-\) hermitian conjugate to each other, i.e.
\[
J_- = (J_+)^\dagger.
\]
(2.1.5)

If we require that \(J_\pm, J_3\) form an \(SO(2,1)\) algebra Eq.(2.1.3) has to be satisfied. It is obvious that for arbitrary differentiable functions \(k_0(x)\) and \(k_1(x)\) the second commutation relation of Eq.(2.1.3) can always be satisfied, i.e.
\[
[J_3, J_\pm] = \pm J_\pm.
\]

But if we require
\[
[J_+, J_-] = -2J_3,
\]
\(k_0(x)\) and \(k_1(x)\) have to satisfy certain conditions.

It can be shown directly that
\[
[J_+, J_-] = 2 \left[ \frac{dk_1(x)}{dx} + k_1^2(x) \frac{\partial}{\partial \phi} \right] + 2 \left[ \frac{dk_0(x)}{dx} + k_0(x)k_1(x) \right].
\]

If this expression is required equal to "\(-2J_3\)" we obtain the equations:
\[
\begin{align*}
\frac{dk_1(x)}{dx} + k_1^2(x) &= 1, \\
\frac{dk_0(x)}{dx} + k_0(x)k_1(x) &= 0.
\end{align*}
\]
(2.1.6)

If \(k_0(x)\) and \(k_1(x)\) satisfy Eq.(2.1.6) the Casimir operator of the \(SO(2,1)\) algebra is
\[
C_2 = \frac{\partial^2}{\partial x^2} + [k_1^2(x) - 1] \frac{\partial^2}{\partial \phi^2} + \frac{1}{4} + 2 \frac{dk_0(x)}{dx} (i \frac{\partial}{\partial \phi}) - k_0^2(x) - \frac{1}{4},
\]
(2.1.7)
where \(C_2\) is defined to be "\(-C_{1,0}\)" i.e.
\[
C_2 = J_3^2 - J_3 - J_+J_-.
\]
(2.1.8)
Differential Realizations of the $SO(2,1)$ Potential Group

It can be shown that the most general solution for $k_1(x)$ is

$$k_1(x) = \begin{cases} \tanh(x - c), & \text{for } k_1(x) < 1; \\ 1, & \text{for } k_1(x) = 1; \\ \coth(x - c), & \text{for } k_1(x) > 1. \end{cases}$$

Then, the second equation of Eq.(2.1.6) is reduced to a linear equation and can be solved accordingly. Now, consider several simple solutions of Eq.(2.1.6).

(1) $k_1(x) = 1$:

It is easy to check that it satisfies the first equation of Eq.(2.1.6) and the second equation of Eq.(2.1.6) is reduced to

$$\frac{dk_0(x)}{dx} + k_0(x) = 0. \quad (2.1.9)$$

Eq.(2.1.9) has a general solution

$$k_0(x) = ce^{-x}.$$ 

Without losing generality we can set $c = 1$. Therefore, we have

$$k_0(x) = e^{-x}.$$ 

and

$$C_2 + \frac{1}{4} = \frac{\partial^2}{\partial x^2} - [e^{-2x} - 2e^{-x}(-i \frac{\partial}{\partial \phi})].$$

If the Hamiltonian of a physical system is

$$H = -(C_2 + \frac{1}{4}),$$

the simultaneous eigenfunction of $H$ and $J_3$

$$\Psi_m = \psi_m(x)e^{im\phi}$$

will lead to an equation for $\psi_m(x)$, i.e.

$$H\Psi_m = E\Psi_m = [-\frac{\partial^2}{\partial x^2} + e^{-2x} - 2e^{-x}(-i \frac{\partial}{\partial \phi})]\psi_m(x)e^{im\phi}.$$
That is
\[
[- \frac{d^2}{dx^2} + (e^{-2x} - 2me^{-x})] \psi_m(x) = E \psi_m(x).
\] (2.1.10)

After a shift of the origin: \( x = x' - ln m \), we have
\[
[- \frac{d^2}{dx'^2} + m^2(e^{-2x'} - 2e^{-x'})] \psi_m(x') = E \psi_m(x'),
\] (2.1.11)

which is just the one-dimensional Schrödinger equation with a Morse potential.

(2) \( k_1(x) = \tanh x \):

This is another simple solution of the first equation of Eq.(2.1.6) and there always exists a trivial solution to the second equation, i.e.
\[
k_0(x) = 0.
\]

Accordingly, we have
\[
C_2 = \frac{\partial^2}{\partial x^2} + \frac{1}{\cosh^2 x}(- \frac{\partial^2}{\partial \phi^2} - \frac{1}{4}) - \frac{1}{4}.
\]

If we set the Hamiltonian equal to \( -(C_2 + \frac{1}{4}) \) and consider the simultaneous eigenfunction of \( H \) and \( J_3 \)
\[
\Psi_m = \psi_m(x)e^{im\phi},
\]
we obtain the one-dimensional Schrödinger equation with Pöschl-Teller potential
\[
[- \frac{d^2}{dx^2} - \frac{m^2 - 1/4}{\cosh^2 x}] \psi_m(x) = E \psi_m(x).
\] (2.1.12)

In the two examples shown above we can easily see the structure of a potential group.

Since the Hamiltonian is related only to the Casimir operator of the \( SO(2,1) \) algebra, all the state vectors in a representation (multiplet) of \( SO(2,1) \) are at the same energy. Different states in a multiplet correspond to states for different potential strengths at a given energy.

For the \( SO(2,1) \) algebra we know that the eigenvalue of the Casimir operator is \( j(j + 1) \), where
\[
j = \begin{cases} 
-1/2 + ik, & 0 < k < \infty \\
-1/2 - n/2, & n = 0, 1, 2, 3, \ldots,
\end{cases}
\]
for the continuous principal series;

for the discrete principal series.
Therefore, when the potential strength is given, the Hamiltonian $H$ has a continuous spectrum, which describes the scattering states, for the continuous principal series, i.e.

$$\langle H \rangle = k^2, 0 < k^2 < \infty,$$

and a discrete spectrum, which describes the bound states, for the discrete principal series, i.e.

$$\langle H \rangle = -\frac{n^2}{4}, n = 0, 1, 2, \ldots$$

In Figs. (2.1.1) and (2.1.2), several Morse and Pöschl-Teller potentials are plotted and the $SO(2, 1)$ multiplets are shown by the horizontal dashed lines. In Fig. (2.1.3) the comparison between a Morse potential and a Pöschl-Teller potential with the same spectrum is shown.

Besides these, there exists another simple solution, which will be of use in the next chapter.

(3) $k_1(x) = \coth x$:

For the trivial solution $k_0(x) = 0$ and the Hamiltonian $H = -(C_2 + 1/4)$, the corresponding one-dimensional Schrödinger equation is

$$[-\frac{d^2}{dx^2} + \frac{m^2 - 1/4}{\sinh^2 x}] \psi_m(x) = E \psi_m(x). \quad (2.1.13)$$

In this special case there exist only scattering states which correspond to the continuous principal series of the $SO(2, 1)$ representations. For a more general $k_0(x)$, which will be discussed in the next chapter, there also exist bound states corresponding to the discrete principal series.
MORSE POTENTIALS $V_m(x) = m^2(e^{-2x} - 2e^x)$

Fig.(2.1.1). Morse potentials $V_m(x) = m^2(e^{-2x} - 2e^x)$ for $m = 2, 3, 4$ are plotted in solid lines. The $SO(2,1)$ multiplets for $j = -2, -3, -1/2 + ik$ are shown by the horizontal dashed lines.
Fig. (2.1.2), Pöschl-Teller potentials $V_m(x) = -\frac{m^2 - 1/4}{\cosh^2 x}$ for $m = 3/2, 5/2, 7/2$ are plotted in solid lines. The $SO(2,1)$ multiplets for $j = -3/2, -5/2, -1/2 + ik$ are shown by the horizontal dashed lines.

Differential Realizations of the $SO(2,1)$ Potential Group
Fig. (2.1.3). Comparison between a Morse potential (left) and a Pöschl-Teller potential (right) with the same spectrum.
2.2. The Generalization of the Differential Realization

The most attractive features of the differential realizations given in the last section are:

(1) the Hamiltonian has a mixed (i.e. continuous plus discrete) spectrum and both bound states and scattering states can be described by unitary representations of the same group;

(2) as we shall see, both the bound state spectrum and the scattering matrix can be generated by purely algebraic manipulations through these realizations.

In order to generalize the discussion we introduce a more general class of realizations. In the last section the Hamiltonian of the physical system was defined as

\[ H = -(C_2 + \frac{1}{4}). \]

In general, when the Hamiltonian is only related to the Casimir operator, it does not have to be restricted to the simple form. We can introduce a factor \( f(z) \) in the Hamiltonian, where \( z \) is a variable related to the differential realization, i.e.

\[-f(z)(C_2 + 1/4)\psi = -(j + 1/2)^2 f(z)\psi.\]

But this is not an eigenequation any more. To restore the form of an eigenequation, we remove the \( z \)-dependent part to the lefthand side of the equation and leave a \( z \)-independent part in the righthand side, i.e.

\[ \{[f(z) - a](j + 1/2)^2 - f(z)(C_2 + 1/4)\}\psi = -a(j + 1/2)^2\psi. \]

Therefore, the more general form of the Hamiltonian is

\[ H = [f(z) - a](j + 1/2)^2 - f(z)(C_2 + 1/4) \quad (2.2.1). \]

In the more general form of Eq.(2.2.1), the eigenvalue of the Hamiltonian is

\[ \langle H \rangle = -a(j + 1/2)^2. \]
It is obvious that, if $a > 0$, we still have a "proper" energy spectrum for both bound states and scattering states. Here, "proper" means $E > 0$ for scattering states and $E < 0$ for bound states. Furthermore, in order not to introduce a $j$-dependence into the Hamiltonian we have to have a prescription of the eigenvalue of $J_3$ involved with $j$ such that the explicit part of the $j$-dependence in Eq.(2.2.1) is canceled by the implicit part of the $j$-dependence in the Casimir operator $C_2$. The details of such a prescription will be given in the following example, the realization of the $SO(2,1)$ group to the first class of Ginocchio potentials\[^{12}\].

Consider the differential realization of the $SO(2,1)$ algebra

\[
\begin{align*}
K_\pm &= e^{\pm i\phi}[\pm (\frac{\partial}{\partial x} + \frac{(\lambda^2 - 1)\tanh x}{2(\lambda^2 + \sinh^2 x)}) + \tanh x(i \frac{\partial}{\partial \phi} + \frac{1}{2})], \\
K_3 &= -i \frac{\partial}{\partial \phi}.
\end{align*}
\] (2.2.2)

It is not difficult to check directly that $K_\pm$ and $K_3$ satisfy the commutation relations of Eq.(2.1.3). Actually, we can prove it by showing that they are related to Example (2) of section 2.1 by a similarity transformation; similarity transformations do not change commutation relations.

The $SO(2,1)$ realization in example (2) of the last section is

\[
\begin{align*}
J_\pm &= e^{\pm i\phi}[\pm \frac{\partial}{\partial x} + \tanh x(i \frac{\partial}{\partial \phi} + \frac{1}{2})], \\
J_3 &= -i \frac{\partial}{\partial \phi}.
\end{align*}
\] (2.2.3)

The realization (2.2.2) is related to (2.2.3) through

\[
\begin{align*}
K_\pm &= F J_\pm F^{-1}, \\
K_3 &= F J_3 F^{-1} = J_3,
\end{align*}
\] (2.2.4)

where

\[
F^{-1} = \frac{\cosh^{1/2} x}{(\lambda^2 + \sinh^2 x)^{1/4}}.
\] (2.2.5)

The Casimir operator $C_2$ is

\[
C_2 = \frac{\partial^2}{\partial x^2} + \tanh x \frac{\lambda^2 - 1}{\lambda^2 + \sinh^2 x} \frac{\partial}{\partial x} + \frac{(\lambda^2 - 1)^2}{4(\lambda^2 + \sinh^2 x)^2} \tanh^2 x \\
+ \frac{\lambda^2 - 1}{2(\lambda^2 + \sinh^2 x)} \frac{1}{\cosh^2 x} - \frac{(\lambda^2 - 1)\sinh^2 x}{(\lambda^2 + \sinh^2 x)^2} + \frac{1}{\cosh^2 x} \frac{-\partial^2}{\partial \phi^2} - \frac{1}{4} - \frac{1}{4}.
\] (2.2.6)
After the following consecutive coordinate transformations:

\[
\begin{align*}
  z &= \tanh x, \\
  y &= \frac{z}{[\lambda^2 + (1 - \lambda^2)x^2]^{1/2}}, \\
  r &= \frac{1}{\lambda^2} \left[ \arctanh y + (\lambda^2 - 1)^{1/2} \arctan((\lambda^2 - 1)^{1/2}y) \right],
\end{align*}
\]  

(2.2.7)

we define the Hamiltonian \( H_G \) to be

\[
H_G = [f(z) - \lambda^4](j + \frac{1}{2})^2 - f(z)(C_2 + \frac{1}{4}),
\]

(2.2.8)

where

\[
f(z) = \frac{\lambda^4}{\lambda^2 + (1 - \lambda^2)x^2}.
\]

(2.2.9)

Notice that \( x \) is related to \( r \) through (2.2.7) and

\[
\frac{\partial}{\partial x} = \frac{(\lambda^2 + \sinh^2 x)^{1/2}}{\lambda^2 \cosh x} \frac{\partial}{\partial r},
\]

and

\[
\frac{\partial^2}{\partial x^2} = \frac{\lambda^2 + \sinh^2 x}{\lambda^4 \cosh^2 x} \frac{\partial^2}{\partial r^2} + \frac{(1 - \lambda^2)}{\lambda^2 \cosh^2 x (\lambda^2 + \sinh^2 x)^{1/2}} \frac{\sinh x}{\cosh x} \frac{\partial}{\partial r}.
\]

With a prescription of the quantum numbers

\[
\begin{align*}
\langle C_2 + 1/4 \rangle &= (j + 1/2)^2 = \mu^2, \\
\langle J_3^2 \rangle &= m^2 = (\nu + 1/2)^2 + \mu^2(1 - \lambda^2),
\end{align*}
\]

(2.2.10)

and the equation

\[
H_G \Psi_m = E \Psi_m,
\]

where

\[
\Psi_m = \psi_m(x)e^{im\phi},
\]

after tedious algebra we have

\[
H_G \psi_m(r) = E \psi_m(r),
\]

(2.2.11)
and

\[ V(r) = -\lambda^2 \nu (\nu + 1)(1 - y^2) \]
\[ + \frac{1 - \lambda^2}{4} [5(1 - \lambda^2)y^4 - (7 - \lambda^2)y^2 + 2](1 - y^2), \]  
\[ (2.2.12) \]

and \( y \) is related to \( r \) as shown in Eq.(2.2.7).

\( H_G \) is just the one-dimensional Hamiltonian with a Ginocchio potential, which is \( j \)-independent. Here, the interesting thing is the natural introduction of the \( j \)-dependence in the prescription (2.2.10) of \( m^2 \) and, finally, the cancellation of the \( j \)-dependence in the Hamiltonian of Ginocchio potential (2.2.11). This example tells us that in a purely algebraic model we can properly introduce the energy-dependence in a prescription if it is needed in the phenomenological fit to the experimental data.

The potential group structure can be easily seen by noticing that the Hamiltonian of the Ginocchio potential is only involved with the Casimir operator and its eigenvalue is

\[ \langle H_G \rangle = -\lambda^4 (j + 1/2)^2 \]
\[ = -\lambda^4 \mu^2. \]  
\[ (2.2.13) \]

The state vectors of a multiplet are at the same energy and the eigenvalue of \( K_3 = J_3 \) is related to the potential strength in a more complicated way.

In Fig.(2.2.1), several Ginocchio potentials are plotted. Since \( V(0) \) is negative, the potential is scaled to \( '-1.0' \) at the origin. The radial coordinate \( r \) is scaled by \( r_m \), where \( r_m \) is the value of \( r \) for which \( y(r_m) = 0.98 \). The value \( \lambda = 1.0 \) corresponds to the Pöschl-Teller potential.
Fig. (2.2.1). Ginochchio potentials for the shape parameter $\lambda = 0.5, 1.0, 0.25$ are plotted. The potentials are scaled to "-1.0" at the origin. The radial coordinate $r$ is scaled by $r_m$, where $r_m$ is the value of $r$ for which $y(r_m) = 0.98$. The value $\lambda = 1.0$ corresponds to the Pöschl-Teller potential.
2.3. The Group Realization and the Factorization Method

In section 2.1 and 2.2 we realized the one-dimensional Schrödinger equation with certain solvable potentials by using the $SO(2,1)$ potential group. It turns out that the potential group approach is an alternative to the "Factorization Method" of Infeld and Hull. In this section we will identify the interrelationship between these two methods and quote the theorems of Infeld and Hull for further reference.

Infeld and Hull call a second order differential equation "factorized" if

$$\frac{d^2 y}{dx^2} + r(m, x)y + \lambda y = 0, \quad (2.3.1)$$

where $m = m_0, m_0 + 1, m_0 + 2, \ldots$, with certain boundary conditions can be replaced by the following two equations:

$$+ H^{m+1} y(\lambda, m) = [\lambda - L(m + 1)]^{1/2} y(\lambda, m),$$

$$- H^m y(\lambda, m) = [\lambda - L(m)]^{1/2} y(\lambda, m), \quad (2.3.2)$$

where

$$\pm H^m = k(x, m) \pm \frac{d}{dx}, \quad (2.3.3)$$

and the same boundary conditions.

Note that the dependence of $y$ on $x$ has been suppressed.

**Theorem 1.** If $y(\lambda, m)$ is a solution of the differential equation, then

$$\begin{cases} y(\lambda, m + 1) = - H^{m+1} y(\lambda, m), \\ y(\lambda, m - 1) = + H^m y(\lambda, m), \end{cases} \quad (2.3.4)$$

are also solutions corresponding to the same $\lambda$ but to the different $m$'s suggested by the notation.

If

$$k(x, m) = k_1(x)(m - \frac{1}{2}) - k_0(x), \quad (2.3.5)$$

then

$$k(x, m + 1) = k_1(x)(m + \frac{1}{2}) - k_0(x).$$
Introducing
\[
\begin{align*}
J_\pm &= -\epsilon_{\mp i}\left[\pm \frac{\partial}{\partial x} + k_1(x)(i \frac{\partial}{\partial \phi} - \frac{1}{2}) + k_0(x)\right], \\
J_3 &= -i \frac{\partial}{\partial \phi},
\end{align*}
\] (2.3.6)
and a set of functions
\[
z(\lambda, m) = y(\lambda, m)e^{im\phi},
\] (2.3.7)
we have
\[
J_3z(\lambda, m) = mz(\lambda, m),
\]
and
\[
J_\pm z(\lambda, m) = -\epsilon_{\mp i}\left[\pm \frac{\partial}{\partial x} + k_1(x)(i \frac{\partial}{\partial \phi} - \frac{1}{2}) + k_0(x)\right]e^{im\phi}y(\lambda, m)
\]
\[
= e^{i(m+1)\phi}\left[-\frac{d}{dx} + k(\lambda, m + 1)\right]y(\lambda, m)
\]
\[
= e^{i(m+1)\phi} - H^{m+1}y(\lambda, m)
\]
\[
= [\lambda - L(m + 1)]^{1/2}e^{i(m+1)\phi}y(\lambda, m + 1)
\]
\[
= [\lambda - L(m + 1)]^{1/2}z(\lambda, m + 1),
\]
and similarly,
\[
J_- z(\lambda, m) = [\lambda - L(m)]^{1/2}z(\lambda, m - 1).
\]
That is to say, we have identified \(z(\lambda, m)\) as the eigenvector of the operator \(J_3\) with the eigenvalue \(m\) and the operators \(J_\pm\) as the shift operators which shift the eigenvalues of \(J_3\) by \(\pm 1\).

We can easily prove: (1) if \(L(m) = m(m + 1)\), then \(J_\pm\) and \(J_3\) form an \(O(3)\) algebra satisfying the commutation relations given below:
\[
\begin{align*}
[J_3, J_\pm] &= \pm J_\pm, \\
[J_+, J_-] &= 2J_3.
\end{align*}
\] (2.3.8)

(2) if \(L(m) = -m(m - 1)\), then \(J_\pm\) and \(J_3\) form an \(SO(2,1)\) algebra satisfying the commutation relations of Eq.(2.1.3).

Since we have identified the correspondence between the potential group approach and the factorization method of Infeld and Hull, we quote their theorems below and, with our interpretation in mind, apply them whenever needed without further proof.
Theorem 2.
\[ \int_{a}^{b} \phi(-H^m f)dx = \int_{a}^{b} (\phi H^m f)dx, \]
if \( \phi f \) vanishes at the ends of the interval and the integrands are continuous in the interval.

Notice that only real functions are discussed here. The analog of Theorem 2. in the algebraic language is that \( J_+ \) and \( J_- \) are hermitian conjugate to each other.

Theorem 3. If \( y(\lambda, m) \) is quadratically integrable over the entire range of \( x \) and \( L(m) \) is an increasing function of \( m \) (0 < \( m \)), then the \( H \) operation (2.3.4) of raising \( m \) produces a function which is also quadratically integrable and which vanishes at the end points. If \( L(m) \) is a decreasing function of \( m \) (0 < \( m \)), then the \( H \) operation (2.3.4) of lowering \( m \) produces a function which is also quadratically integrable and which vanishes at the end points.

In our language, it means that the shift operators \( J_\pm \) preserve the boundary conditions. Just as Infield and Hull commented in their paper, Theorem 3. has to be proved for each factorization type; therefore it has to be proven for each differential realization in the group approach. We will not go into the details of the proof, but will simply take it for granted.

Theorem 4. When \( L(m) \) is an increasing function of integer \( m \) for 0 < \( m \) ≤ \( M \), and \( \lambda \) ≤ the larger of \( L(m) \), \( L(m+1) \), then a necessary condition for quadratically integrable solutions is that \( \lambda = \lambda_l = L(l+1) \), where \( l \) is an integer and \( m = 0, 1, 2, \ldots, l \). If \( L(m) \) is a decreasing function of the integer \( m \) for 0 ≤ \( m \) ≤ \( M \), and \( \lambda \) ≤ \( L(0) \), then a necessary condition of the existence of quadratically integrable solutions is that \( \lambda = \lambda_l = L(l) \), where \( l \) is an integer and \( m = l, l+1, l+2, \ldots \).

Notice that, if \( m_0 \) is not taken to be zero, Theorem 4 obviously will require \(|l - m|\) rather than \( l \) to be zero.

The necessary condition of existence of bound states, in our language, is that \( m \) has either a lower bound or an upper bound, since quadratically normalizable solutions mean bound states if the interval is unbounded. The unitary representations of the \( SO(2,1) \) group where "\( m \)" has a lower bound or an upper bound are the discrete principal series \( D_{f}^{\pm} \). In the series \( D_{f}^{+} \) "\( m \)" has a lower bound and in the series \( D_{f}^{-} \) "\( m \)" has an upper bound.
Theorem 5. The $K$ operators defined as follows

$$\pm K_l^m = \begin{cases} [L(l+1) - L(m)]^{-1/2} H^m, & \text{for } L(m) \text{ is an increasing function,} \\ [L(l) - L(m)]^{-1/2} H^m, & \text{for } L(m) \text{ is an decreasing function,} \end{cases}$$

preserve the normalization of the eigenfunctions, when these functions are normalizable.

That is to say, the shift operators $K_\pm$ of the $SO(2,1)$ group with the $J_\pm$ in place of $^\pm H^m$ preserve the normalization of $z(\lambda, m)$. Because the $K$ operators have $m$-dependence in their definitions, we prefer to use $J_\pm$ as shift operators.

With all these theorems in mind we are able to justify some of the procedures that we are going to take in the following discussion. We also notice the fact that Infeld and Hull did not discuss scattering states in their paper and potential group techniques provide more general discussions.

2.4. The Bound State Spectrum

As we have mentioned in the last section, the necessary condition for bound states is that they lie in a representation in which the quantum number $\mu^m$ has either a lower bound or an upper bound. For the unitary representations of the $SO(2,1)$ group, this condition demands the discrete principal series. This is also generally true for the $SO(2,n)$ group, where $m$ is the eigenvalue of the operator

$$J_0 = -i \frac{\partial}{\partial \phi}$$

and $\phi$ is the rotation angle of the two-space in the $(2, n)$ space. Examples of the $SO(2,n)$ group for $n = 1, 2, 3$ will be discussed in this chapter and the next chapter.

Consider the case that $L(m)$ is a decreasing function of $m$, where $m$ is not necessarily an integer. The necessary condition for bounds becomes requiring $m - l = a$ non-negative integer, i.e.

$$m = l, l+1, l+2, \ldots$$
For the special case of the $SO(2,1)$ group, the eigenvalue of the Casimir operator $C_2$ is $j(j + 1)$. For the representation where $j < -1/2$, note $l = -j$. Theorem 4 of Infeld and Hull gives that the condition for bound states is

$$m + j = n,$$

where

$$n = 0, 1, 2, \ldots.$$

The condition is equivalent to the necessary condition of the discrete principal series $D_j^+$. For a realization of the potential group $SO(2,1)$ where the Hamiltonian is

$$H = -(C_2 + 1/4),$$

and $m$ signifies the potential strength, the bound state spectrum for a given potential corresponds to the possible values of $-(j + 1/2)^2$ for the discrete principal series when $m$ is given. Without losing generality we suppose $m > 0$. As we have shown, the one-dimensional Schrödinger systems with the Morse potential and the Pöschl-Teller potential are such examples. If we take the notation $j = -\frac{1}{2} - \mu$, then $\mu > 0$ since $j < -1/2$. Therefore, the bound state spectrum is

$$\langle H \rangle = -\mu^2$$

$$= -(j + 1/2)^2$$

$$= -(n - m + 1/2)^2$$

$$= -(m - n - 1/2)^2,$$

where

$$n = 0, 1, 2, \ldots.$$

Since $j < -1/2$, we have

$$n = m + j < m - 1/2,$$

i.e.

$$n = 0, 1, 2, \ldots, \{m - 1/2\},$$

(2.4.2)
where by the notation \( \{m - 1/2\} \) we mean the largest integer which is less than \( m - 1/2 \).

This means that, if \( \nu \) is itself an integer, then \( \{\nu\} = \nu - 1 \). The consequence is that there is no bound state if \( 0 < m < 1/2 \). This is something obvious for the Pöschl-Teller potential, but not so obvious for the Morse potential.

Notice that \( j = -1/2 \) is not counted as a bound state here. It is both a special case of the discrete principal series for \( j = -1/2 \) and a special case of the continuous principal series \( j = -1/2 + ik \) for \( k = 0 \). Therefore, it can be counted as half a bound state and half a scattering state, and it is indeed so since

\[
\langle H \rangle = \langle -(C_2 + 1/4) \rangle = -(j + 1/2)^2 = 0.
\]

It is interesting when this method is applied to the first class of Ginocchio potentials for which the group structure was obscure before. With the prescription of Eq.(2.2.6), we have

\[
(H_G) = E_G = -\mu^2 \lambda^4,
\]

where

\[
\mu = -(1/2 + j),
\]

and

\[
m^2 = (\nu + 1/2)^2 + \mu^2(1 - \lambda^2).
\]

Since \( m = n - j \), we have another equation for \( m \), i.e.

\[
m^2 = (n + \mu + 1/2)^2.
\]

Combining Eqs.(2.4.4) and (2.4.5), we obtain an equation for \( \mu \), i.e.

\[
\mu^2(1 - \lambda^2) + (\nu + 1/2)^2 = (n + \mu + 1/2)^2,
\]

or

\[
\lambda^2 \mu^2 + (2n + 1)\mu + (n + 1/2)^2 - (\nu + 1/2)^2 = 0.
\]

The solution of Eq.(2.4.6) is

\[
\mu = \frac{\pm[(2n + 1)^2 + \lambda^2((2\nu + 1)^2 - (2n + 1)^2)]^{1/2} - (2n + 1)}{2\lambda^2}.
\]
Since $\mu > 0$, only the positive sign in Eq.(2.4.7) can be taken and, moreover, $n$ can take integer values less than $\nu$ such that in Eq.(2.4.7) the first term in the numerator is larger than the second term, i.e.

$$n = 0, 1, 2, \ldots, \{\nu\}. \quad (2.4.7')$$

Eqs.(2.4.7) and (2.4.7') are just the results given by Ginocchio$^{[12]}$, and $\nu$ is interpreted as the parameter related to the number of bound states that the potential well has. Even though the bound state energy values do not differ by integer values, the potential group structure is still very clear. The bound state spectrum for the second class of Ginocchio potentials involved with the discrete principal series of representations of the group $SO(2,2)$ and $SO(3,2)$ will be discussed in the next chapter.

2.5. The Algebraic Approach to the Scattering Matrix

This section is devoted to the algebraic approach to the scattering matrix. We will illustrate the algebraic procedure with the example of the Morse potential. The reason for choosing this example is that it is the simplest. Usually, in an one-dimensional scattering problem two outgoing channels are involved, the transmission channel and the reflection channel. But in this example only one outgoing channel, the reflection channel, exists since the Morse potential goes to infinity at one end.

The differential realization of Morse potential is introduced in section 2.1 with two variables $x$ and $\phi$ $(-\infty < x < +\infty, 0 \leq \phi \leq 2\pi)$, i.e.

$$I_\pm = e^{\pm i\phi} (\pm \frac{\partial}{\partial x} + i \frac{\partial}{\partial \phi} + e^{-x} \mp \frac{1}{2}),$$

$$I_3 = -i \frac{\partial}{\partial \phi}. \quad (2.5.1)$$

The generators $I_\pm$ and $I_3$ satisfy the commutation relations

$$[I_3, I_\pm] = \pm I_\pm,$$

$$[I_+, I_-] = -2I_3. \quad (2.5.2)$$

They form an $SO(2,1)$ algebra. The Casimir operator of the $SO(2,1)$ algebra is

$$C_2 = I_3^2 - I_3 - I_+ I_-$$

$$= \frac{\partial^2}{\partial x^2} - [e^{-2x} - 2e^{-x}(-i \frac{\partial}{\partial \phi})] - \frac{1}{4} - 28$$

$$- 28$$
A representation basis $|j, m\rangle$ of the $SO(2,1)$ algebra is characterized by the pair of equations
\begin{align}
C_2 |j, m\rangle &= j(j+1)|j, m\rangle, \\
I_3 |j, m\rangle &= m|j, m\rangle.
\end{align}

In the realization defined by Eqs. (2.5.1) and (2.5.3) the solution of Eq. (2.5.4) is given by
\begin{equation}
|j, m\rangle = R_{jm}(x)e^{im\phi},
\end{equation}
where $R_{jm}(x)$ satisfies the one-dimensional Schrödinger equation
\begin{equation}
[-\frac{d^2}{dx^2} - (e^{-2x} - 2me^{-x})]R_{jm}(x) = k^2 R_{jm}(x)
\end{equation}
\((-\infty < x < +\infty),\)
where we have used the continuous principal series representation $j = -1/2 + ik$ to describe the scattering states. A translation
\begin{equation}\xi = x - \ln m,\end{equation}
will transform Eq. (2.5.6) into the one-dimensional Schrödinger equation with a Morse potential, i.e.
\begin{equation}
[-\frac{d^2}{d\xi^2} - m^2(e^{-2\xi} - 2e^{-\xi})]R_{jm}(\xi) = k^2 R_{jm}(\xi),
\end{equation}
\((-\infty < \xi < \infty).\)

The Morse Hamiltonian is related to the Casimir operator of the $SO(2,1)$ group through
\begin{equation}
H_M = -(C_2 + 1/4).
\end{equation}

The reflection amplitude of the Morse oscillator is defined by
\begin{equation}
\overline{R}_m = \frac{B_m}{A_m},
\end{equation}
where
\begin{equation}
R_{jm}(\xi) \rightarrow_{x \to -\infty} A_me^{-ik\xi} + B_me^{ik\xi}.
\end{equation}
It is easy to show from Eq. (2.5.7) that this amplitude $\overline{R}_m$ is related to the reflection amplitude $R_m$ of the potential $(e^{-2x} - 2me^{-x})$ simply by
\begin{equation}
\overline{R}_m = m^{-2ik}R_m.
\end{equation}
In the following it is convenient for us to treat directly the potential in Eq. (2.5.6) and to present our algebraic approach for the reflection amplitude $R_m$.

Since the scattering matrix is obtained from considering the asymptotic limit as $x \to \infty$, we define asymptotic states $|j, m\rangle^\infty$ for scattering states $|j, m\rangle$, where

$$|j, m\rangle^\infty = \lim_{x \to \infty} |j, m\rangle,$$

(2.5.12)

and asymptotic operators $I_\pm^\infty$ and $I_3^\infty$ in a similar way related to generators $I_\pm$ and $I_3$. In the realization (2.5.1) we have

$$
\begin{align*}
I_\pm^\infty &= e^{\pm i\phi} \left( \mp \frac{\partial}{\partial x} + i \frac{\partial}{\partial \phi} \mp \frac{1}{2} \right), \\
I_3^\infty &= -i \frac{\partial}{\partial \phi} = I_3.
\end{align*}
$$

(2.5.13)

Since $I_3$ does not change in the asymptotic limit we shall drop the superscript when referring to its asymptotic form.

Notice that the commutation relations are preserved in the asymptotic limit so that $I_\pm^\infty$ and $I_3$ still satisfy the $SO(2,1)$ commutation relations, i.e.

$$
\begin{align*}
[I_3, I_\pm^\infty] &= \pm I_\pm^\infty, \\
[I_+^\infty, I_-^\infty] &= -2I_3,
\end{align*}
$$

(2.5.14)

and

$$
I_\pm^\infty |j, m\rangle^\infty = [(m \pm 1/2)^2 - (j + 1/2)^2]^{1/2} |j, m \pm 1\rangle^\infty.
$$

(2.5.15)

In this realization the scattering wave has the asymptotic form

$$
|j, m\rangle^\infty = A_m e^{-ikx} e^{im\phi} + B_m e^{ikx} e^{im\phi},
$$

(2.5.16)

which represents a superposition of an incoming free wave and an outgoing free wave in a two-dimensional space. The group which leaves the free-particle energy invariant is the Euclidean group $E(2)$. It is quite natural to introduce the Euclidean group $E(2)$ to describe the asymptotic states. Its generators are the linear momenta $P_1, P_2$ and the angular momentum $I_3$ [16]. The commutation relations among them are

$$
\begin{align*}
[I_3, P_\pm] &= \pm P_\pm, \\
[P_+, P_-] &= 0.
\end{align*}
$$

(2.5.17)
The Algebraic Approach to the Scattering Matrix

where

$$P_\pm = P_1 \pm i P_2. \quad (2.5.18)$$

In polar coordinates $x$ and $\phi$, the asymptotic $E(2)$ generators have the form

$$\begin{cases} 
P_\pm^\infty = \lim_{x \to \infty} P_\pm = e^{\pm i \phi} (-i \frac{\partial}{\partial x}), \\
I_3^\infty = I_3 = -i \frac{\partial}{\partial \phi}.
\end{cases} \quad (2.5.19)$$

They still obey the $E(2)$ commutation relations.

The irreducible representations\cite{16} of the Euclidean group $E(2)$ are labeled by "++" and "--" and their states $| \pm k, m \rangle$ are defined by

$$\begin{cases} 
P_2 | \pm k, m \rangle = k^2 | \pm k, m \rangle; \\
I_3 | \pm k, m \rangle = m | \pm k, m \rangle;
\end{cases} \quad (2.5.20)$$

where $P^2 = P_1^2 + P_2^2 = P_+ P_-$ is the $E(2)$ Casimir operator. The action of the operators $P_\pm$ in these representations is given by

$$\begin{cases} 
P_\pm | + k, m \rangle = | + k, m \rangle; \\
P_\pm | - k, m \rangle = | - k, m \rangle.
\end{cases} \quad (2.5.21)$$

In the realization (2.5.19), it is easy to check that $| \pm k, m \rangle$ are just the incoming $(-k)$ and outgoing $(+k)$ wave of energy $k^2$ and angular momentum $m$ appearing on the righthand side of Eq.(2.5.16), i.e.

$$| \pm k, m \rangle = e^{\pm ikx} e^{i m \phi}. \quad (2.5.22)$$

Therefore, Eq.(2.5.16) can be written in the form

$$| j, m \rangle^\infty = A_m | - k, m \rangle + B_m | + k, m \rangle. \quad (2.5.23)$$

Eq.(2.5.23) is in a form independent of any specific differential realization. Consequently, this equation, together with the $SO(2, 1)$ algebra for the scattering states and the $E(2)$ equations (2.5.20) and (2.5.21) for the free waves, provides us with the basis for the algebraic approach.
It is clear from Eq.(2.5.13) that the asymptotic generators of the $SO(2, 1)$ group can be rewritten in terms of the $E(2)$ generators

$$I_+^\infty = -[(-1/2 - (\pm ik))P_+^\infty + I_3P_+^\infty]/(\pm k),$$

where "$\pm k$" have to be used for the $E(2)$ representations "$\pm k$", respectively. We will show in the next section that Eq.(2.5.24) is a special case of the general form of the asymptotic $SO(2, 1)$ algebra in terms of the $E(2)$ generators up to quadratic terms.

Furthermore, assume that the order of the action of $I_\pm$ and the asymptotic limit can be reversed, i.e.

$$\lim_{x \to -\infty} (I_+|j,m\rangle) = \lim_{x \to -\infty} \alpha_{j,m+1}|j,m\rangle$$

$$= \alpha_{j,m+1}|j,m+1\rangle^\infty$$

$$= \alpha_{j,m+1}(A_{m+1}| - k, m + 1) + B_{m+1}| k, m + 1\rangle$$

$$= I_+^\infty|j,m\rangle^\infty$$

$$= (-m - 1/2 - ik)A_m| - k, m + 1\rangle + (-m - 1/2 + ik)B_m| k, m + 1\rangle,$$

where

$$\alpha_{j,m} = [(m - 1/2)^2 - (j + 1/2)^2]^{1/2}.$$ 

Thus, we find the recursion relations for the coefficients $A_m$ and $B_m$ in the following

$$\begin{cases} 
\alpha_{j,m+1}A_{m+1} = (-m - 1/2 - ik)A_m, \\
\alpha_{j,m+1}B_{m+1} = (-m - 1/2 + ik)B_m.
\end{cases}$$

Then follows the recursion relation for the reflection-amplitude $\Re_m = B_m/A_m$:

$$\Re_{m+1} = \frac{-m - 1/2 + ik}{-m - 1/2 - ik} \Re_m.$$ 

Solving Eq.(2.5.27) we obtain

$$\Re_m(k) = \frac{\Gamma(-m + 1/2 - ik)}{\Gamma(-m + 1/2 + ik)} \Delta(k),$$

where $\Delta(k)$ is an $m$-independent factor determined by $R_1(k)$. 

In fact, the solution of Eq.(2.5.27) could be

\[ \mathcal{R}_m(k) = \frac{\Gamma(\pm m + 1/2 - ik)}{\Gamma(\pm m + 1/2 + ik)} \Delta(k). \]

Since there is still a factor \( \Delta(k) \) undetermined, the choice of the sign can be guided by the physical results. Notice that the exact value of \( \alpha_{j,m+1} \) is irrelevant to the recursion relation (2.5.27). Therefore, the relative phase factor in defining states with different angular momentum \( m \) which affects the phase of \( \alpha_{j,m} \) has nothing to do with the reflection amplitude (2.5.28). Moreover, the recursion relation (2.5.27) derived for integer value \( m \) can be shown to hold for any real value \( m \), since \( m \) does not have to be restricted to integer values from the viewpoint of the Lie algebra.

Considering Eq.(2.5.11) we can rewrite the reflection amplitude for Morse potential in the form

\[ \mathcal{R}_m(k) = m^{-2ik} \frac{\Gamma(-m + 1/2 - ik)}{\Gamma(-m + 1/2 + ik)} \Delta(k). \quad (2.5.29) \]

From the above procedure we can see that the form of the recursion relation of the reflection amplitude is essentially determined by the asymptotic form of generators of the Lie algebra, which is, in turn, determined by the differential realization. In order to generalize the procedure to cases divorced from any specific realization we need a general form of the asymptotic generators, which will be discussed in the next section.

2.6. The General Form of the Asymptotic \( SO(2,1) \) Algebra

In the last section we derived the recursion relation for the scattering matrix in an algebraic way, but the explicit form of the asymptotic operator \( J_+ \) in terms of the \( E(2) \) generators is still dependent on the specific differential realization. In order to develop a purely algebraic model divorced from any specific potential we need a general form for the asymptotic operator to bridge the gap between the \( SO(2,1) \) group and the Euclidean group \( E(2) \). For the simplest model we restrict ourselves to the general form up to quadratic terms.

Consider a physical system with a structure of the \( SO(2,1) \) potential group and its Hamiltonian is involved with the Casimir operator only. Suppose \( J_\pm, J_3 \) form an \( SO(2,1) \)
The General Form of the Asymptotic $SO(2,1)$ Algebra

algebra satisfying the following commutation relations

\[
\begin{align*}
[J_3, J_{\pm}] &= \pm J_{\pm}, \\
[J_+, J_-] &= -2J_3,
\end{align*}
\]

(2.6.1)

and the asymptotic operators can be expressed by the $E(2)$ generators satisfying the commutation relations

\[
\begin{align*}
[J_3, P_{\pm}] &= \pm P_{\pm}, \\
[P_+, P_-] &= 0.
\end{align*}
\]

(2.6.2)

Notice that the angular momentum operator $J_3$ is supposed to be the same for the $SO(2,1)$ algebra and the $E(2)$ algebra and it does not change in the asymptotic limit, i.e.

\[
J_3^\infty = J_3,
\]

(2.6.3)

and that the scattering energy is $k^2$, the eigenvalue of the $E(2)$ Casimir operator $P^2 = P_+P_-$. Since both the $SO(2,1)$ group and the $E(2)$ group leave the scattering energy $k^2$ invariant the coefficients in the most general form can be functions of $k$. It is not difficult to see that the most general form of the raising operator $J_+$ in terms of the $E(2)$ generators up to quadratic terms is given by

\[
J_+^\infty = a(k)J_3P_+ + b(k)P_+,
\]

(2.6.4)

where $a(k)$ and $b(k)$ are to be determined. Accordingly, the lowering operators $J_-^\infty$ is

\[
J_-^\infty = (J_+^\infty)^\dagger
= a^*(k)P_-J_3 + b^*(k)P_- \\
= a^*(k)J_3P_- + [a^*(k) + b^*(k)]P_-.
\]

(2.6.5)

Since the commutation relations are preserved in the asymptotic limit, we will determine the forms of functions $a(k)$ and $b(k)$ by requiring that $J_{\pm}^\infty$ and $J_3$ form an $SO(2,1)$ algebra.

It is easy to check that

\[
[J_3, J_-^\infty] = \pm J_-^\infty
\]

(2.6.6)
by using the first commutation relation in Eq. (2.6.2). Since the action of the asymptotic operator is on the states of an \( E(2) \) representation, we can replace the \( E(2) \) Casimir operator \( P^2 \) by its eigenvalue \( k^2 \) and obtain the second commutation relation

\[
[J_{+}^{\infty}, J_{-}^{\infty}] = -a(k)a^{*}(k)2k^{2}J_{3} - k^{2}[a(k)(a^{*}(k) + b^{*}(k)) + b(k)a^{*}(k)].
\]  

(2.6.7)

Requiring \( J_{\pm}^{\infty} \) and \( J_{3} \) to form an \( SO(2, 1) \) algebra we set the righthand side of Eq. (2.6.7) to be "\(-2J_{3}\). Then we have

\[
\begin{align*}
   a(k)a^{*}(k) & = \frac{1}{k^2}, \\
   a(k)a^{*}(k) + a(k)b^{*}(k) + b(k)a^{*}(k) & = 0.
\end{align*}
\]

(2.6.8)

The solution of the first equation in Eq. (2.6.8) is as follows:

\[
a(k) = e^{i\alpha(k)}/k,
\]

(2.6.9)

and the second equation of Eq. (2.6.8) is reduced to

\[
b(k)e^{-i\alpha(k)} + b^{*}(k)e^{i\alpha(k)} = -\frac{1}{k},
\]

(2.6.10)

where \( \alpha(k) \) is a real function of \( k \). Introducing \( z = a^{*}(k)b(k) \) we obtain from Eq. (2.6.10)

\[
2Re(z) = -\frac{1}{k^2},
\]

i.e.

\[
z = -\frac{1}{2k^2} + is(k),
\]

where \( s(k) \) is a real function of \( k \), or

\[
b(k) = \frac{e^{i\alpha(k)}}{k}(-1/2 + if(k)),
\]

(2.6.11)

where \( f(k) = k^{2}s(k) \) is a real function of \( k \). Substituting the Eqs. (2.6.9) and (2.6.11) into Eq. (2.6.4) we have the general form of the asymptotic raising operator

\[
J_{+}^{\infty} = \frac{e^{i\alpha(k)}}{k}[J_{3}P_{+} + (-1/2 + if(k))P_{+}].
\]

(2.6.12)
Taking the hermitian conjugate of $J_\pm^\infty$ we obtain

$$
J_-^\infty = \frac{e^{-i\alpha(k)}}{k}[J_3P_- + (1/2 - if(k))P_-]. \quad (2.6.13)
$$

It can be shown that the Casimir invariant of the asymptotic $SO(2,1)$ algebra is

$$
C_2 = J_3^2 - J_3J_+J_-
$$

$$
= -1/4 - [f(k)]^2. \quad (2.6.14)
$$

Since the eigenvalue of the Casimir invariant is "$j(j+1)$", we have

$$
j = -1/2 \pm if(k), \quad (2.6.14')
$$

which means that the representation associated with the general form of the asymptotic algebra is characterized by the function $f(k)$.

As we have seen in the last section, there exist two representations of the Euclidean group corresponding to the same eigenvalue $k^2$ of the $E(2)$ Casimir invariant $P^2$, which are labeled by $\pm k$, respectively. The forms of the asymptotic raising operator in these two representations may not be the same. Since they are asymptotic forms of the same $SO(2,1)$ algebra they must have the same Casimir invariant. That is to say, the characteristic real function $f(k)$ can only differ by an overall sign. Actually, in the non-trivial scattering problem they must differ by the overall sign, otherwise the reflection amplitude would be $\pm 1$. Therefore, we have the general form

$$
J_+^\infty = \frac{e^{i\alpha_\pm(k)}}{k}[J_3P_+ + (1/2 \pm if(k))P_+], \quad (2.6.12')
$$

where $\pm$ refer to the $\pm k$ representations of the $E(2)$ group and $\alpha_\pm(k)$ are real functions.

Taking the algebraic procedure given in the last section we can determine a unique recursion relation for the scattering matrix at a given scattering energy. The reflection amplitude corresponding to the general form (2.6.12') is

$$
R_m(k) = (-1)^m \exp[i(m(\alpha_+(k) - \alpha_-(k)))] \frac{\Gamma(m + 1/2 + if(k))}{\Gamma(m + 1/2 - if(k))} \Delta(k), \quad (2.6.15)
$$

where $\Delta(k)$ is to be determined by the reflection amplitude at some fixed value of $m$, say $R_1(k)$. 

In the two examples, the Morse potential and the Pöschl-Teller potential, we have the same form of the asymptotic algebra

\[ J_+^\infty = -[(\frac{-1}{2} - (\pm ik))P_3 + J_3P_+] / (\pm k), \]

i.e., the characteristic function \( f(k) = -k \) and \( \alpha_\pm(k) = \pm 1 \). But the one-dimensional scattering problem with the Pöschl-Teller potential is more complicated than the Morse potential problem since two outgoing channels are involved. Instead of a reflection amplitude, we have to introduce an \( U(2) \) scattering matrix. Since in higher dimensional scattering problems the one-dimensional radial equation is terminated at the origin and only a reflection amplitude exists we are not going to discuss the solution of the \( U(2) \) scattering matrix.

With the most general form given above we can parametrize the characteristic function \( f(k) \) and cast the whole procedure algebraically, divorced from any specific differential realization. However, in the algebraic solution of the reflection amplitude, there is always an \( m \)-independent factor \( \Delta(k) \) which can not be determined in the algebraic procedure. This is a feature of the algebraic approach to one-dimensional scattering problem. As we shall see, in a three-dimensional scattering problem, an \( l \)-independent factor in the reflection amplitude will contribute a constant phase shift to all partial waves and, therefore, will not contribute to the angular distribution of the scattering problem. The analogue in the algebraic approach to the bound state problem is that a constant contribution to all energy eigenvalues will not affect the energy spectrum so far as the energy spectrum relative to the ground state energy is concerned.

2.7. The Confluent Hypergeometric Equation and \( SO(2,1) \) Potential Group

In section 2.1 we introduced two types of \( SO(2,1) \) realizations. One of them is related to the confluent hypergeometric equation, the other is related to the hypergeometric equation. The Morse potential is an example of the former, the Pöschl-Teller potential is an example of the latter. To complete the discussion of the \( SO(2,1) \) potential group, we will realize the confluent hypergeometric equation with the \( SO(2,1) \) potential algebra. Hence
all potential problems related to the confluent hypergeometric function can be, in some way, cast into differential realizations of the \( SO(2,1) \) potential group. The hypergeometric function has three parameters, while the \( SO(2,1) \) group provides us with only two quantum numbers. Therefore, the \( SO(2,1) \) group structure cannot be a full description of the potential problem related to the hypergeometric equation. We have to appeal to a higher dimensional group structure. The higher dimensional potential group is the \( SO(2,2) \) group and the next chapter is devoted to its discussion.

Consider an \( SO(2,1) \) realization

\[
\begin{align*}
K_\pm &= e^{\pm \phi}(\pm \frac{\partial}{\partial x} + i \frac{\partial}{\partial \phi} + \frac{1}{2} e^{-x} \mp \frac{1}{2}), \\
K_3 &= -i \frac{\partial}{\partial \phi}.
\end{align*}
\]  

(2.7.1)

It is not difficult to check that \( K_\pm \) and \( K_3 \) really satisfy the \( SO(2,1) \) commutation relations of Eq.(2.1.3). In fact, this is obtained from a special solution of Eq.(2.1.9)

\[
k_0(x) = \frac{1}{2} e^{-x}.
\]

After a coordinate transformation \( y = e^{-x} \), we can rewrite Eq.(2.7.1) in the following form

\[
\begin{align*}
K_\pm &= e^{\mp \phi}(\mp y \frac{\partial}{\partial y} + i \frac{\partial}{\partial \phi} + \frac{y}{2} \mp \frac{1}{2}), \\
K_3 &= -i \frac{\partial}{\partial \phi}.
\end{align*}
\]  

(2.7.2)

The Casimir operator of this realization is

\[
C_2 = K_3^2 - K_3 - K_+ K_-
= y^2 \frac{\partial^2}{\partial y^2} + y \frac{\partial}{\partial y} - y(i \frac{\partial}{\partial \phi}) - \frac{y^2}{4} - \frac{1}{4}.
\]  

(2.7.3)

If we define a null operator

\[
Q = \frac{1}{y} (C_2 - j(j + 1))
= y \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial y} - (i \frac{\partial}{\partial \phi}) - \frac{y}{4} - (j + 1/2)^2,
\]  

(2.7.4)

where \( j(j + 1) \) is the eigenvalue of the \( SO(2,1) \) Casimir operator. The action of the operator \( Q \) on the simultaneous eigenfunction of the Casimir operator \( C_2 \) and the angular momentum \( K_3 \), i.e.

\[
\psi_m = e^{im\phi} \psi_m(y),
\]
will lead to a differential equation
\[ Q \psi_m = Q \psi_m(y) e^{im\phi} \]
\[ = e^{im\phi} (y \frac{d^2}{dy^2} + \frac{d}{dy} + m - \frac{y}{4} - (j + 1/2)^2) \psi_m(y) \]
\[ = 0, \]
or
\[ Q \psi_m(y) = (y \frac{d^2}{dy^2} + \frac{d}{dy} + m - \frac{y}{4} - (j + 1/2)^2) \psi_m(y) \]
\[ = 0. \tag{2.7.5} \]

With the help of a similarity transformation
\[ F = e^{au} y^b, \tag{2.7.6} \]
we transform Eq. (2.7.5) into the confluent hypergeometric equation. Notice that
\[ F \frac{d}{dy} F^{-1} = \frac{d}{dy} - \frac{b}{y} - a, \tag{2.7.7} \]
and
\[ F \frac{d^2}{dy^2} F^{-1} = \frac{d^2}{dy^2} + (-2a - \frac{2b}{y}) \frac{d}{dy} \]
\[ + a^2 + \frac{2ab}{y} + \frac{b^2 + b}{y^2}. \tag{2.7.8} \]

Using Eqs. (2.7.7) and (2.7.8) we obtain
\[ H_{CH} = F Q F^{-1} \]
\[ = y \frac{d^2}{dy^2} + (1 - 2b - 2ay) \frac{d}{dy} + \left( b^2 - (j + 1/2)^2 \right) \frac{1}{y} \]
\[ + (a^2 - \frac{1}{4}) y + (2ab - a + m). \tag{2.7.9} \]

Requiring
\[ H_{CH} = y \frac{d^2}{dy^2} + (\gamma - y) \frac{d}{dy} - \alpha, \tag{2.7.10} \]
we have to set
\[ \begin{cases} a = 1/2, \\ b = (1 - \gamma)/2, \end{cases} \tag{2.7.11} \]
and choose a prescription for the quantum numbers
\[ \begin{cases} j = -\frac{\gamma}{2}, \\ m = \gamma/2 - \alpha. \end{cases} \tag{2.7.12} \]
With Eqs. (2.7.11) and (2.7.12) we realize the confluent hypergeometric equation

$$H_{CH} \psi_m'(y) = [y \frac{d^2}{dy^2} + (\gamma - y) \frac{d}{dy} - \alpha] \psi_m'(y) = 0,$$

(2.7.13)

where \( \psi_m'(y) = e^{\alpha y} y^\mu \psi_m(y) \).

We summarize the realization of the confluent hypergeometric equation in the following. We start with an \( SO(2,1) \) realization

$$\begin{align*}
J_\pm &= e^{\pm i \phi} [\mp y (\frac{\partial}{\partial y} - \frac{\gamma}{2y} - \frac{1}{2}) + i \frac{\partial}{\partial \phi} + \frac{y}{2} \mp 1/2], \\
J_3 &= -i \frac{\partial}{\partial \phi}.
\end{align*}$$

(2.7.14)

Notice that \( J_\pm \) and \( J_3 \) are related to \( K_\pm \) and \( K_3 \) in Eq.(2.7.2) through a similarity transformation (2.7.6), and, therefore, they form an \( SO(2,1) \) algebra.

The lefthand side of the confluent hypergeometric equation is related to the realization through

$$H_{CH} = \frac{1}{y} [C_2 - j(j + 1)]$$

$$= y \frac{d^2}{dy^2} + (\gamma - y) \frac{d}{dy} - \alpha,$$

(2.7.15)

where a prescription for the quantum numbers

$$\begin{align*}
 j &= -\frac{\gamma}{2}, \\
m &= \frac{\gamma}{2} - \alpha.
\end{align*}$$

(2.7.16)

has been used in deriving Eq.(2.7.15).

For the bound state problem the discrete principal series \( D_j^+ \) satisfies the condition that \( m \) has a lower bound, i.e.

$$m + j = n$$

where \( n = 0, 1, 2, \ldots \). From Eqs.(2.7.11) and (2.7.12) we have

$$m = \frac{\gamma}{2} - \alpha$$

$$= -j - \alpha.$$

Therefore, for bound states we have

$$\alpha = -n,$$
where

\[ n = 0, 1, 2, \ldots, \]

i.e., \( \alpha \) has to be zero or a negative integer. This condition agrees with the requirement in solving the confluent hypergeometric equation for bound states that the confluent hypergeometric series has to be terminated at some point and becomes a polynomial.
The Potential Group \( SO(2,2) \)

The techniques introduced in chapter 2 for solving the scattering matrix via a potential group are quite general and can be used in connection with groups which are more complicated than \( SO(2,1) \) and also in higher dimensions. Before going to the extension of the potential group approach to higher dimensions we devote this chapter to its generalization to a solvable class of potentials, i.e. the Natanzon potentials, which contain an important subclass rediscovered by Ginocchio. This solvable class is associated with the group \( SO(2,2) \). The \( SO(2,2) \) representations referred to are discussed in the Appendix C. Actually, the theory of the \( SO(m,n) \) \((m,n \geq 2)\) potential group approach could be discussed in parallel. We shall not go into the details, but comments on the further extension will be made whenever needed.

3.1. The Hypergeometric Equation and \( SO(2,2) \) Potential Group

In this section we shall realize the hypergeometric equation with the \( SO(2,2) \) potential group. It is expected that all potential problems associated with hypergeometric function are, in some way, related to the \( SO(2,2) \) group. A large solvable class of potentials, the Natanzon and Ginocchio potentials, are examples.

First, we consider a differential realization of the \( SO(2,2) \) group on the \((2,2)\) hyperboloid \( \mathcal{H}^3 \).

In the four dimensional \((2,2)\) space the metric tensor in the Cartesian coordinates is \( \eta_{ij} \) \((i,j = 1,2,3,4)\) where

\[
\eta_{ij} = (+, +, -, -). \tag{3.1.1}
\]

The three-dimensional hyperboloid \( \mathcal{H}^3 \) of the \((2,2)\) space is defined by

\[
\rho^2 = x_1^2 + x_2^2 - x_3^2 - x_4^2 = \text{constant}. \tag{3.1.2}
\]

On the sheet of the \((2,2)\) hyperboloid where

\[
\sigma = \text{sign}[\rho^2] = +1,
\]

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we can parameterize the hyperboloid with three variables \((\chi, \phi, \alpha)\) as follows:

\[
\begin{align*}
x_1 &= \rho \cosh \chi \cos \phi, \\
x_2 &= \rho \cosh \chi \sin \phi, \\
x_3 &= \rho \sinh \chi \cos \alpha, \\
x_4 &= \rho \sinh \chi \sin \alpha,
\end{align*}
\]

(3.1.3)

where \(\phi\) and \(\alpha\) are the rotation angles in 1-2 and 3-4 two-spaces, respectively.

We introduce bilinear self-adjoint operators denoted by

\[
\begin{align*}
M_{ab} &= x_a P_b - x_b P_a, \\
N_{ab} &= x_a P_b + x_b P_a - i\delta_{ab} I,
\end{align*}
\]

(3.1.4)

for \(a, b = 1, 2, 3, 4\), where

\[P_a = -i \frac{\partial}{\partial x_a},\]

and \(I\) is the unit operator. It can be shown that the six operators, \(M_{12}, M_{34}, N_{13}, N_{14}, N_{23}\) and \(N_{24}\) form an \(SO(2,2)\) algebra. They satisfy the following commutation relations

\[
\begin{align*}
[J_2, J_3] &= iJ_1, \\
[K_2, K_3] &= iJ_1, \\
[J_3, J_1] &= iJ_2, \\
[K_3, K_1] &= iJ_2, \\
[J_1, J_2] &= -iJ_3, \\
[K_1, K_2] &= -iJ_3, \\
[J_2, K_3] &= iK_1, \\
[K_2, J_3] &= iK_1, \\
[J_3, K_1] &= iK_2, \\
[K_3, J_1] &= iK_2, \\
[J_1, K_2] &= -iK_3, \\
[K_1, J_2] &= -iK_3,
\end{align*}
\]

(3.1.5)

where

\[
\begin{align*}
J_1 &= N_{23}, \\
K_1 &= N_{14}, \\
J_2 &= -N_{13}, \\
K_2 &= N_{24}, \\
J_3 &= M_{12} = -i \frac{\partial}{\partial \phi}, \\
K_3 &= M_{34} = -i \frac{\partial}{\partial \alpha}.
\end{align*}
\]

(3.1.5')

It is well known that the \(SO(2,2)\) algebra can be decomposed into two commuting \(SO(2,1)\) algebras, i.e.

\[SO(2,2) = SO_a(2,1) \oplus SO_b(2,1).\]

(3.1.6)
The $SO(2,2)$ decomposition (3.1.6) may be written explicitly in terms of the generators (3.1.5') as

$$
\begin{align*}
A_i &= \frac{1}{2} (J_i + K_i), \\
B_i &= \frac{1}{2} (J_i - K_i),
\end{align*}
$$

where $i = 1, 2, 3$. Introducing raising and lowering operators for the two subalgebras

$$
\begin{align*}
A_\pm &= A_1 \pm iA_2, \\
B_\pm &= B_1 \pm iB_2,
\end{align*}
$$

we can rewrite the $SO(2,2)$ generators explicitly in terms of the coordinate variables

$$
\begin{align*}
A_\pm &= \frac{1}{2} e^{\pm i(\phi + \alpha)} \left[ \mp \frac{\partial}{\partial \chi} + \tanh \chi (-i \frac{\partial}{\partial \phi}) + \coth \chi (i \frac{\partial}{\partial \alpha}) \right], \\
A_3 &= -\frac{i}{2} \left( \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \alpha} \right), \\
B_\pm &= \frac{1}{2} e^{\pm (\phi - \alpha)} \left[ \mp \frac{\partial}{\partial \chi} + \tanh \chi (-i \frac{\partial}{\partial \phi}) + \coth \chi (i \frac{\partial}{\partial \alpha}) \right], \\
B_3 &= -\frac{i}{2} \left( \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \alpha} \right).
\end{align*}
$$

The Casimir operator $C_2$ of this realization is

$$
C_2 = \frac{\partial^2}{\partial \chi^2} + (\tanh \chi + \coth \chi) \frac{\partial}{\partial \chi} + \frac{1}{\cosh^2 \chi} (\frac{\partial^2}{\partial \phi^2}) - \frac{1}{\sinh^2 \chi} (\frac{\partial^2}{\partial \alpha^2}).
$$

Notice that the $SO(2,2)$ realization on the $(2,2)$ hyperboloid is a symmetric realization since the two commuting algebra have the same Casimir operator, i.e.

$$
C_2^{so(2,1)} = C_2^{so(2,1)} = \frac{1}{4} C_2.
$$

(In this chapter we will not use the superscript if $C_2$ is referring to the $SO(2,2)$ Casimir operator.)

In general, for the realization on the sheet $\sigma = +1$ of the $(m,n)$ hyperboloid the Casimir operator $C_2$ has the form

$$
C_2^{(m,n)} = \frac{\partial^2}{\partial \chi^2} + [(m-1)\tanh \chi + (n-1)\coth \chi] \frac{\partial}{\partial \chi} + \frac{1}{\cosh^2 \chi} C_2^{(m)} - \frac{1}{\sinh^2 \chi} C_2^{(n)},
$$

$$
\text{(3.1.12)}
$$
where $C_2^e(m)$ and $C_2^e(n)$ are Casimir operators of the subgroups $O(m)$ and $O(n)$ respectively. For the realization on the sheet $\sigma = -1$ of the $(m, n)$ hyperboloid $\sinh \chi$ and $\cosh \chi$ exchange their places in the above expressions.

After a transformation

$$z = \tanh^2 \chi,$$

we can rewrite the $SO(2, 2)$ Casimir operator as

$$C_2 = 4[z(1 - z)^2 \frac{\partial^2}{\partial z^2} + (1 - z)^2 \frac{\partial}{\partial z} + \frac{1 - z}{4z} (-\frac{\partial^2}{\partial \phi^2}) - \frac{1 - z}{4z} (-\frac{\partial^2}{\partial \alpha^2})].$$

(3.1.14)

The basis $|\omega, m_1, m_2\rangle$ of the symmetric representation of the $SO(2, 2)$ algebra is characterized by the following equations:

$$\begin{cases} C_2|\omega, m_1, m_2\rangle = \omega(\omega + 2)|\omega, m_1, m_2\rangle, \\ J_3|\omega, m_1, m_2\rangle = m_1|\omega, m_1, m_2\rangle, \\ K_3|\omega, m_1, m_2\rangle = m_2|\omega, m_1, m_2\rangle. \end{cases}$$

(3.1.15)

In the realization on the $(2, 2)$ hyperboloid the general form of the solution of Eq.(3.1.15) is given by

$$|\omega, m_1, m_2\rangle = \Psi_{\omega m_1 m_2}$$

$$= \psi_{\omega m_1 m_2}(\chi)e^{im_1 \phi}e^{im_2 \alpha}.$$

(3.1.16)

We are going to show that $\psi_{\omega m_1 m_2}(z)$ is related to the hypergeometric function $\,_{1}F_2(\delta, \beta, \gamma; z)$ through a similarity transformation:

$$M^{-1} = z^a(1 - z)^b,$$

(3.1.17)

where $a$ and $b$ are to be determined.

We define a null operator $Q$ as

$$Q = \frac{1}{4(1 - z)}[C_2 - \omega(\omega + 2)].$$

$$= z(1 - z)\frac{\partial^2}{\partial z^2} + (1 - z)\frac{\partial}{\partial z} - \frac{1}{4} \frac{\partial^2}{\partial \phi^2}$$

$$+ \frac{1}{4z} \frac{\partial^2}{\partial \alpha^2} - \omega(\omega + 2) \frac{1}{4(1 - z)}.$$
Notice that
\[ \psi'_{\omega m_1m_2}(z) = M \psi_{\omega m_1m_2}(z). \] (3.1.19)

and
\[ M \frac{\partial}{\partial z} M^{-1} = \frac{\partial}{\partial z} + \frac{a}{z} - \frac{b}{1-z}, \]

and
\[ M \frac{\partial^2}{\partial z^2} M^{-1} = \frac{\partial^2}{\partial z^2} + \left( \frac{2a}{z} - \frac{2b}{1-z} \right) \frac{\partial}{\partial z} + \frac{a^2}{z^2} + \frac{b^2}{(1-z)^2} - \frac{2ab}{z(1-z)} - \frac{a}{z^2} - \frac{b}{(1-z)^2}. \]

The action of the operator \( Q \) on the simultaneous eigenfunction of \( C_2, J_3 \) and \( K_3 \) leads to a differential equation, i.e.
\[ MQM^{-1}(M \psi_{\omega m_1m_2}) = 0, \]

or equivalently,
\[ H_{HG} \psi'_{\omega m_1m_2}(z) = 0, \]

where
\[ H_{HG} = z(1-z) \frac{\partial^2}{\partial z^2} + [2a + 1 - (2a + 2b + 1)z] \frac{\partial}{\partial z} + \left( a^2 - \frac{m_2^2}{4} \right) \frac{1}{z} \]
\[ + (a^2 - b - \frac{\omega(\omega + 2)}{4}) \frac{1}{1-z} - (b^2 - \frac{m_2^2}{4} + 2ab + a^2). \]

By requiring \( H_{HG} \) to have the form of hypergeometric equation, i.e.
\[ H_{HG} = z(1-z) \frac{\partial^2}{\partial z^2} + [\gamma - (\delta + \beta + 1)z] \frac{\partial}{\partial z} - \delta \beta, \]

we find the proper choice of parameters \( a \) and \( b \), i.e.
\[ \begin{cases} 
  a = \frac{\gamma - 1}{2}, \\
  b = \frac{\delta + \beta - \gamma + 1}{2},
\end{cases} \] (3.1.20)

and the prescription for the quantum numbers
\[ \begin{cases} 
  (\omega + 1)^2 = (\delta + \beta - \gamma)^2, \\
  m_1^2 = (\delta - \beta)^2, \\
  m_2^2 = (\gamma - 1)^2.
\end{cases} \] (3.1.21)
The Hypergeometric Equation and $SO(2,2)$ Potential Group

We, thus realize the hypergeometric equation

$$H_{HG} \psi_{\omega m_1 m_2} (z) = [z(1 - z) \frac{\partial^2}{\partial z^2} + (\gamma - (\delta + \beta + 1) z) \frac{\partial}{\partial z} - \delta \beta] \psi_{\omega m_1 m_2} (z)$$

$$= 0,$$

with an $SO(2,2)$ potential algebra formed by

$$\begin{cases} J'_i = MJ_i M^{-1}, \\ K'_i = MK_i M^{-1}, \end{cases}$$

where $i = 1, 2, 3$, and $H_{HG}$ is related to the Casimir operator $C'_2$ of the $SO(2,2)$ algebra through

$$H_{HG} = \frac{1}{4(1 - z)} [C'_2 - \omega(\omega + 2)].$$

There is an alternative to the realization of the hypergeometric equation in terms of

$$y = 1 - z = \frac{1}{\cosh^2 \chi}.$$ 

Under the transformation $z \rightarrow 1 - z$, the hypergeometric equation is changed to another hypergeometric equation with the replacement:

$$\gamma \rightarrow (\delta + \beta - \gamma + 1).$$

Instead of Eq.(3.1.14), the Casimir operator of the realization (3.1.5) is

$$C'_2 = 4y^2 (1 - y) \frac{\partial^2}{\partial y^2} - 4y^2 \frac{\partial}{\partial y} - y \frac{\partial^2}{\partial \phi^2} + \frac{y}{(1 - y)} \frac{\partial^2}{\partial \alpha^2}.$$  

After a similarity transformation $N$:

$$\begin{cases} J'_i = NJ_i N^{-1} , \\ K'_i = NK_i N^{-1}, \end{cases}$$

where $i = 1, 2, 3$, and

$$C'_2 = NC_2 N^{-1},$$

where

$$N^{-1} = y^{\gamma/2} (1 - y)^{(\delta + \beta - \gamma)/2},$$

we can also realize the hypergeometric equation in terms of $y$:

$$H'_{HG} \psi'_{\omega m_1 m_2} (y) = [y(1 - y) \frac{\partial^2}{\partial y^2} + (\gamma - (\delta + \beta + 1) y) \frac{\partial}{\partial y} - \delta \beta] \psi_{\omega m_1 m_2}$$

$$= 0,$$
\[ \psi_{\omega m_1 m_2}(y) = N \psi_{\omega m_1 m_2}(y), \]

and \( H'_{HG} \) is related to the Casimir operator \( C_2^2 \) through

\[ H'_{HG} = \frac{1}{4y}[C_2^2 - \omega(\omega + 2)]. \quad (3.1.26) \]

The prescription of the quantum numbers for Eq. (3.1.26) is as follows:

\[
\begin{align*}
(\omega + 1)^2 &= (\gamma - 1)^2, \\
m_1^2 &= (\delta - \beta)^2, \\
m_2^2 &= (\delta + \beta - \gamma)^2.
\end{align*}
\]

\[ (3.1.27) \]

As shown by Frank and Wolff\textsuperscript{18}, the generalized Pöschl-Teller potential can be realized on the \((m, n)\) hyperboloid for \(m, n \geq 2\) with the replacement:

\[
\begin{align*}
(\omega + 1)^2 &\to (C_2^o(m,n) + (\frac{m+n-2}{2})^2, \\
m_1^2 &\to (C_2^o(m) + (\frac{m-2}{2})^2, \\
m_2^2 &\to (C_2^o(n) + (\frac{n-2}{2})^2.
\end{align*}
\]

\[ (3.1.28) \]

The hypergeometric equation can also be realized on the \((m, n)\) hyperboloid with the same replacement \((3.1.28)\) in the prescription of the quantum numbers. The \(SO(2, 2)\) realization is the simplest case of the non-trivial \(SO(m, n)\) realization for \(m, n \geq 2\).

\[ \text{3.2. An Exactly Solvable Class of Potentials—Natanzon Potentials} \]

A general investigation was made by Natanzon\textsuperscript{13} into the problem of constructing the bound state spectrum and the scattering matrix for a family of potentials that admit solution of the radial Schrödinger equation by means of hypergeometric functions. The Pöschl-Teller potential\textsuperscript{19}, Eckart potential\textsuperscript{20}, Rosen-Morse potential\textsuperscript{21} and Manning-Rosen potential\textsuperscript{22} turn out to be members of this family. Recently, another important branch of this family rediscovered by Ginocchio has also been studied in detail. We devote this section to the realization of the Natanzon potential with the symmetric representation.
of the $SO(2,2)$ group and the next section to the application of the algebraic techniques developed in the last chapter to this family, which provides us with another example how the closed forms of the bound state spectrum and the scattering matrix of exactly solvable potentials are related to a certain potential group structure.

A specific feature of Natanzon potential is that a transformation $z(r)$, which converts the Schrödinger equation
\[ \left\{ \frac{d^2}{dr^2} + [E - U(r)] \right\} \psi(r) = 0 \]
into the hypergeometric equation, does not carry the coordinate origin (the point $r=0$) into one of the singular points of the hypergeometric equation. The Natanzon potential can be represented in the form

\[ U[z(r)] = \frac{f(z) + h_0(1 - z) + h_1 z}{R(z)} - \frac{1}{2} \{z, r\}, \quad (3.2.1) \]

where

\[ R(z) = az^2 + b_0 z + c_0 \]
\[ = a(z - 1)^2 + b_1(z - 1) + c_1, \quad (3.2.2) \]

and the Schwarz derivative of $z$ is defined by

\[ \{z, r\} = \frac{d^2 z}{dz^2} - \frac{3}{2} \left( \frac{d^2 z}{dr^2} \right)^2. \quad (3.2.3) \]

The transformation $z(r)$ is determined by the differential equation

\[ \left( \frac{dz}{dr} \right)^2 = \frac{4z^2(1 - z)^2}{R(z)}. \quad (3.2.4) \]

The domain of definition of the variable $z$ is chosen to be the interval $(0, 1)$. In order that the potential $U[z(r)]$ has no singularities for $r \geq 0$, we require that the zeros of $R(z)$ do not lie in the above interval. If we rewrite

\[ E - U(r) = \left( \frac{dz}{dr} \right)^2 I(z) + \frac{1}{2} \{z, r\}, \quad (3.2.5) \]

then we have

\[ I(z) = \frac{(1 - \lambda_0^2)(1 - z) + (1 - \lambda_1^2)z + (\mu^2 - 1)z(1 - z)}{4z^2(1 - z)^2}, \quad (3.2.6) \]
where

\[
\begin{align*}
1 - \mu^2 &= aE - f, \\
1 - \lambda_0^2 &= c_0 E - h_0, \\
1 - \lambda_1^2 &= c_1 E - h_1,
\end{align*}
\]  

(3.2.7)

Consider the $SO(2,2)$ realization on the $(2,2)$ hyperboloid given in the last section

\[
\begin{align*}
A_\pm &= \frac{1}{2} e^{\pm i(\phi + \alpha)} \left[ \mp \frac{\partial}{\partial \chi} + \tanh \chi (-i \frac{\partial}{\partial \phi}) + \coth \chi (-i \frac{\partial}{\partial \alpha}) \right], \\
B_\pm &= \frac{1}{2} e^{\pm i(\phi - \alpha)} \left[ \mp \frac{\partial}{\partial \chi} + \tanh \chi (-i \frac{\partial}{\partial \phi}) + \coth \chi (i \frac{\partial}{\partial \alpha}) \right], \\
A_3 &= -\frac{i}{2} (\frac{\partial}{\partial \phi} + \frac{\partial}{\partial \alpha}), \\
B_3 &= -\frac{i}{2} (\frac{\partial}{\partial \phi} - \frac{\partial}{\partial \alpha}).
\end{align*}
\]  

(3.2.8)

A similarity transformation $F$, \[F = (\frac{dz}{dr})^{-1/2} z^{1/2},\]  

(3.2.9)

where $z = \tanh^2 \chi$ is related to $r$ through Eq. (3.2.4), will transform the realization (3.2.8) into

\[
\begin{align*}
A'_\pm &= \frac{1}{2} e^{\pm i(\phi + \alpha)} \left[ \mp \frac{\partial}{\partial \chi} + \frac{1}{2} \frac{dz}{dr} \frac{d^2 z}{dr^2} - \frac{1}{2z} \frac{dz}{dr} \right] \\
&\quad + \tanh \chi (-i \frac{\partial}{\partial \phi}) + \coth \chi (-i \frac{\partial}{\partial \alpha}), \\
B'_\pm &= \frac{1}{2} e^{\pm i(\phi - \alpha)} \left[ \mp \frac{\partial}{\partial \chi} + \frac{1}{2} \frac{dz}{dr} \frac{d^2 z}{dr^2} - \frac{1}{2z} \frac{dz}{dr} \right] \\
&\quad + \tanh \chi (-i \frac{\partial}{\partial \phi}) + \coth \chi (i \frac{\partial}{\partial \alpha}), \\
A'_3 &= -\frac{i}{2} (\frac{\partial}{\partial \phi} + \frac{\partial}{\partial \alpha}) = A_3, \\
B'_3 &= -\frac{i}{2} (\frac{\partial}{\partial \phi} - \frac{\partial}{\partial \alpha}) = B_3.
\end{align*}
\]  

(3.2.10)
The Casimir operator of the \( SO(2, 2) \) algebra is

\[
C'_2 = FC_2F^{-1} = 4C'_{\text{iso}(2,1)} = 4C'_{\text{iso}(2,1)}
\]

\[
= \frac{\partial^2}{\partial \chi^2} + (z^{1/2} + z^{-1/2} + \frac{1}{2} \frac{d^2z}{d\chi^2} - \frac{1}{z \cdot d\chi}) \frac{\partial}{\partial \chi}
\]

\[
+ \frac{1}{4} (\frac{1}{d^{2} \chi} - \frac{1}{z \cdot d\chi})^2 + \frac{1}{2} \frac{\partial}{\partial \chi} (\frac{1}{z \cdot d\chi} - \frac{1}{z \cdot d\chi})
\]

\[
+ (z^{1/2} + z^{-1/2})(\frac{1}{z \cdot d\chi} - \frac{1}{z \cdot d\chi}) + (1 - z)(- \frac{\partial^2}{\partial \phi^2}) + (1 - \frac{1}{z})(- \frac{\partial^2}{\partial \alpha^2}).
\]

(3.2.11)

A basis \( |\omega, m_1, m_2\rangle \) of the symmetric representation of the \( SO(2, 2) \) algebra is characterized by

\[
\begin{align*}
C'_2|\omega, m_1, m_2\rangle &= \omega(\omega + 2)|\omega, m_1, m_2\rangle, \\
J_3|\omega, m_1, m_2\rangle &= m_1|\omega, m_1, m_2\rangle, \\
K_3|\omega, m_1, m_2\rangle &= m_2|\omega, m_1, m_2\rangle.
\end{align*}
\]

(3.2.12)

where

\[
\begin{align*}
J_3 &= A_3 + B_3 = -i \frac{\partial}{\partial \phi}, \\
K_3 &= A_3 - B_3 = -i \frac{\partial}{\partial \alpha}.
\end{align*}
\]

In this realization the basis can be written in the following form:

\[
|\omega, m_1, m_2\rangle = e^{i(m_1 \phi + m_2 \alpha)} R_{\omega m_1 m_2}(\chi).
\]

(3.2.13)

Notice that \( z = \tanh^2 \chi \) is related to \( \chi \) through Eqn.(3.2.4) and

\[
\frac{\partial}{\partial \chi} = 2 \frac{1}{z} z^{1/2} (1 - z) \frac{\partial}{\partial \chi},
\]

and

\[
\frac{\partial^2}{\partial \chi^2} = R(z) \frac{\partial^2}{\partial r^2} + \frac{R(z)}{z^2 (1 - z)} (-z(1 - z) \frac{d^2z}{dr^2} + \frac{1 - 2z}{2} \frac{dz}{dr} \frac{\partial}{\partial r}.
\]

If we introduce the Hamiltonian of a physical system as

\[
H_N = \frac{z}{R(z)} [((\omega + 1)^2 - (C'_2 + 1)) - (\omega + 1)^2,
\]

(3.2.14)
and prescribe the quantum numbers as

\[ \begin{align*}
(\omega + 1)^2 & = \lambda^2, \\
m_1^2 & = \mu^2, \\
m_2^2 & = \lambda^2,
\end{align*} \quad (3.2.15) \]

we find that the one-dimensional Schrödinger equation for \( R_{\omega m_1 m_2}(r) \) is given by

\[ H_N R_{\omega m_1 m_2}(r) = \left[ -\frac{\text{d}^2}{\text{d}r^2} + U(r) \right] R_{\omega m_1 m_2}(r) = ER_{\omega m_1 m_2}(r), \quad (3.2.16) \]

where \( U(r) \) is just the same as shown in Eq.(3.2.1) when \( h_0, h_1, f \) are related to \( \lambda_0, \lambda_1, \mu \) through Eq.(3.2.7). Therefore, using the differential realization (3.2.10) of the \( SO(2,2) \) algebra, we realize the Schrödinger equation with the Natanzon potential; the Natanzon Hamiltonian is related to the Casimir operator of the \( SO(2,2) \) algebra through Eq.(3.2.14).

Since \( R(z) \) contains three parameters \( a, b_0, c_0 \) and since there are three other parameters \( h_0, h_1, f \) involved in \( U(r) \), the Natanzon potentials constitute a very large class of solvable potentials. For example,

1. if \( R(z) = cz \) (c ≠ 0), we have the Pöschl-Teller potential:
   \[ U(r) = \frac{1}{c} \left( \frac{h_0 + 3/4}{\sinh^2(r/\sqrt{c})} - \frac{f + 3/4}{\cosh^2(r/\sqrt{c})} \right); \quad (3.2.17) \]

2. if \( R(z) = cz^2 \) (c ≠ 0), we have the Manning-Rosen potential:
   \[ U(r) = \frac{1}{c} \left[ f + \frac{1}{2} \left( 1 - \coth \frac{r}{\sqrt{c}} \right) + \frac{h_0}{4 \sinh^2(r/\sqrt{c})} \right]; \quad (3.2.18) \]

3. if \( R(z) = c \) (c ≠ 0), we have the Rosen-Morse potential:
   \[ U(r) = \frac{1}{c} \left[ f + \frac{h_0 + 1}{2} \left( 1 - \tanh \frac{r}{\sqrt{c}} \right) + \frac{f}{4 \cosh^2(r/\sqrt{c})} \right]. \quad (3.2.19) \]

The potentials in (2) and (3) are special cases of the Eckart potential. In section 3.4 we will show how the Ginocchio potential is also a branch of this family.

In order to show how these potentials look like, we plot a generalized Pöschl-Teller potential in Fig.(3.2.1), a Manning-Rosen potential in the upper part of Fig.(3.2.2) and a Rosen-Morse potential in the lower part of Fig.(3.2.2).
An Exactly Solvable Class of Potentials—Natanzon Potentials

$$V(x) = \frac{(m+m')^2 - 1/4}{\cosh^2 x} + \frac{(m-m')^2 - 1/4}{\sinh^2 x}$$

Fig. (3.2.1). A generalized Pöschl-Teller potential:
Fig. (3.2.2). The upper part shows a Manning-Rosen potential \( V(x) = \frac{j(j+1)}{\sinh^2 x} - 2mm' \cosh x \)

and the lower part shows a Rosen-Morse potential \( V(x) = -\frac{j(j+1)}{\cosh^2 x} - 2mm' \tanh x \).
3.3. The Algebraic Approach to the Natanzon Potential

Once the realization of the $SO(2, 2)$ potential group is known, the application of the algebraic techniques developed in chapter 2 to the Natanzon potential is straightforward. The differential realization used here is a $SO(2, 2)$ symmetric representation and our knowledge of the $SO(2, 2)$ symmetric representation is summarized in Appendix C for reference.

The symmetric representation of the $SO(2, 2)$ group is characterized by a quantum number $\omega$. The eigenvalue of the $SO(2, 2)$ Casimir operator is given by

$$\langle C_2 \rangle = \omega (\omega + 2).$$

The $SO(2, 2)$ basis of the symmetric representation is given by

$$\begin{align*}
C_2|\omega, m_1, m_2\rangle &= \omega (\omega + 2)|\omega, m_1, m_2\rangle, \\
J_3|\omega, m_1, m_2\rangle &= m_1|\omega, m_1, m_2\rangle, \\
K_3|\omega, m_1, m_2\rangle &= m_2|\omega, m_1, m_2\rangle.
\end{align*}$$

(3.3.1)

The discrete series representations where the eigenvalue $m_1$ of the operator $J_3$ has a lower bound describe the bound states; the continuous series representations describe the scattering states.

1. The Bound state spectrum:

From the $SO(2, 2)$ representation theory, the discrete series of symmetric representations are characterized by a negative integer

$$\omega = -1 - s$$

where $s = 0, 1, 2, \ldots$. The lower bound of $m_1$ is $\omega$. For a given value of $m_2$, $m_1$ can take those values that satisfy the following equation:

$$m_1 - |m_2| = -\omega + 2n,$$  (3.3.2)

where $n = 0, 1, 2, \ldots$. Eq.(3.3.2) can be rewritten as

$$m_1 - |m_2| - [- (\omega + 1)] = 2n + 1.$$  (3.3.2')
The prescription for the quantum numbers for the Natanzon potential is as follows

\[
\begin{align*}
(\omega + 1)^2 &= \lambda_1^2, \\
\mu^2 &= \lambda_0^2, \\
\lambda_1^2 &= \lambda_0^2,
\end{align*}
\]  

(3.3.3)

and \( \mu, \lambda_0 \) and \( \lambda_1 \) are related to the parameters of the Natanzon potential through

\[
\begin{align*}
1 - \mu^2 &= aE - f, \\
1 - \lambda_0^2 &= c_0E - h_0, \\
1 - \lambda_1^2 &= c_1E - h_1.
\end{align*}
\]  

(3.3.4)

Since \( m_1 \) and \( |m_2| \) are positive and \( (\omega + 1) \) is negative we have

\[
\begin{align*}
-(\omega + 1) &= [h_1 + 1 - c_1E]^{1/2}, \\
m_1 &= [f + 1 - aE]^{1/2}, \\
|m_2| &= [h_0 + 1 - c_0E]^{1/2}.
\end{align*}
\]  

(3.3.5)

In terms of Natanzon's parameters, Eq.(3.3.2) is reduced to the bound state spectrum equation given by Natanzon, i.e.

\[
\sqrt{f + 1 - aE_n} - \sqrt{h_0 + 1 - c_0E_n} - \sqrt{h_1 + 1 - c_1E_n} = 2n + 1,
\]

(3.3.6)

where the subscript \( n \) is introduced to label the energy level corresponding to the integer value \( n \).

(2) The scattering matrix:

For the reflection amplitude we have to consider the decomposition of the \( SO(2,2) \) algebra, i.e.

\[
SO(2,2) \cong SO_\sigma(2,1) \oplus SO_\sigma(2,1),
\]

which provides us with shift operators for these two commuting \( SO(2,1) \) algebras. In the differential realization for the Natanzon potential the raising operators in these two algebras are given in Eq.(3.2.10), i.e.

\[
\begin{align*}
A_+ &= e^{i\beta}[(\frac{\partial}{\partial r} + \frac{1}{2z} \frac{dz}{d\beta} - \frac{1}{2z} \frac{dz}{d\beta}) + \coth \beta (i \frac{\partial}{\partial \gamma} + \frac{i}{\sinh \beta} \frac{\partial}{\partial \gamma})], \\
B_+ &= e^{i\theta}[(\frac{\partial}{\partial r} + \frac{1}{2z} \frac{dz}{d\beta} - \frac{1}{2z} \frac{dz}{d\beta}) + \coth \beta (i \frac{\partial}{\partial \theta} + \frac{i}{\sinh \beta} \frac{\partial}{\partial \gamma})],
\end{align*}
\]  

(3.3.7)
where

\[
\begin{align*}
\beta &= -2\chi, \\
\gamma &= \phi + \alpha, \\
\theta &= \phi - \alpha,
\end{align*}
\] (3.3.8)

and we drop the prime in the notation.

Recall that

\[
z = \tanh z^2 \chi = \tanh^2 \left( \frac{\beta}{2} \right),
\]

and

\[
\left( \frac{dz}{dr} \right)^2 = \frac{4z^2(1-z)^2}{R(z)}.
\]

In what follows, we adopt Natanzon's assumptions that the transformation \( r(z) \) carries the points \( z = 0 \) and \( z = 1 \) to \( r = 0 \) and \( r = +\infty \), i.e. \( z_0 = 0 \) and \( c_1 > 0 \), and that the potential \( U(r) \to 0 \) as \( r \to \infty \), i.e. \( h_1 = -1 \) and \( \lambda_1 = -i\sqrt{c_1}k \). Therefore, as \( r \to \infty \), we have

\[
1 - z \sim \rho \exp\left[-2r/\sqrt{c_1}\right] \\
\equiv \exp\left[-2(r + r_0)/\sqrt{c_1}\right].
\] (3.3.9)

The asymptotic behavior in terms of \( \beta \) is, as \( r \to \infty \), \( \beta \to \infty \) and

\[
1 - z = \frac{1}{\cosh^2 \beta/2} \sim 4e^{-\beta}.
\] (3.3.10)

Therefore, as \( r \to \infty \), we have

\[
\frac{dz}{d\beta} \sim -4e^{-\beta}
\]
i.e.

\[
\frac{1}{z} \frac{dz}{d\beta} \to 0,
\] (3.3.11)

and

\[
\frac{dz}{dr} \sim \frac{2(1-z)}{\sqrt{c_1}} \sim \frac{8}{\sqrt{c_1}} e^{-\beta}
\]
i.e.

\[
\frac{1}{2} \frac{d^2z}{dr^2} \to -1/2,
\] (3.3.12)

Using Eqs. (3.3.11) and (3.3.12) we obtain the asymptotic raising operators

\[
\begin{align*}
A_+^\infty &= e^{i\gamma} \left( \frac{\partial}{\partial \beta} + i \frac{\partial}{\partial \gamma} - \frac{1}{2} \right), \\
B_+^\infty &= e^{i\theta} \left( \frac{\partial}{\partial \beta} + i \frac{\partial}{\partial \theta} - \frac{1}{2} \right).
\end{align*}
\] (3.3.13)
It is not difficult to see from Eqs. (3.3.9) and (3.3.10) that the reflection amplitude in the r-frame is related to the reflection amplitude in the β-frame through

\[ R_r(k) = (\rho/4)^{-i2\pi/2}R_\beta(\sqrt{\beta}k/2) \]

\[ = (\rho/4)^{-i\pi/2}R_\beta(\sqrt{\beta}k/2). \] (3.3.14)

In the following we shall find it more convenient to treat it in the β-frame than in the r-frame, then convert the reflection amplitude in the β-frame to the reflection amplitude in the r-frame.

In this realization the symmetric representation basis \(|j,m_1,m_2\rangle\) of the \(SO(2,2) \cong SO_\alpha(2,1) \oplus SO_\beta(2,1)\) is characterized by the equations:

\[
\begin{align*}
C_2^*|j,m_1,m_2\rangle &= j(j+1)|j,m_1,m_2\rangle, \\
A_3|j,m_1,m_2\rangle &= m_a|j,m_1,m_2\rangle, \\
B_3|j,m_1,m_2\rangle &= m_2|j,m_1,m_2\rangle,
\end{align*}
\] (3.3.15)

where

\[ C_2^* = \frac{1}{4}C_2 = C_2^{so_{2,1}} = C_2^{so_{2,1}}, \]

and

\[
\begin{align*}
A_3 &= -i\frac{\partial}{\partial \gamma}, \\
B_3 &= -i\frac{\partial}{\partial \theta},
\end{align*}
\] (3.3.16)

and

\[ j = \frac{1}{2}\omega = -\frac{1}{2} + ik. \]

The solution of Eq. (3.3.15) can be written in the following form

\[ |j,m_1,m_2\rangle = e^{i(m_0^a + m_0^b)}R_{j,m_a,m_b}[\beta(r)], \] (3.3.17)

where \(R_{j,m_a,m_b}[\beta(r)]\) satisfies the one-dimensional Schrödinger equation with the Natanzon potential. Notice that \(\beta \to \infty\) as \(r \to \infty\) and we shall work in the β-frame first.

The reflection amplitude is defined by

\[ R_\beta(k,m_a,m_b) = \frac{B(m_a,m_b)}{A(m_a,m_b)}, \] (3.3.18)
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where

$$R_{j m a m b} (\beta) \rightarrow_{\beta \rightarrow \infty} A(m_a, m_b) e^{-ik\beta} + B(m_a, m_b) e^{ik\beta}. \quad (3.3.19)$$

In the $\beta, \gamma, \theta$ realization the asymptotic scattering wave has the form

$$|j, m_a, m_b\rangle_{\infty} = A(m_a, m_b) e^{-ik\beta} e^{i(m_a \gamma + m_b \theta)} + B(m_a, m_b) e^{ik\beta} e^{i(m_a \gamma + m_b \theta)}. \quad (3.3.20)$$

In parallel with the procedure in section 2.5, we introduce the symmetric representation of the $E_a(2) \otimes E_b(2)$ group for the asymptotic states. The generators of the group are $P_{a1}, P_{a2}, A_3, P_{b1}, P_{b2}$ and $B_3$, which satisfy the following commutation relations:

$$\begin{cases} 
[P_{a+}, P_{a-}] = 0, & [P_{b+}, P_{b-}] = 0, \\
[A_3, P_{a\pm}] = \pm P_{a\pm}, & [B_3, P_{b\pm}] = \pm P_{b\pm},
\end{cases} \quad (3.3.21)$$

where

$$\begin{cases} 
P_{a\pm} = P_{a1} \pm iP_{a2}, \\
P_{b\pm} = P_{b1} \pm iP_{b2}.
\end{cases} \quad (3.3.22)$$

All other commutators vanish. In polar coordinates $\beta, \gamma, \theta$ the asymptotic $E_a(2) \otimes E_b(2)$ generators have the form

$$\begin{cases} 
P_{a\pm\infty} = e^{\pm\gamma} (-i \frac{\partial}{\partial \beta}), & P_{b\pm\infty} = e^{\pm\theta} (-i \frac{\partial}{\partial \beta}), \\
A_3\infty = A_3 = -i \frac{\partial}{\partial \gamma}, & B_3\infty = B_3 = -i \frac{\partial}{\partial \theta}.
\end{cases} \quad (3.3.23)$$

They still obey the $E(2) \otimes E(2)$ commutation relations. The symmetric representation of the $E(2) \otimes E(2)$ group is characterized by

$$P_a^2 = P_b^2 = P^2, \quad (3.2.24)$$

where $P_a^2 = P_{a1}^2 + P_{a2}^2$ and $P_b^2 = P_{b1}^2 + P_{b2}^2$ are the Casimir operators of the two $E(2)$ groups, respectively. The symmetric representations of the $E_a(2) \otimes E_b(2)$ group are labeled by $+k$ and $-k$ and their states $|\pm k, m_a, m_b\rangle$ are defined by

$$\begin{cases} 
P^2 |\pm k, m_a, m_b\rangle = k^2 |\pm k, m_a, m_b\rangle, \\
A_3 |\pm k, m_a, m_b\rangle = m_a |\pm k, m_a, m_b\rangle, \\
B_3 |\pm k, m_a, m_b\rangle = m_b |\pm k, m_a, m_b\rangle.
\end{cases} \quad (3.3.25)$$
The action of $P_{a\pm}, P_{b\pm}$ in these representations are given by

$$\begin{align*}
P_{a\pm}|k,m_a,m_b\rangle &= +k|k,m_a \pm 1,m_b\rangle, \\
P_{a\pm}|k,m_a,m_b\rangle &= -k|k,m_a \pm 1,m_b\rangle, \\
P_{b\pm}|k,m_a,m_b\rangle &= +k|k,m_a,m_b \pm 1\rangle, \\
P_{b\pm}|k,m_a,m_b\rangle &= -k|k,m_a,m_b \pm 1\rangle,
\end{align*}$$

(3.3.26)

In the realization (3.3.23) the incoming $-k$ and outgoing $+k$ waves can be written as

$$|\pm k,m_a,m_b\rangle = e^{\pm ik\theta}e^{i(m_a\gamma + m_b\delta)}.$$  

(3.3.27)

Thus, Eq.(3.3.19) is reduced to

$$|j,m_a,m_b\rangle^\infty = A(m_a,m_b)|-k,m_a,m_b\rangle + B(m_a,m_b)|+k,m_a,m_b\rangle.$$  

(3.3.28)

From Eqs.(3.3.13) and (3.3.19) the asymptotic raising operators for the two $SO(2,1)$ algebras are written as

$$\begin{align*}
A_+^\infty &= -[(-\frac{1}{2} - (\pm ik))(P_{a+}^\infty + A_3P_{a+}^\infty)]/(\pm k), \\
B_+^\infty &= -[(-\frac{1}{2} - (\pm ik))(P_{b+}^\infty + B_3P_{b+}^\infty)]/(\pm k),
\end{align*}$$

(3.3.29)

where $\pm k$ have to be used for the $E_a(2) \otimes E_b(2)$ representations $\pm k$, respectively. As we did in section 2.5, we exchange the order of the action of the raising operators and the asymptotic limit, and obtain the recursion relations for the coefficients $A(m_a,m_b)$ and $B(m_a,m_b)$ as follows:

$$\begin{align*}
\alpha_{j,m_a+1}A(m_a+1,m_b) &= -(m_a + \frac{1}{2} + ik)A(m_a,m_b), \\
\alpha_{j,m_a+1}A(m_a,m_b+1) &= -(m_b - \frac{1}{2} - ik)A(m_a,m_b), \\
\alpha_{j,m_a+1}B(m_a+1,m_b) &= -(m_a + \frac{1}{2} - ik)B(m_a,m_b), \\
\alpha_{j,m_a+1}B(m_a,m_b+1) &= -(m_b - \frac{1}{2} + ik)B(m_a,m_b),
\end{align*}$$

(3.3.30)

where

$$\alpha_{j,m} = [(m - 1/2)^2 - (j + 1/2)^2]^{1/2}.$$
The solution of Eq. (3.3.31) can be written as

\[
\Re_{\beta}(k, m_a, m_b) = \frac{\Gamma(m_a + 1/2 - ik)\Gamma(-m_b + 1/2 - ik)\Delta(k)}{\Gamma(m_a + 1/2 + ik)\Gamma(-m_b + 1/2 + ik)}
\]

where \(\Delta(k)\) is a factor independent of \(m_a, m_b\) and determined by the reflection amplitude for some fixed values \(m_{a0}, m_{b0}\).

Recalling Eq. (3.3.14), we find the reflection amplitude in the \(r\)-frame, i.e.

\[
\Re_{r}(k, m_a, m_b) = (\rho/4)^{-i\sqrt{c_1}}\frac{\Gamma(m_a + 1/2 - ik\sqrt{c_1}/2)\Gamma(-m_b + 1/2 - ik\sqrt{c_1}/2)}{\Gamma(m_a + 1/2 + ik\sqrt{c_1}/2)\Gamma(-m_b + 1/2 + ik\sqrt{c_1}/2)}\Delta(k).
\]

Notice that

\[
\begin{align*}
\{ m_a &= \frac{1}{2}(m_1 + m_2) = \frac{1}{2}(\mu + \lambda_0), \\
m_b &= \frac{1}{2}(m_1 - m_2) = \frac{1}{2}(\mu - \lambda_0).
\end{align*}
\]

We have

\[
\Re_{r}(k, m_a, m_b) = (\rho/4)^{\lambda_1}\frac{\Gamma(\frac{\lambda_0 + \lambda_1 + 1 + \mu}{2})\Gamma(\frac{\lambda_0 + \lambda_1 + 1 - \mu}{2})}{\Gamma(\frac{\lambda_0 - \lambda_1 + 1 + \mu}{2})\Gamma(\frac{\lambda_0 - \lambda_1 + 1 - \mu}{2})}\Delta(k).
\]

From Natanzon's result,[13] we can identify

\[
\Delta(k) = \frac{\Gamma(i\sqrt{c_1}k)}{\Gamma(-i\sqrt{c_1}k)}4^{-i\sqrt{c_1}k}.
\]

Generally, we leave it undetermined in the algebraic approach.

### 3.4. The Ginocchio Potential Problem

The generalized Ginocchio potential is a function of five parameters: the scale parameter \(s\), the depth parameter \(\nu\), the shape parameter \(\lambda\), the effective mass parameter \(\alpha\), and the centrifugal parameter \(\alpha_t\). After a scale transformation the radial equation of
The three-dimensional Schrödinger equation is written in a form of a dimensionless radial equation

\[ \left\{ - \frac{1}{\mu(r)} \frac{d}{dr} \left( \frac{d}{dr} \right) + \frac{l(l+1)}{r^2} + U_i(r) \right\} \psi(r) = E \psi(r) , \] (3.4.1)

where the effective mass is given by

\[ \mu(r) = 1 - a(1 - y^2) , \] (3.4.2)

and the potential is given in two parts

\[ U_i(r) = \frac{1}{\mu(r)} [V_i(r) + C_i(r)] , \] (3.4.3)

\[ \begin{align*}
V_i(r) &= -\lambda^2 \nu_i (\nu_i + 1) (1 - y^2) \\
&\quad + \frac{1 - \lambda^2}{4} (1 - y^2) (2 - (7 - \lambda^2) y^2 + 5(1 - \lambda^2) y^4) \\
&\quad + \frac{a}{\mu^2} (1 - y^2) (1 - (1 - \lambda^2) y^2) [1 - a + (a(4 - 3\lambda^2) - 3(2 - \lambda^2)) y^2 + 5(1 - \lambda^2)(1 - a) y^4 + 2a(1 - \lambda^2) y^6] ,
\end{align*} \] (3.4.4)

\[ C_i(r) = (\alpha_i^2 - \frac{1}{4}) \frac{1 - y^2}{y^2} (1 + (\lambda^2 - 1) y^2) - \frac{l(l+1)}{r^2} , \]

and \( y (0 \leq y < 1) \) is related to \( r (0 \leq r < \infty) \) through

\[ r = \frac{1}{\lambda^2} [\text{arctanh} + (\lambda^2 - 1)^{1/2} \text{arctan}((\lambda^2 - 1)^{1/2} y)] . \] (3.4.5)

This potential is negative and finite at the origin and goes to zero as \( r \) approaches to infinity. The finiteness of the potential determines the value of \( \alpha_i \), i.e.

\[ \alpha_i = l + \frac{1}{2} . \] (3.4.6)

In this section we shall prove that the Ginocchio potential with a constant effective mass, i.e. \( a = 0 \), belongs to a more general class of potentials, Natanzon potentials. Using a similarity transformation \([\mu(r)]^{1/2}\) and the technique that introduces an additional factor \(1/\mu(r)\) to the Hamiltonian, we shall further prove that Ginocchio potential with a variable effective mass can be realized in the \(SO(2,2)\) potential group approach with a replacement

\[ (1 - \lambda^2) \rightarrow [1 - \lambda^2(1 - a)] \]
in the prescription for the quantum numbers.

In section 3.2 a transformation \( z(r) \) was introduced to convert the Schrödinger equation with the Natazon potential into the hypergeometric equation. The transformation \( z(r) \) is determined by the differential equation

\[
(dz/dr)^2 = \frac{4z^2(1-z)^2}{R(z)}, \tag{3.4.7}
\]

where

\[
R(z) = az^2 + b_0z + c_0. \tag{3.4.8}
\]

In the case where

\[
c_0 = 0, \tag{3.4.9}
\]

up to a scaling factor, we can write

\[
\begin{cases}
  a = \frac{1 - \lambda^2}{\lambda^4}, \\
  b_0 = \frac{1}{\lambda^2}.
\end{cases} \tag{3.4.9'}
\]

The above parametrization can be regarded as a combination of two consecutive transformations:

1. a scaling transformation to \( R(z) \) such that \( a + b_0 \) is normalized to 1 and \( b_0 \) is written as a new parameter \( \lambda^2 \);

2. a scaling transformation introducing an over all factor \( \lambda^{-4} \) to \( R(z) \).

Therefore, Eq.(3.4.7) is reduced to

\[
(dz/dr)^2 = \frac{4\lambda^4z(1-z)^2}{\lambda^2 + (1 - \lambda^2)z}. \tag{3.4.10}
\]

Introducing Ginocchio's notation \( y \) through the transformation

\[
z = \frac{\lambda^2 y^2}{1 + (\lambda^2 - 1)y^2}, \tag{3.4.11}
\]

we have

\[
\frac{dz}{dy} = \frac{2\lambda^2 y}{[1 + (\lambda^2 - 1)y^2]^2}. \tag{3.4.12}
\]
The Ginocchio Potential Problem

From Eq. (3.4.7) and

\[
\frac{dz}{dr} = \frac{dz}{dy} \frac{dy}{dr},
\]

we obtain

\[
\left( \frac{dy}{dr} \right)^2 = \frac{(1 + (\lambda^2 - 1)y^2)^4}{4\lambda^4y^2} \frac{(dz)^2}{dr} = [1 + (\lambda^2 - 1)y^2]^2(1 - y^2)^2,
\]

i.e.

\[
\frac{dr}{dy} = \pm \frac{1}{[1 + (\lambda^2 - 1)y^2](1 - y^2)} = \pm \frac{1}{\lambda^2} \left[ \frac{1}{1 - y^2} + \frac{\lambda^2 - 1}{1 + (\lambda^2 - 1)y^2} \right].
\]

(3.4.13)

Taking the plus sign in (3.4.13) and using the initial condition \( r(y = 0) = 0 \) we solve Eq. (3.4.13) and obtain

\[
r = \frac{1}{\lambda^2} \left[ \arctanh y + (\lambda^2 - 1)^{1/2} \arctan((\lambda^2 - 1)^{1/2}y) \right],
\]

which is exactly the same as the Ginocchio transformation (3.4.5)\(^{[12],[14]}\).

Further, by relating Ginocchio’s parameters to Natanzon’s parameters through

\[
\begin{cases}
    h_0 &= \alpha^2 - 1, \\
    h_1 &= -1, \\
    f &= (\nu_1 + \frac{1}{2})^2 - 1,
\end{cases}
\]

(3.4.14)

and straightforward algebra, we can identify this branch of the Natanzon potential with the Ginocchio potential with effective mass \( \mu(r) = 1 \), i.e.

\[
H_G\psi_{\omega m_1 m_2}(r) = \left[ -\frac{d^2}{dr^2} + U_G(r) \right]\psi_{\omega m_1 m_2}(r) = E\psi_{\omega m_1 m_2}(r),
\]

(3.4.15)

where

\[
U_G(r) = \frac{fz(z - 1) + h_0(1 - z) + h_1 z}{\lambda^4 z^2 + (1 - \lambda^2)z^2} - \frac{1}{2} \{z, r\} \\
= -\lambda^2(1 - y^2)v_1(v_1 + 1) \\
+ \frac{1 - \lambda^2}{4}(1 - y^2)[5(1 - \lambda^2)y^4 + (\lambda^2 - 7)y^2 + 2] \\
+ \frac{\alpha^2 - \frac{1}{2}}{y^2}(1 - y^2)(1 + (\lambda^2 - 1)y^2),
\]

(3.4.16)
and the prescription for the corresponding quantum numbers

\[
\begin{align*}
(\omega + 1)^2 &= -\frac{E}{\lambda^4}, \\
\lambda^4 &= (\nu + 1)^2 - \frac{E}{\lambda^4}(1 - \lambda^2), \\
\lambda_2^4 &= \alpha_i^2.
\end{align*}
\]  

(3.4.17)

has been used in deriving (3.4.16).

The more general case of the Ginnocchio potential can be realized by performing a similarity transformation \( F = [\mu(r)]^{1/2} \) and using the technique given in section 2.2 to introduce an additional factor \( \frac{1}{\mu(r)} \). That is to say, the more general Ginnocchio Hamiltonian \( H_G' \) is related to the Hamiltonian \( H_G \) in Eq.(3.4.15) through

\[
H_G' = \frac{1}{\mu(r)} F H_G F^{-1} - E \left( \frac{1}{\mu(r)} - 1 \right),
\]

(3.4.18)

and the corresponding prescription for the quantum numbers is modified to

\[
\begin{align*}
(\omega + 1)^2 &= -\frac{E}{\lambda^4}, \\
\lambda^4 &= (\nu + 1)^2 - \frac{E}{\lambda^4}[1 - \lambda^2(1 - a)], \\
\lambda_2^4 &= \alpha_i^2.
\end{align*}
\]  

(3.4.19)

As we have pointed out in section 2.2, the modification of the prescription just cancels the energy dependent term, i.e. the last term in Eq.(3.4.18). Hence the new Hamiltonian can be written as

\[
H_G' = \frac{1}{\mu(r)} F \left[ \frac{-d^2}{dr^2} + U_G(r) \right] F^{-1}
\]

\[
= \frac{1}{\mu(r)} \left[ \frac{-d^2}{dr^2} + \frac{d\mu(r)}{\mu(r)} \frac{d}{dr} - \frac{1}{4(\mu(r))^2} \left( 3 \left( \frac{d\mu(r)}{dr} \right)^2 - 2\mu(r) \frac{d^2\mu(r)}{dr^2} \right) \right]
\]

\[
+ \frac{d}{dr} \frac{1}{\mu(r)} U_G(r)
\]

\[
= - \frac{d}{dr} \frac{1}{\mu(r)} \frac{d}{dr} + U_G'(r),
\]

(3.4.20)

where

\[
U_G'(r) = \frac{1}{\mu(r)} U_G(r) + \frac{1}{4(\mu(r))^3} \left[ 2\mu(r) \frac{d^2\mu(r)}{dr^2} - 3 \left( \frac{d\mu(r)}{dr} \right) \right]
\]

\[
= \frac{1}{\mu(r)} U_G(r) + \frac{a}{(\mu(r))^3} \{ (1 - y^2)[1 + (\lambda^2 - 1)y^2] \{ 1 - a + [a(4 - 3\lambda^2) + (3\lambda^2 - 2)]y^2 
\]

\[
+ 5(1 - \lambda^2)(1 - a)y^4 + 2a(1 - \lambda^2)y^6 \}.
\]

(3.4.21)
is the same as the potential given in Eqs. (3.4.3) and (3.4.4)\textsuperscript{[14]}.

Since the Ginocchio potential involves only even powers of $y$ as an one-dimensional potential, it is symmetric with respect to the origin $r = 0$. From the viewpoint of Nazon's transformation this is also a consequence of the fact that $x$ is invariant under the transformation $r \rightarrow -r$. Since the Ginocchio potential for $(\alpha^2 - 1/4) \neq 0$ has a pole at the origin $r = 0$, the two channels in the one-dimensional scattering problem are separated and we can view the one-dimensional Schrödinger equation in the interval $0 \leq r < \infty$ as a three-dimensional radial equation for the $s$-partial wave. Even with an $l$-dependence in the parametrization, the Ginocchio potential still can not reflect the asymptotic behavior of the centrifugal potential. But this is not important in the bound state problem, where bound state wave functions tend to vanish exponentially as $r$ tends to infinity. However, it is rather important to the scattering problem where the asymptotic behavior of the potential has significant influence on the phase shift.

Since the Ginocchio potential is interpreted as a three-dimensional potential problem, we need a parameter $l$ in the algebraic approach to label the three-dimensional angular momentum. The way to introduce angular momentum naturally is to realize the Ginocchio potential with the $SO(2,3)$ realization on the sheet $\sigma = +1$ of the $(2,3)$ hyperboloid, where there is a three-space involved. We will not go into the details of the realization, but rather make some comments.

As we mentioned in section 3.1, the $SO(2,3)$ realization on the $(2,3)$ hyperboloid leads to the replacement of $m_2^2$ with $(C_2^{so(3)}) + \frac{(3-2)^2}{4}$, i.e.

$$m_2^2 \rightarrow l(l + 1) + \frac{1}{4} = (l + \frac{1}{2})^2.$$  

From the prescription (3.4.19) it simply means that the parameter $\alpha_l$ takes the value $l + \frac{1}{2}$, which is just the value Ginocchio obtained by requiring the potential, apart from the centrifugal part $\frac{l(l+1)}{r^2}$, to be finite at the origin $r = 0$.

From the representation theory of the $SO(2,3)$ group, the discrete series of unitary irreducible representations are characterized by two quantum numbers, $l_0$ and $m_0$, where $l_0$ gives the lower bound for the angular momentum $l$ and $m_0$ gives the lower bound for $m_1$. The $SO(2,3)$ realization on the $(2,3)$ hyperboloid corresponds to $l_0 = 0$ and $m_0 = -\omega$, \textsuperscript{[14]}
where $\omega(\omega + 3)$ is the eigenvalue of the $SO(2,3)$ Casimir operator and $-(\omega + \frac{3}{2})^2 \lambda^4$ is the energy eigenvalue of the Ginocchio potential problem. For each fixed $l$, $m_1$ can take the values: $2n + m_0 + l$, where $n = 0, 1, 2, \ldots$. In Ginocchio's notation, $\beta = -(\omega + \frac{3}{2})$. Therefore, when $m_1$ and $\omega$ are fixed, $l$ can take values such that $2n + l$ remains a constant $m_1 - m_0 = m_1 + \omega$. In other words, if, in the prescription (3.4.19), $\nu$ is independent of $l$, bound states with different angular momentum $l$ but the same value $2n + l$ are degenerate. Consequently, we conclude that the degeneracy for bound states with the same value $2n + l$ does not have to imply an $SU(3)$ symmetry; it also happens in the $SO(2,3)$ potential group structure.

Since the algebraic approach to the Ginocchio potential is no more than a replica of section 3.3, we shall not pursue it here.

### 3.5. The Relation between $SO(2,1)$ and $SO(2,2)$ Realizations

In section 2.1 we have seen two types of $SO(2,1)$ differential realizations. One is related to the confluent hypergeometric equation; the other is related to the hypergeometric equation. But in this chapter the differential realization of the solvable class of potentials is related to the hypergeometric equation only. To conclude the discussion of the $SO(2,2)$ potential group approach we would like to see the relation between $SO(2,2)$ and $SO(2,1)$ realizations.

It is easy to see the relation between the $SO(2,2)$ and $SO(2,1)$ realizations which are associated with the hypergeometric equation. Since the hypergeometric function involves three free parameters and the $SO(2,1)$ group provides us with only two quantum numbers the $SO(2,1)$ realization can not fully describe the variety of the parameters in the hypergeometric function. For a proper description of hypergeometric equation one has to appeal to the realization of a higher dimensional group, i.e. the $SO(2,2)$ group. Unsurprisingly, with a special parametrization $SO(2,2)$ realizations lead to the same potentials as those given by $SO(2,1)$ realizations. For example, in the generalized Pöschl-Teller potential, the fixed value $m_1^2 = 1/4$ converts it to the Pöschl-Teller potential; in the second class of Ginocchio's potential, the same fixed value $m_1^2 = \alpha^2 = 1/4$ leads us back to the first class of Ginocchio's
potential. In the following example we shall see in detail how the fixed value appears as the integral constant in Eq. (2.1.6).

At the end of section 2.1 we introduced a trial solution to the first equation of (2.1.6), i.e.

\[ k_1(x) = \coth x, \]  

(3.5.1)

and a trivial solution \( k_0(x) = 0 \) to the second equation of (2.1.6). In general, \( k_0(x) \) has to satisfy the first order differential equation

\[ \frac{dk_0(x)}{dx} + k_0(x)c\coth x = 0. \]  

(3.5.2)

A transformation \( y = \ln|k_0(x)| \) will transform Eq. (3.5.2) to

\[ \frac{dy(x)}{dx} = -\coth x. \]  

(3.5.3)

The general solution to Eq. (3.5.3) is given by

\[ y(x) = -\ln(\sinh x) + c. \]  

(3.5.4)

The corresponding solution for \( k_0(x) \) is

\[ k_0(x) = \frac{c}{\sinh x}. \]  

(3.5.4')

With the general solution for \( k_0(x) \), we can write down the \( SO(2,1) \) realization as follows:

\[
\begin{align*}
A_\pm &= e^{\pm i\gamma}[\pm \frac{\partial}{\partial x} + \coth x(i \frac{\partial}{\partial \gamma} + \frac{1}{2}) + \frac{c}{\sinh x}], \\
A_3 &= -i \frac{\partial}{\partial \gamma}.
\end{align*}
\]

(3.5.5)

The \( SO(2,1) \) Casimir operator is given by

\[
C_2 = \frac{\partial^2}{\partial x^2} + \frac{1}{\sinh^2 x} \left( \frac{\partial^2}{\partial \gamma^2} + \frac{1}{4} \right) - 2c \frac{\cosh x}{\sinh^2 x} (i \frac{\partial}{\partial \gamma}) - \frac{c^2}{\sinh^2 x} - \frac{1}{4}.
\]

(3.5.6)

Consider a physical system whose Hamiltonian is related to the \( SO(2,1) \) Casimir operator through

\[ H = -4(C_2 + \frac{1}{4}). \]  

(3.5.7)
Introducing a new variable $\beta = \frac{\chi}{2}$, we obtain an effective Hamiltonian with the generalized Pöschl-Teller potential

$$H = -\frac{d^2}{d\beta^2} - \frac{(m + c)^2 - \frac{1}{4}}{\cosh^2 \beta} + \frac{(m - c)^2 - \frac{1}{4}}{\sinh^2 \beta},$$

(3.5.8)

where $m$ is the eigenvalue of $A_3$ or $m = (A_3)$.

If we define

$$\begin{cases} (m + c) = m_1 = (-i \frac{\partial}{\partial \phi}) , \\ (m - c) = m_2 = (-i \frac{\partial}{\partial \alpha}) , \end{cases}$$

(3.5.9)

then we have

$$\begin{cases} c = (-i \frac{\partial}{\partial \theta}) = \frac{1}{2} (m_1 - m_2) \\ = \frac{1}{2} \left( \left[-i \frac{\partial}{\partial \phi} \right] - \left[-i \frac{\partial}{\partial \alpha} \right] \right) . \\ m = (-i \frac{\partial}{\partial \gamma}) = \frac{1}{2} (m_1 + m_2) \\ = \frac{1}{2} \left( \left[-i \frac{\partial}{\partial \phi} \right] + \left[-i \frac{\partial}{\partial \alpha} \right] \right) . \end{cases}$$

(3.5.9')

It can be realized if we introduce two associated angular variables $\phi$ and $\alpha$ through

$$\begin{cases} \gamma = \phi + \alpha , \\ \theta = \phi - \alpha . \end{cases}$$

(3.5.10)

Further, we introduce $x = -2\chi$. The $SO(2, 1)$ realization (3.5.5) can be rewritten as

$$\begin{cases} A_\pm = e^{\pm i(\phi + \alpha)} \left[ \frac{1}{2} \frac{\partial}{\partial \chi} - \frac{1}{2} \coth 2\chi (i \frac{\partial}{\partial \phi} + i \frac{\partial}{\partial \alpha} \mp 1) \\ + \frac{1}{2 \sinh 2\chi} (i \frac{\partial}{\partial \phi} - i \frac{\partial}{\partial \alpha}) \right] \\ = \frac{1}{2} e^{\pm i(\phi + \alpha)} \left[ \frac{\partial}{\partial \chi} - \tanh \chi (i \frac{\partial}{\partial \phi}) \\ - \coth \chi (i \frac{\partial}{\partial \alpha}) \pm \coth \chi \right] , \\ A_3 = -\frac{i}{2} (\frac{\partial}{\partial \phi} + \frac{\partial}{\partial \alpha}) . \end{cases}$$

(3.5.11)

Up to a similarity transformation $F = \sinh \chi$, the $SO(2, 1)$ realization (3.5.5) agrees with the $SO_4(2, 1)$ algebra of the $SO(2, 2)$ realization (3.1.9) and the integral constant $c$ appearing in
the general solution to the $SO(2,1)$ realization turns out to be the fixed quantum number $m_b$ in the $SO(2,2)$ realization, i.e.

$$m_b = (-i \frac{\partial}{\partial \theta}) = \text{constant}.$$ 

In the above discussion, we have shown that, as far as the hypergeometric equation is concerned, the $SO(2,2)$ potential group gives a proper description. But for the other type of $SO(2,1)$ realization, the relation between the $SO(2,1)$ and $SO(2,2)$ realizations is not trivial. The well known limiting process from the hypergeometric function to the confluent hypergeometric function suggests a contraction process connecting the two realizations.

Let us consider a hypergeometric equation

$$[z(1 - z) \frac{d^2}{dz^2} + (\gamma - (\alpha + \beta + 1)z) \frac{d}{dz} - \alpha \beta]f(z) = 0. \quad (3.5.12)$$

Substituting $z$ by $\frac{z}{b}$ into Eq.(3.5.12), we have

$$[z(1 - \frac{z}{b}) \frac{d^2}{dz^2} + (\gamma - (\alpha + \beta + 1)\frac{z}{b}) \frac{d}{dz} - \alpha \beta \frac{1}{b}]f(z) = 0. \quad (3.5.13)$$

Letting $b = \beta$ and taking the limit of Eq.(3.5.13) as $b = \beta \rightarrow \infty$, we obtain the confluent hypergeometric equation

$$[z \frac{d^2}{dz^2} + (\gamma - z) \frac{d}{dz} - \alpha]f(z) = 0. \quad (3.5.14)$$

A similar procedure can be applied to the $SO(2,2)$ realization (3.1.9). With the following substitution

$$\begin{cases} 
\gamma = \phi + \alpha, \\
\theta = \phi - \alpha, \\
\beta = -2\chi,
\end{cases} \quad (3.5.15)$$

in Eq.(3.1.9), we have the $SO_\alpha(2,1)$ realization

$$\begin{cases} 
A_\pm = e^{\pm i \gamma}[\pm \frac{\partial}{\partial \beta} + \coth(\beta) \frac{i}{\sinh\beta} \frac{\partial}{\partial \gamma}], \\
A_3 = -i \frac{\partial}{\partial \gamma}.
\end{cases} \quad (3.5.16)$$
The Relation between $SO(2, 1)$ and $SO(2, 2)$ Realizations

for the $SO(2, 2)$ decomposition

$$SO(2, 2) \cong SO_a(2, 1) \oplus SO_b(2, 1).$$

In the realization of the hypergeometric equation we prefer to use the variable $y = \frac{1}{\cosh \gamma}$. After the transformation of the variable from $\chi$ to $y$, Eq.(3.5.16) is reduced to

$$A_\pm = e^{\pm i\gamma} [\pm y (1 - y) \frac{\partial}{\partial y} + (1 - y)^{-1/2} (1 - \frac{y}{2}) (-i \frac{\partial}{\partial \gamma}) + (1 - y)^{-1/2} \frac{y}{2} (-i \frac{\partial}{\partial \theta})],$$

$$A_3 = -i \frac{\partial}{\partial \gamma}.$$  

(3.5.17)

A substitution $y \rightarrow \frac{1}{y}$ will transform Eq.(3.5.17) into

$$A_\pm = e^{\pm i\gamma} [\pm y (1 - \frac{y}{b}) \frac{\partial}{\partial y} + (1 - \frac{y}{b})^{-1/2} (1 - \frac{y}{2b}) (-i \frac{\partial}{\partial \gamma}) + (1 - \frac{y}{b})^{-1/2} \frac{y}{2b} (-i \frac{\partial}{\partial \theta})],$$

$$A_3 = -i \frac{\partial}{\partial \gamma}.$$  

(3.5.18)

While keeping $\frac{1}{b} (-i \frac{\partial}{\partial \theta}) = 1$, we take the limit $b \rightarrow \infty$ and obtain

$$A_\pm = e^{\pm i\gamma} [\pm y \frac{\partial}{\partial y} + (-i \frac{\partial}{\partial \gamma}) + \frac{y}{2}],$$

$$A_3 = -i \frac{\partial}{\partial \gamma}.$$  

(3.5.19)

After a transformation $\phi = -\gamma$, we have

$$I_\pm = A_\mp = e^{\pm i\phi} [\pm y \frac{\partial}{\partial y} + (i \frac{\partial}{\partial \phi}) + \frac{y}{2}],$$

$$I_3 = -A_3 = -i \frac{\partial}{\partial \phi}.$$  

(3.5.20)

Up to a similarity transformation $F = y^{1/2}$, Eq.(3.5.20) is the same as the realization (2.7.2), where we started with the discussion of the confluent hypergeometric equation.

The above procedure tells us that the contraction from the $SO(2, 2)$ potential group structure to the other type of $SO(2, 1)$ potential group structure which is associated with the confluent hypergeometric equation is achieved through taking the limit

$$m_b = (-i \frac{\partial}{\partial \theta}) \rightarrow \infty$$

in a proper way.
Since Natanzon’s transformation leads to a solvable class of potentials related to the $SO(2,2)$ potential algebra we may expect that applying the same limit to Natanzon’s transformation may lead to a solvable class of potentials associated with the $SO(2,1)$ potential algebra. This is really so, but we obtain nothing new. What we find is

\[
(\frac{dy}{dr})^2 = \frac{y^2}{c_0},
\]

which will lead to the transformation we used the in the $SO(2,1)$ realization of Morse potential.
Chapter 4.

Coulomb Scattering Problem

Coulomb scattering\textsuperscript{[25]} is the only known solvable three-dimensional scattering problem. Three-dimensional Coulomb scattering has $SO(3, 1)$ symmetry. An algebraic approach to Coulomb scattering problem was first presented by Zwanziger\textsuperscript{[10]}. However, Zwanziger’s method cannot be generalized to other problems. In this chapter we shall use the Coulomb problem to show how the algebraic procedure for scattering problems developed in chapter 2 can be generalized to higher dimensional problems and to symmetry group problems, where the group operation leaves the scattering energy invariant. Then we will discuss the general form of $SO(3, 1)$ scattering.

4.1. The Two-dimensional Coulomb Scattering Problem

All the potentials we have discussed in the last two chapters are essentially one-dimensional potentials, even though some of them may be used as fake three-dimensional potentials. In these problems the main feature of the potential group structure is that the Hamiltonian involves only the Casimir operator of the potential group, so that different states in the same multiplet are at the same energy but correspond to different potential strengths. The symmetry problems in higher dimensional space also possess such a feature if we view different partial waves as different states in a multiplet with the effective potential defined by

$$U_i(r) = V(r) + \frac{l(l + 1)}{r^2}. \quad (4.1.1)$$

Operations of the symmetry group leave the energy invariant but connect states of different angular momentum, which are associated with different strengths of the centrifugal potential. Therefore, the algebraic procedure developed in chapter 2 can be applied to symmetry group problems.

It is well known that the two-dimensional Coulomb problem has $SO(3)$ symmetry\textsuperscript{[25]} for bound states and $SO(2, 1)$ symmetry for scattering states. The two-dimensional Coulomb
The Two-dimensional Coulomb Scattering Problem

Hamiltonian

\[ H = \frac{1}{2M} p^2 - \frac{\alpha}{r} \] (4.1.2)

commutes with three operators, the angular momentum \( L_3 \) and the Lenz vector \( A_1, A_2 \):

\[
\begin{cases}
L_3 = -i \frac{\partial}{\partial \phi}, \\
A_1 = -\frac{ax_1}{r} + \frac{1}{2M} (L_3 P_2 + P_2 L_3), \\
A_2 = -\frac{ax_2}{r} - \frac{1}{2M} (L_3 P_1 + P_1 L_3).
\end{cases}
\] (4.1.3)

Here \( x_i (i = 1, 2) \) are the Cartesian coordinates of the two-space and \( \phi \) is the polar angle.

We have

\[
\begin{cases}
[H, A_i] = 0, (i = 1, 2), \\
[H, L_3] = 0.
\end{cases}
\] (4.1.4)

So far as the scattering states at certain energy are concerned the energy \( E > 0 \) is invariant under the three operators. If \( K_i = \frac{\hbar}{k} A_i, (i = 1, 2) \), where \( E = \frac{\hbar^2}{2M} \), then \( K_1, K_2 \) and \( L_3 \) form an \( SO(2,1) \) algebra satisfying the commutation relations

\[
\begin{cases}
[L_3, K_\pm] = \pm K_\pm, \\
[K_+, K_-] = -2L_3,
\end{cases}
\] (4.1.5)

where \( K_\pm = K_1 \pm iK_2 \) are the shift operators of the \( SO(2,1) \) algebra. Notice that

\[ K_\pm = -\alpha' \hat{x}_\pm \mp i(P_\pm L_3 + L_3 P_\pm), \] (4.1.6)

where

\[
\begin{cases}
\alpha' = \alpha M/k, \\
P_\pm = P_1 \pm iP_2, \\
\hat{x}_\pm = (x_1 \pm ix_2)/r,
\end{cases}
\] (4.1.6')

and the \( SO(2,1) \) Casimir invariant is

\[ C_2 = L_3^2 - K_1^2 - K_2^2 = -\frac{1}{4} - \alpha'^2. \] (4.1.7)

This realization is characterized by the quantum number

\[ j = -1/2 + i\alpha'. \] (4.1.7')
The difference between the $SO(2,1)$ potential group and the $SO(2,1)$ symmetry group, as we have seen from this example, is that in the symmetry problem the Euclidean group $E(2)$, which is formed by $L_3$ and $P_\pm$, is built in because it is a scattering problem in the two-space, while the two-dimensional Euclidean group in the potential group problem is introduced by combining the one-dimensional $E(1)$ operator $P = -i\frac{\partial}{\partial \varphi}$ with an $E(2)$ realization formed by $\phi \pm i\varphi$ and $-i\frac{\partial}{\partial \varphi}$. This feature will be used in constructing a three-dimensional scattering model, where an $SO(3,2)$ potential group can be formed by combining the $E(3)$ generators of the three-space with the same $E(2)$ realization. Such a model will be discussed in the next chapter.

For a symmetry problem, once the symmetry generators are constructed, the algebraic procedure follows in parallel to that given in chapter 2. First, we denote the asymptotic limits of the operators and the state vectors by superscript "$\infty$", i.e.

$$\lim_{r \to \infty} K_\pm = K_\pm^\infty,$$

$$\lim_{r \to \infty} |j, m\rangle = |j, m\rangle^\infty$$

where $|j, m\rangle$ is the $SO(2,1)$ state vector and $|\pm k, m\rangle$ is the asymptotic $E(2)$ state. Asymptotic $E(2)$ states are characterized by

$$L_3|\pm k, m\rangle = m|\pm k, m\rangle,$$

$$P_+|\pm k, m\rangle = \pm k|\pm k, m + 1\rangle,$$

$$P_-|\pm k, m\rangle = \pm k|\pm k, m - 1\rangle.$$  

Since $x_+$ can be written equivalently in terms of the asymptotic generators as $P_+/(\pm k)$, where $\pm k$ refer to the $E(2)$ representations $\pm k$ to be used, the $SO(2,1)$ raising operator has the asymptotic form

$$K_+^\infty = -\alpha' x_+ - i\frac{\partial}{\partial k}(2L_3P_+ - P_+)$$

$$= -\frac{i}{k}[L_3P_+ + (-1/2 - i(\pm \alpha'))]P_+].$$

Eq.(4.1.10) obeys the general asymptotic form (2.6.12) for the $SO(2,1)$ asymptotic structure. Here the characteristic function takes the form

$$f(k) = -\alpha' = -M\alpha/k,$$
and the phase factors $e^{i\alpha \pm}$ are the same for the $E(2)$ representations $\pm k$, i.e. $-i$. The whole procedure to reverse the order of the action of the $SO(2,1)$ raising operator and performing the asymptotic limit and to derive the recursion relation for the reflection amplitude follows as in section 2.6. We obtain the reflection amplitude

$$R_m(k) = (-1)^m \frac{\Gamma(m+1/2-i\alpha')}{\Gamma(m+1/2+i\alpha')} \Delta(k). \quad (4.1.11)$$

Notice that in the symmetry group case the angular momentum $m$ is meaningful and takes only integer values.

The scattering matrix is defined in such a way that the $m$-partial wave tends to

$$\sin(kr - \frac{\pi}{2}(m-1/2) + \delta_m(k)),$$

as $r \to \infty$. Therefore, we have

$$S = e^{2i\delta_m} = e^{i(m+1/2)x} R_m(k). \quad (4.1.12)$$

Then we have

$$S_m(k) = \frac{\Gamma(m+1/2-i\alpha')}{\Gamma(m+1/2+i\alpha')} \Delta'(k), \quad (4.1.13)$$

where $\Delta'(k) = i\Delta(k)$. Except for the explicit form of the raising operator the whole procedure is cast in a purely algebraic fashion. It is easy to see, divorced from any specific realization, that the general form of the scattering matrix in a two-dimensional $SO(2,1)$ symmetry problem is

$$S_m(k) = \frac{\Gamma(m+1/2+i\alpha(k))}{\Gamma(m+1/2-i\alpha(k))} \Delta(k). \quad (4.1.14)$$

### 4.2. The Connection between the Dynamical Group and Euclidean Group

Up to now the Euclidean group we have been using is $E(2)$. Since there is no ambiguity in the choice of the $E(2)$ representations, we deferred the discussion of the connection between the $S(2,1)$ group and the $E(2)$ group. But when higher dimensional groups are introduced into the discussion there may exist quite different notations in the representation theory. In order that the representations describing the scattering states and the
asymptotic states are consistent with each other we have to clarify the connection between them. Actually, the $E(2)$ representations describing the asymptotic states can be directly constructed from the $SO(2,1)$ representations induced by the scattering states. This is accomplished by a limiting process known as contraction\cite{26}. We will discuss the contraction from $SO(2,1)$ to $E(2)$ first.

Consider an $SO(2,1)$ dynamical algebra formed by three generators $J_\pm$ and $J_3$. They satisfy the following commutation relations

\[
\begin{align*}
[J_3, J_\pm] &= \pm J_\pm, \\
[J_+, J_-] &= -2J_3.
\end{align*}
\]

The contraction from the $SO(2,1)$ algebra to the $E(2)$ algebra can be performed by a limiting process. Defining $P_\pm^\epsilon = e^{\epsilon J_\pm}$, we see that in the limit $\epsilon \to 0$, $P_\pm^\epsilon$ and $J_3$ satisfy the $E(2)$ commutation relations

\[
\begin{align*}
[J_3, P_\pm^\epsilon] &= \epsilon [J_3, J_\pm] \\
&= \epsilon (\pm J_\pm) = \pm P_\pm^\epsilon, \\
[P_+^\epsilon, P_-^\epsilon] &= \epsilon^2 [J_+, J_-] \\
&= \epsilon^2 (-2J_3) \to_{\epsilon \to 0} 0.
\end{align*}
\]

The suitable limiting process for the representations is easily found by inspecting the Casimir invariants which characterize the representations. The $SO(2,1)$ continuous series representations describing the scattering states are labeled by $j = -1/2 + is, (0 < s < \infty)$. The eigenvalue of the Casimir invariant $C_2$ is $j(j+1)$, i.e.

\[
\langle C_2 \rangle = \langle J_3^2 - J_3 - J_+J_- \rangle = -1/4 - s^2.
\]

The $E(2)$ representations are labeled by $\pm k(0 < k < \infty)$, and the corresponding eigenvalue of the $E(2)$ Casimir invariant $P^2$ is $k^2$. In the limiting process $\epsilon \to 0$, the $SO(2,1)$ representation $j$ is required to reproduce the $E(2)$ representations $\pm k$. Therefore, the limiting
process has to be subjected to some constraints. That is
\[ \langle P^2 \rangle = \langle P_+^2 P_-^2 \rangle = \varepsilon^2 (J_+ J_-) = \varepsilon^2 [\frac{1}{4} + s^2 + \langle J_3^2 \rangle - \langle J_3 \rangle] \to \varepsilon \to 0 \ k^2. \]  

Eq.(4.2.4) can be satisfied by letting \( s \to \infty \) in such a way that
\[ \varepsilon s = \pm k \]

is kept constant.

In such a process the \( SO(2,1) \) representation \( j \) produces definite \( E(2) \) representations \( \pm k \). For example, the action of the raising operator \( P_+ \) is well defined by the action of \( J_+ \). If in the \( SO(2,1) \) representation we have
\[ J_+ |j, m\rangle = [(m + 1/2)^2 - (j + 1/2)^2]^{1/2} |j, m\rangle \]

accordingly, we have
\[ P_+ |\pm k, m\rangle = \lim_{\varepsilon \to 0} \varepsilon J_+ |j, m\rangle = \lim_{\varepsilon \to 0} \varepsilon [(m + 1/2)^2 + s^2]^{1/2} |j, m\rangle \]
\[ = \lim_{\varepsilon \to 0} (\pm k) |j, m\rangle = \pm k |\pm k, m\rangle, \]
in the asymptotic \( E(2) \) representations \( \pm k \), which is exactly the same as we did before.

Three-dimensional Coulomb scattering has \( SO(3,1) \) symmetry. Before generalizing our procedure to this case we have to discuss the contraction from an \( SO(3,1) \) representation to an \( E(3) \) representation. Since different authors may use quite different notations, it is not so obvious how to work consistently without performing the contraction procedure.

The \( SO(3,1) \) algebra is composed of six generators, \( L_i \) and \( K_i \); \( i = 1, 2, 3 \). They satisfy the following commutation relations:
\[ \begin{align*}
[L_i, L_j] &= i\varepsilon_{ijk} L_k, \\
[L_i, K_j] &= i\varepsilon_{ijk} K_k, \\
[K_i, K_j] &= -i\varepsilon_{ijk} L_k,
\end{align*} \]  

(4.2.7)
where $\epsilon_{ijk}$ is the unit anti-symmetric tensor and $i, j, k = 1, 2, 3$. Define $P_i^\epsilon = \epsilon K_i$ for $i = 1, 2, 3$. The limiting process $\epsilon \to 0$ brings us from the $SO(3,1)$ algebra to the $E(3)$ algebra:

\[
\begin{align*}
[L_i, [L_j, P_i^\epsilon]] &= i\epsilon_{ijk}L_k, \\
[L_i, P_j^\epsilon] &= \epsilon [L_i, K_j], \\
&= \epsilon \epsilon_{ijk}K_k = i\epsilon_{ijk}P_k^\epsilon, \\
[P_i^\epsilon, P_j^\epsilon] &= -i\epsilon^2 \epsilon_{ijk}L_k \to 0. 
\end{align*}
\]

The $SO(3,1)$ representations are labeled by a pair of numbers $(l_0, c)$ which are related to the two Casimir invariants $L^2 - K^2$ and $L \cdot K$ by

\[
\begin{align*}
L^2 - K^2 &= l_0^2 + c^2 - 1, \\
L \cdot K &= il_0c. 
\end{align*}
\]

The $E(3)$ representations are labeled by a pair of numbers $(k_0, \pm k)$ which are related to the two Casimir invariants $P^2$ and $L \cdot P$ by

\[
\begin{align*}
P^2 &= k^2, \\
L \cdot P &= -k_0(\pm k). 
\end{align*}
\]

It is easy to see the correspondence between $l_0$ and $k_0$ since they serve as the lower bounds for the angular momentum $l$ in the $SO(3,1)$ and $E(3)$ representations respectively. If the two representations are related by a contraction process the lower bounds of the angular momentum for these two representations coincide, i.e.

\[
l_0 = k_0.
\]

From the second equation of Eq.(4.2.9) we have

\[
\epsilon K \cdot L = \epsilon i l_0 c \\
= P^\epsilon \cdot L = -k_0(\pm k).
\]

Then we obtain that, as $\epsilon \to 0$, $\epsilon c$ has to be kept constant, i.e.

\[
\epsilon c = i(\pm k).
\]
Eqs. (4.2.11) and (4.2.11') give the correspondence between the $SO(3,1)$ and $E(3)$ representations in the asymptotic limit. As to explicit expressions for the matrix elements of the $SO(3,1)$ and $E(3)$ generators, we have to be careful about conventions for the phase factors of the states with different angular momentum $l$. Usually, we do not care what convention to choose as long as we stay with the same one. But here in our discussion the representations of two groups are involved; the phase conventions for these two representations have to be matched so that explicit expressions for matrix elements for the two groups can be put in the same equation. The matched phase conventions have been chosen for the contraction from $SO(3,1)$ to $E(3)$ as we summarizes the explicit forms of matrix elements for the generators in the appendices.

4.3. The Three-dimensional Coulomb Scattering Problem

The three-dimensional Coulomb problem plays an important role in the history of modern quantum mechanics. The solution of hydrogen atom was the first successful example of the non-relativistic quantum mechanics. Later on it was shown that the bound state spectrum of the Coulomb problem could also be obtained from the commutation relations of the angular momentum vector $L$ and the Runge-Lenz vector $A$\cite{1}. Not long after that it was recognized that this algebra generates the $O(4)$ symmetry group for bound states and $O(3,1)$ symmetry group for scattering states\cite{2} and it was further realized\cite{3} that the separability of the wave equation in spherical and parabolic coordinates is due to this symmetry. In this section we shall show that our algebraic procedure to the scattering problem provides us with an alternative to the algebraic calculation of non-relativistic Coulomb phase shifts due to the same symmetry.

Let the Hamiltonian of a Coulomb system be

$$H = \frac{p^2}{2M} - \frac{\alpha}{r}.$$ \hspace{1cm} (4.3.1)

The angular momentum vector and Runge-Lenz vector are defined by

$$\begin{align*}
L &= r \times P, \\
A &= \frac{1}{2M} (P \times L - L \times P) - \alpha \hat{r},
\end{align*}$$ \hspace{1cm} (4.3.2)
The Three-dimensional Coulomb Scattering Problem

where \( \hat{r} = \frac{r}{r} \), and they satisfy

\[
\begin{align*}
[L, H] &= 0, \\
[A, H] &= 0,
\end{align*}
\]

(4.3.3)

and

\[
\begin{align*}
[L_i, L_j] &= i\epsilon_{ijk}L_k, \\
[L_i, A_j] &= i\epsilon_{ijk}A_k, \\
[A_i, A_j] &= -i\epsilon_{ijk}L_k(2H/M),
\end{align*}
\]

(4.3.4)

and

\[
\begin{align*}
L \cdot A &= 0, \\
A^2 &= (L^2 + 1)(2H/M) + \alpha^2.
\end{align*}
\]

(4.3.5)

When we restrict ourselves to the subspace where \( H \) has a definite value \( k^2/2M (k > 0) \), we can renormalize \( A \) to

\[
K = (M/k)A,
\]

(4.3.6)

and obtain an \( SO(3,1) \) algebra:

\[
\begin{align*}
[L_i, L_j] &= i\epsilon_{ijk}L_k, \\
[L_i, K_j] &= i\epsilon_{ijk}K_k, \\
[K_i, K_j] &= -i\epsilon_{ijk}L_k.
\end{align*}
\]

(4.3.7)

The \( SO(3,1) \) representation is characterized by a pair of numbers

\[
\begin{align*}
l_0 &= 0, \\
c &= i\alpha',
\end{align*}
\]

(4.3.8)

where \( \alpha' = M\alpha/k \), since from Eq.(4.3.5) the two \( SO(3,1) \) Casimir invariants are

\[
\begin{align*}
L \cdot K &= 0, \\
L^2 - K^2 &= -1 - \alpha'^2.
\end{align*}
\]

(4.3.9)

The \( SO(3,1) \) basis states \(|\alpha', l, m\rangle \) are given by

\[
\begin{align*}
(L^2 - K^2)|\alpha', l, m\rangle &= -(1 + \alpha'^2)|\alpha', l, m\rangle, \\
L^2|\alpha', l, m\rangle &= l(l + 1)|\alpha', l, m\rangle, \\
L_3|\alpha', l, m\rangle &= m|\alpha', l, m\rangle,
\end{align*}
\]

(4.3.10)
where the quantum numbers \( l \) and \( m \) take the following values:
\[
\begin{align*}
\begin{cases}
m = -l, -l + 1, -l + 2, \ldots, l - 1, l, \\
l = 0, 1, 2, \ldots.
\end{cases}
\end{align*}
\] (4.3.11)

In the Coulomb problem, the \( E(3) \) group is well defined and the \( SO(3,1) \) generators are expressed in terms of the \( E(3) \) generators \( L, P \). The contraction process tells us that the corresponding \( E(3) \) representation describing the asymptotic states is labeled by a pair of numbers
\[
(k_0 = 0, \pm k),
\] (4.3.12)

and the basis states \( |\pm k, l, m\rangle \) are given by
\[
\begin{align*}
P^2|\pm k, l, m\rangle &= k^2|\pm k, l, m\rangle, \\
L^2|\pm k, l, m\rangle &= l(l + 1)|\pm k, l, m\rangle, \\
L_3|\pm k, l, m\rangle &= m|\pm k, l, m\rangle,
\end{align*}
\] (4.3.13)

where \( l \) and \( m \) take the same values as in (4.3.11).

The Coulomb scattering states are related to the asymptotic states through
\[
\lim_{r \to \infty} |\alpha', l, m\rangle = A_{lm}(k)|-k, l, m\rangle + B_{lm}(k)|+k, l, m\rangle.
\] (4.3.14)

It can be easily shown by using the \( O(3) \) raising and lowering operators on both sides of Eq.(4.3.14) that \( A_{lm}(k) \) and \( B_{lm}(k) \) are independent of \( m \), and we shall drop the subscript \( m \) in the following discussion. In fact, this is the consequence of the \( O(3) \) symmetry, and we shall keep this convention without further notice whenever the \( O(3) \) symmetry is assumed.

For any of the \( SO(3,1) \) representations we can define a \( l \)-raising operator \( X_+(l) \) and a \( l \)-lowering operator \( X_-(l) \) such that
\[
\begin{align*}
\begin{cases}
X_+(l) &= K_- L_+ + K_3 L_3 + (l + 1) K_3 + \frac{i\lambda_0 c}{l + 1} L_3 + i\lambda_0 c I, \\
X_-(l) &= K_+ L_- + K_3 L_3 + l K_3 + \frac{i\lambda_0 c}{l} L_3 + i\lambda_0 c I,
\end{cases}
\end{align*}
\] (4.3.15)

where \( K_\pm = K_1 \pm i K_2 \) and \( L_\pm = L_1 \pm i L_2 \). If we write the \( SO(3,1) \) basis vector as \( |l, m\rangle \), then we have (see Appendix E)
\[
\begin{align*}
\begin{cases}
X_+(l)|l, m\rangle &= \left(2l + 1\right)\left[(l + 1)^2 - m^2\right]l\left[(l + 1)^2 - l_0^2\right]\left[(l + 1)^2 - c^2\right]\left[l + 1, m\right], \\
X_-(l)|l, m\rangle &= \left(2l + 1\right)\left[l^2 - m^2\right]l\left[l^2 - l_0^2\right]\left[l^2 - c^2\right]\left[l, m\right].
\end{cases}
\end{align*}
\] (4.3.16)
Note the difference between the $SO(3, 1)$ $l$-raising and $l$-lowering operators $X_\pm(l)$ and the $SO(2, 1)$ $m$-raising and $m$-lowering operators $J_\pm$. The latter is independent of $m$, but the former is dependent on $l$. Further, $X_\pm(l)$ do not directly correspond to representations of $SO(3, 1)$, but rather to representations of the universal enveloping algebra of $SO(3, 1)$. In the $SO(3, 1)$ representation describing Coulomb scattering states, i.e. $(l_0 = 0, c = i\alpha')$, Eq.(4.3.16) is reduced to

$$
\begin{align*}
X_+(l)|\alpha', l, m) &= \left[\frac{(2l + 1)((l + 1)^2 - m^2)((l + 1)^2 + \alpha'^2)}{2l + 3}\right]^{1/2}(-i)|\alpha', l + 1, m), \\
X_-(l)|\alpha', l, m) &= \left[\frac{(2l + 1)((l^2 - m^2)((l^2 + \alpha'^2)}{2l + 1}\right]^{1/2}i|\alpha', l - 1, m).
\end{align*}
$$

(4.3.16')

From Eq.(4.3.2) and (4.3.6) we have

$$
\begin{align*}
K_\pm &= -\frac{i}{k} [\pm (P_\pm L_3 - P_3 L_\pm) - (-1 - i(\pm \alpha'))P_\pm], \\
K_3 &= \frac{i}{k} [\frac{1}{2}(P_+ L_- - P_- L_+) + (-1 \pm i\alpha')P_3],
\end{align*}
$$

(4.3.6')

where $P_\pm = P_1 \pm iP_2$. Hence $K$ is an operator in the universal enveloping algebra of $E(3)$. So is $X_+(l)$. From Appendix F, it can be shown that

$$
X_+(l)|\pm k, l, m) = -\left[\frac{(l + 1)^2 - m^2}{(2l + 1)(2l + 3)}\right]^{1/2}(2l + 1)(l + 1 - i(\pm \alpha')(\pm 1)|\pm k, l, m),
$$

(4.3.17)

where $\pm$ have to be used in the $E(3)$ representations $\pm k$. Assuming that the order of performing the action of the $SO(3, 1)$ operator and taking the asymptotic limit $r \to \infty$ can be reversed, i.e.

$$
\lim_{r \to \infty} X_+(l)|\alpha', l, m) = A_i(k)X_+(l)| - k, l, m) + B_i(k)X_+(l)| + k, l, m),
$$

(4.3.18)

we obtain the recursion relations for $A_i(k)$ and $B_i(k)$

$$
\begin{align*}
-\left[\frac{(2l + 1)((l + 1)^2 + \alpha'^2)}{2l + 3}\right]^{1/2}iA_{i+1}(k) &= \left[\frac{(2l + 1)}{(2l + 3)}\right]^{1/2}(l + 1 + i\alpha')A_i(k), \\
-\left[\frac{(2l + 1)((l + 1)^2 + \alpha'^2)}{2l + 3}\right]^{1/2}iB_{i+1}(k) &= -\left[\frac{(2l + 1)}{(2l + 3)}\right]^{1/2}(l + 1 - i\alpha')B_i(k).
\end{align*}
$$

(4.3.19)

The recursion relation for the reflection amplitude, which is defined as $R_i(k) = B_i(k)/A_i(k)$, is then

$$
R_{i+1}(k) = -\frac{l + 1 - i\alpha'}{l + 1 + i\alpha'}R_i(k).
$$

(4.3.20)
Recalling that the radial wave function of the \( l \)-partial wave tends to 

\[
\sin(kr - \frac{l\pi}{2} + \delta_l(k))
\]
as \( r \to \infty \), we have

\[
S_l(k) = e^{2i\delta_l(k)} = e^{i(l+1)\pi} R_l(k).
\]

Finally, we obtain the scattering matrix element for the \( l \)-partial wave

\[
S_l(k) = \frac{\Gamma(l + 1 - i\alpha')}{\Gamma(l + 1 + i\alpha')} \Delta'(k),
\]

where \( \Delta'(k) = -\Delta(k) \) is independent of \( l \) and to be determined by the s-wave phase shift.

Notice that the recursion relation for the scattering matrix elements is the same as given by Zwanziger\cite{Zwanziger10}, but in Zwanziger's notation the difference between the reflection amplitude and the scattering matrix element is obscure.

4.4. The General Form of the SO(3,1) Symmetry

In order that the discussion can be divorced from any specific differential realization, as in section 2.6, we are going to derive in this section a general form for the SO(3,1) symmetry to bridge the gap between the SO(3,1) and the \( E(3) \) representations. For the sake of simplicity we still restrict ourselves to the general form up to quadratic terms.

Consider a physical system with an SO(3,1) symmetry. The Hamiltonian \( H \) of the system is commuting with the six SO(3,1) generators \( J, K \), i.e.

\[
[H, J] = [H, K] = 0,
\]

and

\[
\begin{align*}
[J_i, J_j] &= i\epsilon_{ijk} J_k, \\
[J_i, K_j] &= i\epsilon_{ijk} K_k, \\
[K_i, K_j] &= -i\epsilon_{ijk} J_k.
\end{align*}
\]
The asymptotic states of the $SO(3,1)$ scattering states form the representations $\pm k$ of an $E(3)$ realization, which consists of six generators $L, P$ satisfying the commutation relations

\[
\begin{align*}
[L_i, L_j] &= i\epsilon_{ijk}L_k, \\
[L_i, P_j] &= i\epsilon_{ijk}P_k, \\
[P_i, P_j] &= 0,
\end{align*}
\] (4.4.3)

The $E(3)$ representations are labeled by a pair of numbers $(k_0, k)$ which are related to the two $E(3)$ Casimir invariants through

\[
\begin{align*}
P^2 &= P \cdot P = k^2, \\
P \cdot L &= -k_0k.
\end{align*}
\] (4.4.4)

Since $P^2 = k^2$ leads to two $E(3)$ representations $\pm k$ we do not have to impose the condition $k > 0$.

Suppose the asymptotic $SO(3,1)$ generators can be expressed in a general form in terms of the Euclidean generators up to quadratic terms. Since the angular momentum $J$ does not change in the asymptotic limit $r \to \infty$, we drop the superscript "$\infty$" and identify $J$ with $L$, i.e.

\[
J^\infty = J = L.
\] (4.4.5)

The second commutation relation in Eq.(4.4.2) shows that $K$ is a three-vector, and so is $K^\infty$. The most general form of a hermitian three-vector in terms of two three-vectors $P, L$ is a linear combination of $L, P, P \times L$ and $L \times P$. When we restrict ourselves to the subspace where $H$ has a definite value $k^2$, the coefficients in the general form can be functions of $k$. Therefore, the general form for the vector $K^\infty$ is as follows:

\[
K^\infty = a(k)P \times L + d(k)L \times P + b(k)P + c(k)L,
\] (4.4.6)

where $a(k), d(k), b(k)$ and $c(k)$ are to be determined.

Since $K^\infty$ is hermitian and the hermitian conjugate of $P \times L$ is $- (L \times P)^\dagger$, i.e.

\[
P \times L = -(L \times P)^\dagger,
\] (4.4.7)
The General Form of the $SO(3,1)$ Symmetry

we find that $b(k), c(k)$ are real and that

$$d(k) = -a^*(k). \quad (4.4.8)$$

Eq.(4.4.6) reduces to

$$K^\infty = a(k)P \times L - a^*(k)L \times P + b(k)P + c(k)L. \quad (4.4.9)$$

In any of the $SO(3,1)$ unitary irreducible representations $K \cdot L$ does not change in the asymptotic limit $r \to \infty$, i.e.

$$\lim_{r \to \infty} K \cdot L = K^\infty \cdot L = il_0c. \quad (4.4.10)$$

From Eqs.(4.4.4) and (4.4.9) we rewrite Eq.(4.4.10) as

$$K^\infty \cdot L = i[a(k) - a^*(k)](-k_0k) + b(k)(-k_0k) + c(k)L^2 = il_0c. \quad (4.4.10')$$

In order that Eq.(4.4.10') holds true for all possible $l$ in the $SO(3,1)$ representation we find that $c(k)$ has to be zero. Therefore, Eq.(4.4.9) further reduces to

$$K^\infty = a(k)P \times L - a^*(k)L \times P + b(k)P. \quad (4.4.11)$$

Since the commutation relations do not change in the asymptotic limit, $\{K^\infty, L\}$ still obey the $SO(3,1)$ commutation relations (4.4.2). In the following we shall determine the forms of the functions $a(k)$ and $b(k)$, where $a(k)$ is complex and $b(k)$ is real, by imposing the $SO(3,1)$ commutation relations and inspecting the two $SO(3,1)$ Casimir invariants.

It is easy to see that the first two commutation relations in Eq.(4.4.2) are satisfied automatically if $K^\infty$ takes the form (4.4.11). Recalling that $P, L$ satisfy the $E(3)$ commutation relations (4.4.3) and performing straightforward and tedious algebra, we obtain

$$[K_i^\infty, K_j^\infty] = -[a(k) + a^*(k)]^2i\epsilon_{ijk}k^2L_k, \quad (4.4.12)$$

where $P^2 = k^2$ has been used in the derivation. Requiring the third commutation relation in (4.4.2) to be satisfied we have

$$[a(k) + a^*(k)]^2k^2 = 1$$
Eq. (4.4.13) simply tells us that $a(k)$ has to have the form

$$a(k) = -\frac{1}{2k} + is(k), \quad (4.4.14)$$

where $s(k)$ is a real function of $k$. Here $b(k)$ does not appear in the commutation relation (4.4.12), we have to discuss it by inspecting the two $SO(3,1)$ Casimir invariants.

The $SO(3,1)$ unitary irreducible representations are labeled by a pair of numbers $(l_0, c)$ and the $E(3)$ representations are labeled by another pair of numbers $(k_0, k)$, where $l_0$ and $k_0$ serve as the lower bounds of the angular momentum $l$ in the $SO(3,1)$ and $E(3)$ representations respectively. Since the $SO(3,1)$ representation and the $E(3)$ representations are related by the asymptotic limit $r \to \infty$, angular momentum is not changed by this limiting process. The lower bounds of angular momentum for the $SO(3,1)$ and $E(3)$ representations should coincide with each other, i.e.

$$l_0 = k_0. \quad (4.1.15)$$

Notice that, from Eq. (4.4.11), we have

$$K^{\infty 2} = L^2 - k_0^2 + 4k^2a(k)a^*(k) + k^2b^2(k) + 2ik^2b(k)[a(k) - a^*(k)]. \quad (4.4.16)$$

One of the $SO(3,1)$ Casimir invariants is

$$L^2 - K^{\infty 2} = k_0^2 - 4k^2a(k)a^*(k) - k^2b^2(k) - 2ik^2b(k)[a(k) - a^*(k)] \quad (4.4.16')$$

and the other Casimir invariant is

$$L \cdot K^{\infty} = b(k)(-kk_0) + i[a(k) - a^*(k)](-kk_0) \quad (4.4.17)$$

From Eqs. (4.4.17) and (4.4.14) we find

$$c = ik(b(k) - 2s(k)). \quad (4.4.18)$$
Substituting Eq. (4.4.18) into Eq. (4.4.11) we obtain

\[
K^\infty = \pm \frac{1}{2k} (P \times L - L \times P) + is(k)(P \times L + L \times P) + b(k)P
\]

\[
= \pm \frac{1}{2k} (P \times L - L \times P) + (b(k) - 2s(k))P
\]

\[
= \pm \frac{1}{2k} (P \times L - L \times P) + \frac{c}{ik} P. \tag{4.4.19}
\]

Introducing the notation

\[c = if(k)\tag{4.4.20}\]

where \(f(k)\) is a real function of \(k\), we reach the general asymptotic form

\[
K^\infty = \pm \frac{1}{k} \frac{1}{2} (P \times L - L \times P) \pm f(k)P. \tag{4.4.21}
\]

The representation given by Eq. (4.4.21) is characterized by the pair of numbers

\[(l_0, c = if(k)).\]

In the Coulomb problem the characteristic function takes the special form

\[f(k) = -\alpha' = -M\alpha/k.\tag{4.4.22}\]

Here we do not have to restrict ourselves to \(l_0 = 0\). The reason for not doing so is that, we expect, the representations for \(l_0 \neq 0\) may be applied to particles with non-zero spin.

In the general form (4.4.21) the overall sign remains undetermined, while in the general form for the asymptotic \(SO(2, 1)\) structure there exists an arbitrary phase factor. In other words, in the general form of the \(SO(3, 1)\) symmetry the phase factor for \(K^\infty\) is not arbitrary and the phase angle can only take two values, 0 and \(\pi\).

Once the general form (4.4.21) is reached, the algebraic procedure for the scattering matrix is straightforward and parallel to the Coulomb case except for the change:

\[(l_0 = 0, i\alpha') \rightarrow (l_0, if(k)).\]

Assuming that

\[
\lim_{r \to \infty} |f, l, m\rangle = A_l(k) - k, l, m\rangle + B_l + k, l, m\rangle, \tag{4.4.23}
\]
where \( |f, l, m) \) is the \( SO(3,1) \) scattering state and \( |\pm k, l, m) \) are the asymptotic \( E(3) \) states, and that \( K^\infty \) has the general form

\[
K^\infty = \frac{1}{k^2} (P \times L - L \times P) \pm f(k)P, \tag{4.4.24}
\]

where \( \pm \) have to be used in the \( E(3) \) representations \( \pm k \), we can construct the \( SO(3,1) \) \( l \)-raising operator \( X_+(l) \) given in Eq.(4.3.15). By reversing the order of the action of \( X_+(l) \) and the asymptotic limit, we obtain the recursion relations for \( A_l(k), B_l(k) \)

\[
\begin{align*}
\gamma_{lm} A_{l+1}(k) &= \beta_{lm} A_l(k), \\
\gamma_{lm} B_{l+1}(k) &= \beta_{lm} B_l(k),
\end{align*} \tag{4.4.25}
\]

where

\[
\gamma_{lm} = \left( \frac{(l+1)(l+1)^2 - m^2)(l+1)^2 - l_0^2}{2l + 3} \right)^{1/2} \frac{-i}{l+1},
\]

and

\[
\beta_{lm} = \mp \left[ \frac{(l+1)^2 - l_0^2}{4(l+1)^2 - 1} \right]^{1/2} \frac{2l+1}{l+1} (l+1 \pm if(k)).
\]

The recursion relation for the reflection amplitude follows:

\[
R_{l+1}(k) = -\frac{l+1+if(k)}{l+1-if(k)} R_l(k). \tag{4.4.26}
\]

The solution to the recursion relation (4.4.26) is given by

\[
R_l(k) = (-1)^l \frac{\Gamma(l+1+if(k))}{\Gamma(l+1-if(k))} \Delta(k). \tag{4.4.27}
\]

Relating the scattering matrix elements to the reflection amplitudes, finally, we have

\[
S_l(k) = \frac{\Gamma(l+1+if(k))}{\Gamma(l+1-if(k))} \Delta'(k), \tag{4.4.28}
\]

where \( \Delta'(k) \) is independent of \( l \) and to be determined by the s-wave phase shift. The last comment to conclude this chapter is that the scattering matrix elements given by Eq.(4.4.28) are independent of \( l_0 \).
In the last chapter we generalized the algebraic approach to the three-dimensional scattering problem with $SO(3, 1)$ symmetry, where the scattering matrix contains a parameter which is one of the pair labeling the $SO(3, 1)$ representations. In the Coulomb problem this parameter is known as "Sommerfeld parameter". However, it does not provide us with enough degrees of freedom to describe atomic or nuclear elastic scattering. In this chapter we attempt to extend the algebraic approach in order to involve more parameters. We construct some preliminary algebraic models for elastic scattering.

5.1. Phase shifts and Angular Distribution

In the algebraic approach what is unique is the recursion relations for the phase shifts. The solution to the recursion relation always contains an undetermined factor. In this section we shall illustrate that this undetermined factor can be ignored in so far as the angular distribution is concerned, such that the algebraic approach may be used to construct phenomenological models.

In nuclear physics we may study nuclear structure by observing the radiation from radioactive decay. However, more information can be obtained by studying the interaction between a nucleus and an external field. One way to study this interaction is to make the nucleus interact with another particle in an experiment. In usual nuclear experiments collimated beams are used to bombard a target containing the nucleus to be studied and the reaction products and their distribution in angle and energy are observed. Artificial acceleration is commonly used to obtain a much greater intensity of radiation and much greater control over the energy of particles in the concentrated beam.

In heavy ion collision we can study the interaction between two nuclei. The interaction of two nuclei is actually a complicated many-body problem. In the potential approach the optical model is constructed to describe elastic scattering. In this model the many-body
problem is replaced by the problem of two structureless particles interacting through a potential. When the spin dependence is ignored this model potential is assumed to depend only on the distance \( r \) between the centers of mass of the two nuclei, i.e.

\[
V = V(r).
\]

This potential represents an effective average of the complicated interaction. When inelastic channels are open the change of flux from the elastic scattering is achieved by making the potential complex.

In the center-of-mass frame, the corresponding problem is to find the scattering solution to the time-independent Schrödinger equation with the boundary condition

\[
\psi(r) \sim r \rightarrow \infty e^{ikz} + f(\theta) \frac{e^{ikr}}{r},
\]

where the collimated beam is along the \( z \)-direction. Since the projectile is moving in a central potential angular momentum is a good quantum number and the wave function can be expanded on the eigenfunctions of angular momentum. Because of the axial symmetry the eigenfunctions involved are just the Legendre polynomials, i.e.

\[
\Psi(r, \theta) = \sum_{l=0}^{\infty} \psi_l(r) P_l(cos \theta).
\]

If we write the radial wave function as

\[
\psi_l(r) = \frac{u_l(r)}{r},
\]

then \( u_l(r) \) satisfies the radial equation

\[
\frac{d^2 u_l(r)}{dr^2} + \frac{k^2 - U_l(r)}{r} u_l(r) = 0,
\]

where \( k^2 = (2ME/h^2) \) and

\[
U_l(r) = \frac{2M}{h^2} \left[ V(r) + \frac{l(l+1)}{r^2} \right].
\]

Notice that the central potential can be further generalized to contain \( l \)-dependence, i.e. \( V_l(r) \).
When $V_i(r) = 0$, no scattering appears. Each partial wave is the sum of an incoming wave $e^{-ikr}$ and an outgoing wave $e^{ikr}$ with equal amplitudes, i.e.

$$\psi_i(r) = i^l(2l + 1)j_l(kr) \rightarrow r \rightarrow -\infty \frac{(2l + 1)}{2ikr} [e^{ikr} - (-1)^l e^{-ikr}]. \quad (5.1.6)$$

But if a well-behaved potential $V_i(r)$ exists, the effect of the potential is to shift the phase of the out-going wave and the asymptotic behavior of the radial wave function is

$$u_l(r) \rightarrow r \rightarrow -\infty \frac{(2l + 1)}{2ik} [S_l e^{ikr} - (-1)^l e^{-ikr}]. \quad (5.1.7)$$

When the inelastic channels are not open the amplitude of the outgoing wave is just a phase factor

$$S_l = e^{2i\delta_l(k)}, \quad (5.1.8)$$

where $\delta_l(k)$ is known as the phase shift of the $l$-th partial wave. If the inelastic channels are open then we have

$$S_l = \rho_l(k)e^{2i\delta_l(k)}, \quad (5.1.9)$$

where $\rho_l < 1$ corresponds to absorption and $\rho_l > 1$ corresponds to creation. For a collimated incident beam the scattering angular amplitude is

$$f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1) (S_l - 1) P_l(cos\theta), \quad (5.1.10)$$

and the angular distribution is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2. \quad (5.1.11)$$

In the group theory approach to scattering Eqs. (5.1.10) and (5.1.11) still hold true and they are regarded as the basic equations that connect the experimental data and phenomenological models in which phase shifts are determined in an algebraic fashion.

Since

$$\sum_{l=0}^{\infty} (2l + 1) P_l(cos\theta) = \delta(cos\theta - 1),$$

for $\theta \neq 0$, Eq. (5.1.10) can be written as

$$f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1) S_l P_l(cos\theta). \quad (5.1.12)$$
If there exists a common factor $e^{2i\delta}$ undetermined for all the partial waves

$$S_i = e^{2i(\delta + \Delta_i)}, \quad (5.1.13)$$

where $\Delta_i = \delta_i - \delta$, then the angular distribution for $\theta \neq 0$ is independent of the undetermined common factor, i.e.

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

$$= \frac{1}{4k^2} \sum_{l=0}^{\infty} (2l + 1)P_l(\cos \theta)e^{2i\sigma_i} |\Delta(k)|$$

$$(\theta \neq 0). \quad (5.1.14)$$

For example, in the scattering problem with the $SO(3,1)$ symmetry we have

$$S_i = e^{2i\sigma_i} = \frac{\Gamma(l + 1 + if(k))}{\Gamma(l + 1 - if(k))} \Delta(k),$$

where $|\Delta(k)| = 1$. For $\theta \neq 0$, summing up all the partial waves we obtain the angular amplitude

$$f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1)P_l(\cos \theta)e^{2i\sigma_i} \Delta(k)$$

$$= -\frac{f(k)}{2k \sin^2 \frac{k}{2}} e^{-if(k)\ln[\sin^2 \frac{k}{2}]+2i\sigma_0} \Delta(k),$$

where

$$\sigma_i = \text{Arg}[\Gamma(l + 1 + if(k))].$$

Therefore, the angular distribution for the $SO(3,1)$ scattering can be put in the closed form:

$$\frac{d\sigma}{d\Omega} = \frac{f^2(k)}{4k^2 \sin^4 \frac{k}{2}}, \quad (\theta \neq 0). \quad (5.1.15)$$

Even though there exists an undetermined factor in the solution to the recursion relation, the angular distribution is definite and comparison can be made to experimental results by means of Eq.(5.1.15). Moreover, the energy dependence of the angular distribution can also provide information for the characteristic function $f(k)$ in the general form of the $SO(3,1)$ symmetry.

In general, in the group theory approach to scattering such problems always exist, but can be overcome in one way or another in constructing phenomenological models.
5.2. The SU(3,1) Scattering Model

In the last chapter we presented a general form for SO(3,1) scattering in which we have only one parameter to play with. In this section we shall enlarge the group to SU(3,1) such that SO(3,1) is contained as a subgroup and the SO(3,1) general form can be applied to the dynamical symmetry SU(3,1) \supset SO(3,1). In this SU(3,1) scattering model more parameters are introduced to describe the scattering.

In each SU(3,1) irreducible unitary representation, every irreducible representation of the maximum compact subgroup U(3) occurs at most once. We can write symbolically

$$H_{[SU(3,1)]} = \bigoplus_{U(3)} H_{[U(3)]},$$

where $H_{[SU(3,1)]}$ is the Hilbert space of the SU(3,1) irreducible representation and $H_{[U(3)]}$ is the Hilbert space of the U(3) irreducible representation. In practical calculations it is convenient to use boson realizations. In a boson realization we introduce an s-boson and three p-bosons. The sixteen generators

$$\begin{align*}
  &s^+s, s^+p_1^+, s^+p_0^+, s^+p_{-1}^+, \\
  &p_1s, p_1^+p_1, p_1^+p_0, p_1^+p_{-1}, \\
  &p_0s, p_0^+p_1, p_0^+p_0, p_0^+p_{-1}, \\
  &p_{-1}s, p_{-1}^+p_1, p_{-1}^+p_0, p_{-1}^+p_{-1},
\end{align*}
$$

form an U(3,1) algebra. The SU(3,1) boson representations can be characterized by the boson number

$$N = N_p - N_s = p_1^+p_1 + p_0^+p_0 + p_{-1}^+p_{-1} - s^+s,$$

where $\tilde{\rho}_\mu = (-1)^{1+\mu}p_{-\mu}$ is a tensor operator. Since the boson number operator $N$ commutes with all the sixteen U(3,1) generators the SU(3,1) algebra consists of fifteen generators:

$$\begin{align*}
  &((p^+ \times \tilde{\rho})(0) + s^+s, [(s^+p^+) (1)] + (s\tilde{p})(1)]) (1), i[(s^+p^+) (1) - (s\tilde{p})(1)] (1), \\
  &((p^+ \times \tilde{\rho})(1), (p^+ \times \tilde{\rho})(2)).
\end{align*}$$

The SU(3,1) basis can be obtained through the group chain

$$SU(3,1) \supset U(3) \supset SU(3) \supset O(3) \supset O(2).$$
The SU(3,1) Scattering Model

Therefore, we have an SU(3,1) basis $|N, N_p, l, m\rangle$ characterized by

$$H_N = \bigoplus_{N_p = \overline{N}}^{\infty} \bigoplus_{l=0}^{\infty} \bigoplus_{m=-l}^{l} |N, N_p, l, m\rangle,$$  \hspace{1cm} (5.2.4)

where $\overline{N} = \max(0, N)$.

Since we are going to apply the SO(3,1) general form to the SU(3,1) states we have to consider another group chain

$$SU(3,1) \supset O(3,1) \supset O(3) \supset O(2).$$  \hspace{1cm} (5.2.5)

The generators of the subgroups in the group chain (5.2.5) are as follows:

$$SO(3,1) : [(s^+p^+)^{(1)} + (s\hat{p})^{(1)}], (p^+ \times \hat{p})^{(1)};$$  \hspace{1cm} (5.2.6)

$$O(3) : (p^+ \times \hat{p})^{(1)};$$  \hspace{1cm} (5.2.7)

$$O(2) : (p^+ \times \hat{p})_0^{(1)}.$$  \hspace{1cm} (5.2.8)

But this SU(3,1) representation is labeled by an integer value $N$. In the potential group approach to scattering the quantum number corresponding to the scattering energy should have a continuum spectrum. We have to appeal to another SU(3,1) boson realization which contains the same SO(3,1) algebra as a subalgebra.

It is known\cite{18} that $N$ boson operators can form an $Sp(2N;\mathbb{R})$ algebra and in the $Sp(2N;\mathbb{R})$ algebra there exists a subalgebra chain

$$Sp(2N;\mathbb{R}) \supset Sp(2;\mathbb{R}) \oplus SO(n; m),$$

where $N = m + n$. In the $Sp(8;\mathbb{R})$ algebra formed by the four pairs of creation and annihilation boson operators we have a subalgebra chain

$$Sp(8;\mathbb{R}) \supset Sp(2;\mathbb{R}) \oplus SO(3; 1).$$  \hspace{1cm} (5.2.9)

The $Sp(2;\mathbb{R})$ consists of three generators

$$\begin{cases} 
J_1 = (\sqrt{3}(p^+ \times p^+)(0) + (\hat{p} \times \hat{p})(0)](0) + s^+s^++ ss)/4, \\
J_2 = i(-\sqrt{3}(p^+ \times p^+)(0) - (\hat{p} \times \hat{p})(0)](0) + s^+s^+ - ss)/4, \\
J_3 = [\sqrt{3}(p^+ \times \hat{p})](0) - s^+s + 1)/4,
\end{cases}$$  \hspace{1cm} (5.2.10)
The subalgebra which commutes with $J_5$ is the $SU(3,1)$ algebra given by (5.2.3). The subalgebra which commutes with $J_1$ is another $SU(3,1)$ algebra denoted by $SU_p(3,1)$, which contains the same $SO(3,1)$ subalgebra given by (5.2.6). This $SU_p(3,1)$ algebra consists of fifteen generators:

$$\left\{ \begin{align*}
[(p^\dagger \times p^\dagger)^{(2)} + (\bar{p} \times \bar{p})^{(2)}]^{(2)},
(p^\dagger \times p^\dagger)^{(1)},
\sqrt{3}[(p^\dagger \times p^\dagger)^{(0)} + (\bar{p} \times \bar{p})^{(0)}]^{(0)} - s^s s^s - ss,

[(p^\dagger s^s)^{(1)} + (s\bar{p})^{(1)}],
i[(p^\dagger s)^{(1)} - (s^s \bar{p})^{(1)}]^{(1)}.
\end{align*} \right.$$  

(5.2.11)

The $SU_p(3,1)$ representation is characterized by the eigenvalue of $J_1$. Since $J_1$ is a non-compact generator of the $Sp(2; \mathbb{R})$ algebra, the $SU_p(3,1)$ representation can be labeled by a quantum number with a continuum spectrum.

In this $SU(3,1)$ representation the two Casimir operators of the $SO(3,1)$ subalgebra (5.2.6) are

$$\left\{ \begin{align*}
L^2 - K^2 &= \omega (\omega + 1), \\
K \cdot L &= 0,
\end{align*} \right.$$  

(5.2.12)

where

$$\left\{ \begin{align*}
L &= \sqrt{2}(p^\dagger \times \bar{p})^{(1)}, \\
K &= (p^\dagger s^s)^{(1)} + (s\bar{p})^{(1)}.
\end{align*} \right.$$  

(5.2.13)

Therefore, the $SO(3,1)$ representation describing the scattering states is labeled by $(l_0 = 0, c = is)$ or only $s$, where $s$ is related to the quantum number $\omega$ through

$$\omega = -1 + is.$$  

The $SU_p(3,1)$ basis $|\beta, s, l, m\rangle$ is characterized by the eigenequations:

$$\left\{ \begin{align*}
J_1|\beta, s, l, m\rangle &= \beta|\beta, s, l, m\rangle, \\
(L^2 - K^2)|\beta, s, l, m\rangle &= (-s^2 - 1)|\beta, s, l, m\rangle, \\
L^2|\beta, s, l, m\rangle &= l(l + 1)|\beta, s, l, m\rangle, \\
L_3|\beta, s, l, m\rangle &= m|\beta, s, l, m\rangle.
\end{align*} \right.$$  

(5.2.14)

Assuming that the $SO(3,1)$ states have the general form of the $SO(3,1)$ asymptotic behavior we can obtain the scattering matrix for any $SU(3,1)$ state by using the expansion on the basis (5.2.14).
The basic assumption of this $SU(3,1)$ model is that

$$\lim_{r \to \infty} |\beta, s, l, m\rangle = A_{s,l}(k) |k, s, l, m\rangle + B_{s,l}(k) |k, s, l, m\rangle$$

where

$$\beta = \beta(k), \quad (5.2.15)$$

and

$$\begin{cases} 
A_{s,l} = \Gamma(l + 1 - is), \\
B_{s,l} = \Gamma(l + 1 + is).
\end{cases} \quad (5.2.16)$$

Notice that $\beta$ is a function of $k$. All the states in the same $SU(3,1)$ representation stay at the same energy. Eq.(5.2.16) is an important assumption. In the general $SO(3,1)$ form there exist an undetermined factor and an undetermined function form $f(k)$. Now, we do not have to determine the function form since it appears as an $SO(3,1)$ label $s$ for the $SO(3,1)$ representation. In general, the undetermined factor may depend on both $k$ and $s$. But if we require the $SO(3,1)$ dynamical symmetry to correspond to Coulomb scattering this factor can be so determined such that the $SO(3,1)$ label $s$ only appears in the Gamma function. Thus, in this basis everything is well-defined by Eq.(5.2.16) and Coulomb scattering turns out to be the special case of the $SU(3,1) \supset SO(3,1)$ dynamical symmetry.

In the algebraic language the $SO(3,1)$ dynamical symmetry can be realized by assuming that the effective potential operator be

$$V_{\text{eff}} = -(C_2^{so(3,1)} + 1), \quad (5.2.17)$$

where

$$C_2^{so(3,1)} = L^2 - K^2.$$ 

Coulomb scattering is characterized by the potential strength

$$\langle V_{\text{eff}} \rangle = \alpha^2. \quad (5.2.18)$$

For the more general case where only the $O(3)$ symmetry is assumed we have the group chain

$$SU(3,1) \supset SO(3,1) \supset O(3). \quad (5.2.19)$$
The effective potential operator is an $O(3)$ scalar in terms of the boson operators. If we further assume that the $s$-boson and $p$-boson operators carry the parity "+1" and "−1", respectively, and that the parity is conserved in the scattering process, only those terms in which the $p$-boson operators appear an even number of times remain. For the simplicity, we still restrict ourselves to up to quadratic terms in the $SU(3,1)$ generators. With these restrictions the most general form for the effective potential $V_{\text{eff}}$ is

$$V_{\text{eff}} = a_1 \left\{ \sqrt{3} \left[ (p^\dagger \times p^\dagger)^{(0)} + (\vec{p} \times \vec{p})^{(0)} \right] - s^t s - ss \right\}$$
$$+ a_2 \left\{ \sqrt{3} \left[ (p^\dagger \times p^\dagger)^{(0)} + (\vec{p} \times \vec{p})^{(0)} \right] - s^t s - ss \right\}$$
$$\times \left\{ \sqrt{3} \left[ (p^\dagger \times p^\dagger)^{(0)} + (\vec{p} \times \vec{p})^{(0)} \right] - s^t s - ss \right\}$$
$$+ a_3 ((p^\dagger \times \vec{p})^{(1)} \times (p^\dagger \times \vec{p})^{(1)})^{(0)}$$
$$+ a_4 \left\{ \left[ (p^\dagger s^t)^{(1)} + (s \vec{p})^{(1)} \right] \times \left[ (p^\dagger s^t)^{(1)} + (s \vec{p})^{(1)} \right] \right\}^{(0)}$$
$$+ a_5 \left\{ \left[ (p^\dagger s^t)^{(1)} + (s \vec{p})^{(1)} \right] \times \left[ (p^\dagger s^t)^{(1)} + (s \vec{p})^{(1)} \right] \right\}^{(0)}$$
$$+ a_6 \left\{ i \left[ (p^\dagger s)^{(1)} - (s \vec{p})^{(1)} \right] \times \left[ (p^\dagger s)^{(1)} - (s \vec{p})^{(1)} \right] \right\}^{(0)}$$
$$+ a_7 \left\{ \left[ (p^\dagger \times p^\dagger)^{(2)} + (\vec{p} \times \vec{p})^{(2)} \right] \times \left[ (p^\dagger \times p^\dagger)^{(2)} + (\vec{p} \times \vec{p})^{(2)} \right] \right\}^{(0)}.$$

The scattering states $|\beta, v, l, m\rangle$ are defined by

$$J_1 |\beta, v, l, m\rangle = \beta |\beta, v, l, m\rangle,$$
$$V_{\text{eff}} |\beta, v, l, m\rangle = v |\beta, v, l, m\rangle,$$
$$L^2 |\beta, v, l, m\rangle = l(l + 1) |\beta, v, l, m\rangle,$$
$$L_3 |\beta, v, l, m\rangle = m |\beta, v, l, m\rangle.$$

We can diagonalize the effective potential operator in the basis given by (5.2.14). Since $O(3)$ symmetry is assumed the effective potential operator is block diagonalized for the quantum number $l$. The diagonalization can be restricted to the fixed $l$ subspace for each partial wave.

Generally, we have an expansion

$$|\beta, v, l, m\rangle = \int ds \ c(s) |\beta, s, l, m\rangle + \sum_s c_s |\beta, s, l, m\rangle,$$

where $c(s)$ is the scattering amplitude.
for each partial wave. The asymptotic behavior of the $SO(3,1)$ basis $|\beta, s, l, m\rangle$ will determine the asymptotic behavior of the scattering states defined by Eq.(5.2.21), i.e.

$$
\lim_{r \to \infty} |\beta, v, l, m\rangle = \int ds \; c(s) A_{s,l} | - k, s, l, m\rangle + \int ds \; c(s) B_{s,l} | + k, s, l, m\rangle.
$$

(5.2.23)

Since the discrete states correspond to bound states, they have no contribution to the asymptotic limit. The diagonalization in the fixed $l$ subspace usually involves an infinite number of linear equations for the general form of the effective potential operator $V_{\text{eff}}$ given by (5.2.20). This is the main feature of the diagonalization of operators in the universal enveloping algebra of a non-compact group. The numerical method for the diagonalization has to be developed when no dynamical symmetry is concerned. If this can be done the amplitude $c(s)$ is known numerically and the scattering matrix is straightforward, i.e.

$$
S_{v,l}(k) = \frac{\int ds \; c(s) \Gamma(l + 1 + is)}{\int ds \; c(s) \Gamma(l + 1 - is)}.
$$

(5.2.24)

5.3. An $SO(3,2)$ Scattering Model

In the last section use is made of the $SO(3,1)$ general form to develop an $SU(3,1)$ scattering model. In this section we are going to present an alternative which may be of use in constructing phenomenological models.

In the one-dimensional scattering problems discussed in Chapter 2 an $SO(2,1)$ realization was introduced, where the two-dimensional angular momentum just described the potential strength and the potential group structure provided us with a connection between states with different potential strengths. In order to obtain the scattering matrix we introduce an $E(2)$ group to describe the asymptotic states. Actually, the asymptotic states of a one-dimensional problem are the $E(1)$ states which are the eigenstates of the only $E(1)$ generator $-i\frac{\partial}{\partial x}$. The $E(2)$ angular momentum does not provide us with any additional information about the asymptotic states. If we reexamine the asymptotic form, it is not difficult to see that the asymptotic operators $J_{\pm}\infty$ are expressed in terms of the $E(1)$ generator $-i\frac{\partial}{\partial x}$ and three generators of an $E(2)$ realization: $e^{\pm i\phi}$ and $-i\frac{\partial}{\partial \phi}$, i.e., the asymptotic form is written in terms of the $E(1) \otimes E(2)$ generators. Following this idea we
can generalize the potential group structure to three-dimensional scattering problems. In a three-dimensional scattering problem the asymptotic states are the $E(3)$ eigenstates. It is expected that the asymptotic form can be written in terms of the $E(3) \otimes E(2)$ generators, where the three-dimensional asymptotic states are the $E(3)$ eigenstates.

The $E(3)$ algebra consists of six generators $L_i, P$ satisfying the following commutation relations:

\[
\begin{align*}
[L_i, L_j] &= i\epsilon_{ijk}L_k, \\
[L_i, P_j] &= i\epsilon_{ijk}P_k, \\
[P_i, P_j] &= 0.
\end{align*}
\]

(5.3.1)

Suppose that in this representation the two $E(3)$ Casimir invariants have the values

\[
\begin{align*}
P^2 &= k^2, \\
P \cdot L &= 0,
\end{align*}
\]

(5.3.2)

In order to introduce an additional quantum number to label scattering states from different "potential strengths" we consider the $E(2)$ realization

\[
\begin{align*}
p_{\pm} &= e^{\pm i\psi}, \\
J_3 &= -i\frac{\partial}{\partial \psi},
\end{align*}
\]

(5.3.3)

where $p_{\pm}, J_3$ satisfy the $E(2)$ commutation relations

\[
\begin{align*}
[J_3, p_{\pm}] &= \pm p_{\pm}, \\
|p_+, p_-| &= 0.
\end{align*}
\]

(5.3.4)

In the universal enveloping algebra of $E(3) \otimes E(2)$, we can find an $SO(3,2)$ realization formed by ten generators: $L, A^\infty, B^\infty$ and $J_3$, where

\[
\begin{align*}
A^\infty &= \frac{1}{k}[\cos \psi(\frac{1}{2}(P \times L - L \times P) \pm f(k)P) + \frac{1}{2}\{\sin \psi, J_3\}P], \\
B^\infty &= \frac{1}{k}[\sin \psi(\frac{1}{2}(P \times L - L \times P) \pm f(k)P) - \frac{1}{2}\{\cos \psi, J_3\}P],
\end{align*}
\]

(5.3.5)

where $\pm$ have to be used in the $E(3) \otimes E(2)$ representation $\pm k$ and "{ }" is the anticommutator, i.e.

\[
\{a, b\} = ab + ba.
\]
It is not difficult to check that they satisfy the $SO(3,2)$ commutation relations:

\[
\begin{align*}
\{&L_i, L_j\} = i\epsilon_{ijk}L_k, \\
\{&L_i, J_3\} = 0, \\
\{&L_i, A_j^\infty\} = i\epsilon_{ijk}A_k^\infty, \\
\{&L_i, B_j^\infty\} = i\epsilon_{ijk}B_k^\infty, \\
\{&A_i^\infty, B_j^\infty\} = -i\delta_{ij}J_3, \\
\{&A_i^\infty, A_j^\infty\} = -i\epsilon_{ijk}L_k, \\
\{&B_i^\infty, B_j^\infty\} = -i\epsilon_{ijk}L_k, \\
\{&A_i^\infty, J_3\} = -iB_i^\infty, \\
\{&B_i^\infty, J_3\} = iA_i^\infty,
\end{align*}
\]

\((i,j,k = 1,2,3).\)

The two $SO(3,2)$ Casimir invariants are

\[
\begin{align*}
C_2^\infty &= L^2 + J_3^2 - (A^\infty)^2 - (B^\infty)^2 \\
&= -(f^2(k) - \frac{9}{4}) \\
P_4^\infty &= -\epsilon^a \epsilon^b L^{bc} L_{de} \\
&= 0,
\end{align*}
\]

where

\[\epsilon^a \epsilon^b L^{bc} L_{de}\]

and $\epsilon^{abcde}$ is the unit anti-symmetric tensor and $L_{ab}, (a,b = 1,2,3,4,5)$ are the angular momentum operators of the $(3,2)$ space:

\[
\begin{align*}
L_{ij} &= \epsilon_{ijk}L_k, \\
L_{45} &= J_3, \\
B_i &= L_{i4}, \\
A_i &= L_{i6}, (i,j,k = 1,2,3).
\end{align*}
\]

The center of the universal enveloping algebra of the $SO(3,2)$ algebra is generated by $C_2$ and $P_4$. 
This SO(3,2) representation can be characterized by a quantum number $\omega$, which is related to the Casimir invariant $C_2^{\infty}$ through

$$C_2^{\infty} = \omega(\omega + 3). \quad (5.3.8)$$

Therefore, we have

$$\omega = \frac{-3}{2} + if(k). \quad (5.3.9)$$

We can regard Eqs.(5.3.5) and (5.3.6) as the asymptotic structure of an SO(3,2) potential algebra formed by $L, A, B$, and $J_3$. We do not have to know the exact forms of $A, B$, and they are meaningful only through their asymptotic forms in our scattering model. Using the algebraic procedure we have so far developed we can obtain the scattering matrix for three-dimensional scattering, where the three-dimensional angular momentum $l$ is the quantum number for the $l$-th partial wave and the two-dimensional angular momentum $m' = \langle J_3 \rangle$ is introduced to label the "potential strength". Since the Casimir invariants remain the same values in the asymptotic limit $r \to \infty$ we drop the superscript "\infty" in the discussion.

This SO(3,2) representation is a singleton representation (one with no $(l, m, m')$ multiplicities greater than 1). The SO(3,2) basis $|\omega, l, m, m'\rangle$ is characterized by the eigen-equations

$$
\begin{align*}
C_2|\omega, l, m, m'\rangle &= \omega(\omega + 3)|\omega, l, m, m'\rangle, \\
L_2|\omega, l, m, m'\rangle &= l(l + 1)|\omega, l, m, m'\rangle, \\
L_3|\omega, l, m, m'\rangle &= m|m, l, m, m'\rangle, \\
J_3|\omega, l, m, m'\rangle &= m'|\omega, l, m, m'\rangle.
\end{align*}
$$

The asymptotic states are defined as

$$
\lim_{r \to \infty} |\omega, l, m, m'\rangle = |\omega, l, m, m'\rangle^{\infty} = A_{lm'}| - k, l, m, m'\rangle + B_{lm'}| + k, l, m, m'\rangle,
$$

where $|\pm k, l, m, m'\rangle$ are the $E(3) \otimes E(2)$ states satisfying the following equations

$$
\begin{align*}
P^2|\pm k, l, m, m'\rangle &= k^2|\pm k, l, m, m'\rangle, \\
L_2^2|\pm k, l, m, m'\rangle &= l(l + 1)|\pm k, l, m, m'\rangle, \\
L_3^2|\pm k, l, m, m'\rangle &= m|\pm k, l, m, m'\rangle.
\end{align*}
$$
An $SO(3,2)$ Scattering Model

and

\[
\begin{align*}
J_3 | \pm k, l, m, m' \rangle &= m' | \pm k, l, m, m' \rangle, \\
p_+ | \pm k, l, m, m' \rangle &= | \pm k, l, m, m' + 1 \rangle, \\
p_- | \pm k, l, m, m' \rangle &= | \pm k, l, m, m' - 1 \rangle.
\end{align*}
\]

(5.3.13)

In order to introduce the $SO(3,2)$ shifting operators we define a pair of vector operators

\[
D^\pm = A \pm iB.
\]

(5.3.14)

Their asymptotic forms are given by

\[
(D^\pm)^\infty = A^\infty \pm iB^\infty = e^{\pm i\psi} \frac{k}{k} [K \mp i(J_3 \pm \frac{1}{2})P],
\]

(5.3.16)

where

\[
K = \frac{1}{2}(P \times L - L \times P) \pm f(k)P.
\]

(5.3.17)

In Eq.(5.3.17) "±" have to be used for the $E(3) \otimes E(2)$ representations ±$k$, respectively. In the $E(3) \otimes E(2)$ asymptotic representation we find the shifting operators

\[
\begin{align*}
X_{++}^\infty (l) &= (D_{-}^+)^\infty L_+ + (D_3^+)^\infty L_3 + (l + 1)(D_3^+)^\infty, \\
X_{+-}^\infty (l) &= (D_{-}^-)^\infty L_+ + (D_3^-)^\infty L_3 + (l + 1)(D_3^-)^\infty, \\
X_{-+}^\infty (l) &= (D_{+}^+)^\infty L_- + (D_3^+)^\infty L_3 + l(D_3^+)^\infty, \\
X_{--}^\infty (l) &= (D_{+}^-)^\infty L_- + (D_3^-)^\infty L_3 + l(D_3^-)^\infty,
\end{align*}
\]

(5.3.18)

where

\[
\begin{align*}
D_{\pm}^+ &= D_1^+ \pm iD_2^+, \\
D_{\pm}^- &= D_1^- \pm iD_2^-.
\end{align*}
\]

(5.3.19)

The shifting operators $X_{\pm}^\infty (l)$ shift both $l$ and $m'$ at the same time, but $m$ remains invariant under their actions. The first subscripts ± indicate $l$-raising and $l$-lowering, while the second subscripts ± indicate $m'$-raising and $m'$-lowering. Their actions in the $E(3) \otimes$
$E(2)$ asymptotic representation can be summarized in the following

$$
\begin{align*}
X_{++}^\infty(l) | \pm k, l, m, m' \rangle &= \omega^l_{mm'+} | \pm k, l, m, m' + 1 \rangle, \\
X_{+-}^\infty(l) | \pm k, l, m, m' \rangle &= \omega^l_{mm'-} | \pm k, l - 1, m, m' + 1 \rangle, \\
X_{+-}^\infty(l) | \pm k, l, m, m' \rangle &= \omega^l_{mm'+} | \pm k, l + 1, m, m' - 1 \rangle, \\
X_{--}^\infty(l) | \pm k, l, m, m' \rangle &= \omega^l_{mm'-} | \pm k, l - 1, m, m' - 1 \rangle,
\end{align*}
$$

(5.3.20)

The matrix elements of $X_{\pm\pm}(l)$ in the $SO(3,2)$ representation are irrelevant to the recursion relations for the scattering matrix and we can write them symbolically as follows:

$$
\begin{align*}
X_{++}(l)|\omega, l, m, m'\rangle &= \alpha_{\omega mm'+} | \omega, l + 1, m, m' + 1 \rangle, \\
X_{+-}(l)|\omega, l, m, m'\rangle &= \alpha_{\omega mm'-} | \pm k, l - 1, m, m' + 1 \rangle, \\
X_{+-}(l)|\omega, l, m, m'\rangle &= \alpha_{\omega mm'+} | \omega, l + 1, m, m' - 1 \rangle, \\
X_{--}(l)|\omega, l, m, m'\rangle &= \alpha_{\omega mm'-} | \pm k, l - 1, m, m' - 1 \rangle,
\end{align*}
$$

(5.3.21)

Reversing the order of the action of $X_{\pm\pm}(l)$ and the asymptotic limit we obtain the recursion relations for the reflection amplitude $R_{l,m'}(k) = B_{lm'}(k)/A_{lm'}(k)$:

$$
\begin{align*}
R_{l+1,m'+1}(k) &= -\frac{l + 1 + m' + 1/2 + if(k)}{l + 1 + m' + 1/2 - if(k)} R_{l,m'}(k), \\
R_{l-1,m'+1}(k) &= -\frac{l - m' - 1/2 - if(k)}{l - m' - 1/2 + if(k)} R_{l,m'}(k), \\
R_{l+1,m'-1}(k) &= -\frac{l + 1 - m' + 1/2 + if(k)}{l + 1 - m' + 1/2 - if(k)} R_{l,m'}(k), \\
R_{l-1,m'-1}(k) &= -\frac{l + m' - 1/2 - if(k)}{l + m' - 1/2 + if(k)} R_{l,m'}(k),
\end{align*}
$$

(5.3.22)

It is easy to check that there exists a solution to the recursion relations (5.3.22), i.e.

$$
R_{l,m'}(k) = (-1)^l \frac{\Gamma(l+m'+3/2+if(k))\Gamma(l-m'+3/2-if(k))}{\Gamma(l+m'+3/2-if(k))\Gamma(l-m'+3/2+if(k))} \Delta(k)
$$

(5.3.23)
We find the scattering matrix

\[ S_{l,m'}(k) = \frac{\Gamma\left(\frac{l+m'+3/2+if(k)}{2}\right) \Gamma\left(\frac{l-m'+3/2-if(k)}{2}\right)}{\Gamma\left(\frac{l+m'+3/2-if(k)}{2}\right) \Gamma\left(\frac{l-m'+3/2+if(k)}{2}\right)} \Delta'(k), \] (5.3.24)

where \( \Delta'(k) = -\Delta(k) \).

Notice that

\[ \Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\pi^{1/2}\Gamma(2z). \]

If we want to include the \( SO(3,1) \) scattering matrix as a special case where \( m' = 1/2 \), we can write

\[ \Delta'(k) = 2^{2if(k)} \Delta_1(k) \] (5.3.25)

such that

\[ S_{l,1/2}(k) = \frac{\Gamma(l + 1 + if(k))}{\Gamma(l + 1 - if(k))} \Delta_1(k). \] (5.3.26)

For

\[
\begin{cases}
  f(k) = \pm \frac{\alpha M}{k}, \\
  \Delta_1(k) = 1,
\end{cases}
\] (5.3.27)

Eq.(5.3.26) describes Coulomb scattering. It can be shown that, even for \( m' \neq 1/2 \), the phase shifts given by Eqs.(5.3.24) and (5.3.27) tend to Coulomb phase shifts as \( l \to \infty \), since

\[ S_{l,m}(k) \sim_{l \to \infty} e^{2if(k)ln(l+1)} + o\left(\frac{1}{l^2}\right). \]

This means that under the assumption (5.3.27) the scattering matrix given by Eq.(5.3.24) describes modified Coulomb scattering. The long range part of the interaction gives the large \( l \) behavior of the phase shifts, which is of the nature of Coulomb scattering; but the short range part is modified. In the group theory approach we do not have to restrict ourselves to the case where the eigenvalue of \( J_3 \) describes the potential strength. To be more general we can assume that the potential of certain shape can be described by an operator denoted by "\( V_{e\text{ff}} \)", whose eigenvalue describes the potential strength. For example, we can take

\[ V_{e\text{ff}} = [a + (L^2 - b)^2](J_3 - 1/2), \] (5.3.28)
where the eigenvalue of $V_{\text{eff}}$, denoted by $v$, describes the potential strength. Thus we have

$$m' = v \frac{a + [l(l + 1) - b]^2} + \frac{1}{2},$$  \hspace{1cm} (5.3.29)$$

which has an $l$-dependence. The phase shifts given by Eqs.(5.3.29) tend to Coulomb phase shifts very rapidly as $l$ increases and, therefore, the modified part is really of short range nature.

In the optical model a complex potential strength can be introduced to describe the absorption. The same idea can also be used in the algebraic approach. From numerical calculations I identified that a negative imaginary component of $v$ in (5.3.29) corresponds to absorption, while a positive imaginary component corresponds to creation. As we have seen in the Ginocchio potential, energy dependence can also be introduced into the potential strength. Therefore, in general, we can assign a complex function of $l$ and $k$ to the eigenvalue of $J_3$, i.e.

$$m' = m'(l,k).$$

The form of this function may be subject to some restrictions required by the phenomenology. Numerical results with simple prescriptions in such a model have been obtained. The experimental data \cite{31} obtained in the heavy ion collision of $O^{16} + O^{16}$ are shown in Figs.(5.3.1) and (5.3.2). Fig.(5.3.1) shows the excitation function at $\theta_{CM} = 90^\circ$. Fig.(5.3.2) shows the angular distribution at $E_{CM} = 20.5MeV$. Fig.(5.3.3) and (5.3.4) show the model calculation with the following parametrization in (5.3.27) and (5.3.29):

$$\begin{align*}
    f(k) &= 28.511E^{-1/2}, \\
    Re(v) &= 3000.0, \\
    Im(v) &= -176.0E, \\
    a &= 30.0, \\
    b &= 34.0 + 6.5E^{1/2},
\end{align*}$$  \hspace{1cm} (5.3.30)$$

where $E$ is the C.M. energy in units of MeV. Fig.(5.3.3) gives the excitation function at $\theta_{CM} = 90^\circ$. Fig.(5.3.4) is the angular distribution at $E_{CM} = 25MeV$. They just show us the qualitative behavior of the simple scattering model.
Fig.(5.3.1). The experimental excitation function of $O^{16} + O^{16}$ elastic scattering$^{[31]}$ at

$\theta_{CM} = 90^\circ$. 
Fig.(5.3.2). The experimental angular distribution of $^{16}O + ^{16}O$ elastic scattering at $E_{CM} = 20.5\text{MeV}$ is plotted. The solid lines are optical model calculations$^{31}$, the dotted lines are a guide to the eye.
Fig.(5.3.3). The calculated excitation function for identical particles at $\theta = 90^\circ$ is shown. The parametrization in Eq.(5.3.29) is given by $f(k) = 28.511E^{1/2}, \nu = 3000.0 - (176.0E)i, a = 30.0, b = 34.0 + 6.5E^{1/2}$, where E is C.M. energy in units of MeV.
Fig. (5.3.4). The calculated angular distribution for identical particles at $E_{CM} = 25\, MeV$ is shown. The parametrization in Eq. (5.3.29) is the same as that given in Fig. (5.3.2).
Chapter 6.

Conclusion

When we are dealing with many-body problems such as atoms, nuclei, electrons in metals, etc., an exact and complete solution can never be achieved. Even though the electromagnetic forces are known in great detail, the positions of the spectral lines of a complex atom can be measured far more accurately than they can be calculated. Nuclear phenomena are much more complicated, and even the nuclear forces cannot be well described in detail. Models remain as important tools to our understanding of many-body systems.

Recently, algebraic models have been successfully used in a variety of problems in atomic and nuclear physics to describe the collective behavior of many-body systems. These models deal with bound state problems characterized by a discrete and finite spectrum. However, many problems in physics are characterized by a spectrum with both a discrete, finite part and a continuous part. The continuous spectra are associated with the scattering problems. In this thesis, we have presented the second steps towards an extension of the algebraic techniques to cover scattering problems.

The group theory approach to bound state problems consists in writing the Hamiltonian $H$ and the transition operators in terms of the generators of a compact group, $G$. For the case where a dynamical symmetry exists the Hamiltonian $H$ can be written in terms of the Casimir operators of a subgroup chain

$$G \supset \cdots \supset G'$$

and the transition operators can be constructed in terms of the generators of the subgroup $G'$. The energy spectrum and the transition amplitudes can be put in closed forms. In general, the energy spectrum and transition amplitudes are calculated numerically. The algebraic techniques provide a more economical and efficient framework for numerical calculations.

In the group theory approach to scattering we propose a purely algebraic procedure to obtain recursion relations for the scattering matrix. When the Hamiltonian $H$ of a physical
system is related to the Casimir invariant $C$ of a non-compact group, $G$, which is interpreted as a symmetry or a potential group, this procedure is outlined in the following:

1. Identify the representation associated with each energy $H = k^2$. One of the continuous labels for the representations of the group $G$ is a function of $k$. This function is determined by the relation connecting $H$ and $C$.

2. Construct the incoming and outgoing asymptotic representations. These representations are the representations of a certain Euclidean group $E$.

3. Choose an allowed form for the asymptotic limit of the generators of $G$ in terms of the generators of $E$. The allowed form is restricted by the requirement that the commutation relations of $G$ remain unchanged at the asymptotic limit. For the sake of simplicity we can further restrict the form, for example, to terms up to second order in the generators of $E$.

4. Assuming that the order of performing the action of an operator and taking the asymptotic limit can be reversed, apply the asymptotic form to the asymptotic representations and obtain recursion relations for the scattering matrix.

Once the general form of the scattering matrix is obtained for the group $G$, we can imbed the group $G$ in a larger group $G'$ and assume that the states for the dynamical symmetry $G' \supset \cdots \supset G$ have the general asymptotic behavior. These states provide a basis for the representation of $G'$ with which we can diagonalize an arbitrary effective potential operator $V_{\text{eff}}$ and obtain the scattering matrix accordingly.

Since the Coulomb problem is the only solvable example of a three-dimensional scattering problem we have no guideline as to which group can provide a reasonable structure for describing realistic scattering processes. This is an open problem which has to be investigated in order that the algebraic framework we proposed can find applications. Moreover, all the cases in which the scattering matrices have been obtained are special cases with a certain dynamical symmetry. Realistic scattering processes may correspond to cases without dynamical symmetry. The eigenvectors for the scattering states are to be obtained by the diagonalization of the effective potential operator $V_{\text{eff}}$ in the basis where there exists a dynamical symmetry. This can only be done numerically. However, the diagonalization of an operator in the representation of a non-compact group usually corresponds to finding...
a solution to a recursion relation involving an infinite number of states. The numerical method for the direct diagonalization can be very complicated. Therefore, the numerical method to obtain the scattering matrix has to be well-designed, otherwise the algebraic technique will lose its advantage over the potential approach.

Work is in progress along several directions. One recent development is the discovery of an algebraic way to parametrize the resonant contribution to the scattering matrix[28]. As we have seen in the Ginocchio potential, the energy dependence finds its way into the parametrization of the quantum numbers in the algebraic approach. A similar thing happened when we were dealing with the $SO(3,2)$ scattering model, where we introduced a general prescription for the quantum number $m'$

$$m' = m'(l,k).$$

This tells us that the prescription can be guided by the resonant contribution in a realistic scattering process when the phenomenological fit is to be done to the excitation function.

The appearance of resonances is an important aspect of many scattering processes. In modern quantum theory these resonances are identified as poles of the scattering matrix below the real positive energy axis. For a pole $E = E_0 - i\Gamma/2$, the real part, $E_0$, is interpreted as the resonance energy and $\Gamma$ is the resonance width. In a certain sense resonances can be regarded as quasi-bound states whose energies lie in the continuum. If bound states are considered as states with zero width, i.e. $\Gamma = 0$, bound states and quasi-bound states can be treated in a unified way. This idea can also be formulated in the algebraic language. Both bound states and quasi-bound states can be obtained as a basis of a discrete series representation. Since the bound state spectrum is real, the corresponding representation is unitary; the quasi-bound state spectrum is complex and so the corresponding representation is non-unitary. When this idea is introduced to the algebraic approach to scattering, the energy dependence of the quantum number describing the “potential strength” should be prescribed in such a way that the scattering matrix reproduces the resonant contribution through its singularities.

In this paper I intended to present a comprehensive framework on which purely algebraic scattering models could be constructed. Work is still in progress, but a realistic
scattering model has not been established. Many open problems are still left. I view the results here as an important extension of the group techniques to scattering and am hopeful that we are closer to an algebraic description of atomic and nuclear collisions similar to that developed for bound state problems.
Appendix A.

Unitary Representations of SO(2,1)

Unitary representations of $SO(2,1)$ have been discussed by many authors. What we summarize here is primarily from reference [29].

The linearly independent elements of the Lie algebra of $SO(2,1)$, denoted by $J_1, J_2, J_3$, satisfy the commutation relations:

\begin{align*}
\{ J_3, J_1 \} &= i J_2, \\
\{ J_3, J_2 \} &= -i J_1, \\
\{ J_1, J_2 \} &= -i J_3.
\end{align*}

The ladder operators

\begin{equation}
J_\pm = J_1 \pm i J_2
\end{equation}

satisfy the relations

\begin{align*}
\{ J_3, J_\pm \} &= \pm J_\pm, \\
\{ J_+, J_- \} &= -2 J_3.
\end{align*}

The Casimir invariant is given by

\begin{align*}
C_2 &= J_3^2 - J_1^2 - J_2^2 \\
&= J_3^2 - J_3 - J_+ J_-
\end{align*}

The unitary irreducible representations of $SO(2,1)$ can be grouped into three classes according to the spectra of $C_2$ and $J_3$. In all cases we can choose a standard basis \{\ket{j, m}\} where

\begin{align*}
\langle j, m\mid j, m' \rangle &= \delta_{mm'}, \\
C_2\ket{j, m} &= j(j + 1)\ket{j, m}, \\
J_3\ket{j, m} &= m\ket{j, m}, \\
J_+\ket{j, m} &= [(m + 1/2)^2 - (j + 1/2)^2]^{1/2}\ket{j, m + 1}, \\
J_-\ket{j, m} &= [(m - 1/2)^2 - (j + 1/2)^2]^{1/2}\ket{j, m - 1},
\end{align*}

The three series of the unitary irreducible $SO(2,1)$ representations are as follows:

1. The continuous principal series $C^\beta$:

\begin{equation}
j = -\frac{1}{2} + ik, (0 < k < \infty)
\end{equation}
Unitary Representations of \( SO(2,1) \)

\[ m = 0, \pm 1, \pm 2, \ldots, \quad \text{or} \quad m = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots; \]

where \( \delta = 0 \) and \( 1 \), respectively.

(2) The supplementary series \( E_j \):

\[ -\frac{1}{2} < j < 0, \quad m = 0, \pm 1, \pm 2, \ldots \]

notation \( E_j \).

(3) The discrete principal series \( D_j^\varepsilon \):

\[ j = -\frac{1}{2}, -1, -\frac{3}{2}, \ldots, \]

\[ m = -j, -j + 1, -j + 2, \ldots \quad \text{or} \quad m = j, j - 1, j - 2, \ldots, \]

where \( \varepsilon \) is "+" and "-" respectively.

There exists an outer automorphism of the Lie algebra \( J_l \rightarrow J'_l \) where

\[ (J'_1, J'_2, J'_3) = (-J_1, J_2, -J_3). \quad \text{(A.1.6)} \]

The possible values of the quantum number \( j \) for the three classes of the \( SO(2,1) \) representations are plotted in Fig.(A.1.1).
Fig. (A.1.1). The unitary representation of $SO(2,1)$ are shown in the complex $j$-plane. The wavy line is the continuous series, the x's belong to the discrete series and the shaded interval is the supplementary series.
Appendix B.

The Representation Theory of E(2)

Consistent with the notation in this thesis, the representation theory of $E(2)$ [16] is reformulated in this appendix.

A set of linearly independent elements $(p_1, p_2, J_3)$ which satisfy the commutation relations

\[
\begin{align*}
[p_1, p_2] &= 0, \\
[J_3, p_1] &= ip_2, \\
[J_3, p_2] &= -ip_1,
\end{align*}
\] (A.2.1)

form an $E(2)$ algebra. Setting

\[
p_{\pm} = p_1 \pm ip_2,
\] (A.2.2)

we rewrite the commutation relations as

\[
\begin{align*}
[p_+, p_-] &= 0, \\
[J_3, p_{\pm}] &= \pm p_{\pm}.
\end{align*}
\] (A.2.3)

The Casimir invariant of $E(2)$ is

\[
C_2 = p_1^2 + p_2^2 = p_+ p_-.
\] (A.2.4)

We will classify the $E(2)$ representations with properties:

(1) The spectrum of the operator $J_3$ is countable and discrete. Each eigenvalue of $J_3$ is non-degenerate and eigenvectors of $J_3$ form a basis for the complex vector space carrying the representation.

(2) The representation is irreducible.

(3) The operators $J_{\pm}$ are non-zero.

Theorem A.2.1. Every representation of $E(2)$ satisfying conditions (1), (2) and (3) is isomorphic to a representation $(k, m_0)$, where $k \neq 0$ and $m_0$ ($0 \leq Re(m_0) < 1$) are complex numbers. For each representation $(k, m_0)$, the spectrum of $J_3$ is given by

\[S = \{m_0 + n : n \text{ integer}\},\]

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and there exists a standard basis \( \{|k, m\}\), \( m \in S \), such that

\[
\begin{align*}
J_3|k, m\rangle &= m|k, m\rangle, \\
p_+|k, m\rangle &= k|k, m + 1\rangle, \\
p_-|k, m\rangle &= k|k, m - 1\rangle, \\
C_2|k, m\rangle &= k^2|k, m\rangle.
\end{align*}
\]  

Notice that each such representation is infinite and unbounded. The requirement that the representation \((k, m_0)\) is unitary leads to the conclusion that \(k\) and \(m_0\) are real. If \(k^2 > 0\), then there exist two \(E(2)\) representations characterized by \(\pm k\), respectively.
Appendix C.

The Symmetric Representations of SO(2,2)

The $SO(2,2)$ algebra consists of six linearly independent operators, denoted by $J_i$, $K_i$, $(i = 1, 2, 3)$, which satisfy the following commutation relations:

\[
\begin{align*}
[J_2, J_3] &= iJ_1, \\
[K_2, K_3] &= iJ_1, \\
[J_3, J_1] &= iJ_2, \\
[K_3, K_1] &= iJ_2, \\
[J_1, J_2] &= -iJ_3, \\
[K_1, K_2] &= -iJ_3, \\
[J_2, K_3] &= iK_1, \\
[K_2, J_3] &= iK_1, \\
[J_3, K_1] &= iK_2, \\
[K_3, J_1] &= iK_2, \\
[J_1, K_2] &= -iK_3, \\
[K_1, J_2] &= -iK_3.
\end{align*}
\] (A.3.1)

Define the linear operators $A_i, B_i, (i = 1, 2, 3)$ and $A_\pm, B_\pm$ by means of

\[
\begin{align*}
A_i &= \frac{1}{2}(J_i + K_i), \\
B_i &= \frac{1}{2}(J_i - K_i),
\end{align*}
\] (A.3.2)

and

\[
A_\pm = A_1 \pm iA_2,
\]

\[
B_\pm = B_1 \pm iB_2.
\] (A.3.3)

Then we have

\[
\begin{align*}
[A_3, A_\pm] &= \pm A_\pm, \\
[B_3, B_\pm] &= \pm B_\pm,
\end{align*}
\] (A.3.4)

\[
[A_+, A_-] = -2A_3, \\
[B_+, B_-] = -2B_3,
\]

and

\[
[A_i, B_j] = 0, (i, j = 1, 2, 3).
\] (A.3.4')

Eqs. (A.3.4) and (A.3.4') indicate the decomposition of $SO(2,2)$ into two commuting $SO(2,1)$ algebras, i.e

\[
SO(2,2) = SO_a(2,1) \oplus SO_b(2,1).
\] (A.3.5)

Therefore, the representation theory of the $SO(2,2)$ group is no other than a theory of the representations of $SO(2,1) \otimes SO(2,1)$. The unitary representations of $SO(2,2)$ are labeled
by a pair of numbers \((j_a, j_b)\). The basis \(|j_a, m_a; j_b, m_b\rangle\) of the \(SO(2,2)\) representation \((j_a, j_b)\) is characterized by

\[
\begin{align*}
C_2^{so(2,1)}|j_a, m_a; j_b, m_b\rangle &= j_a(j_a + 1)|j_a, m_a; j_b, m_b\rangle, \\
A_3|j_a, m_a; j_b, m_b\rangle &= m_a|j_a, m_a; j_b, m_b\rangle, \\
C_2^{so(2,1)}|j_a, m_a; j_b, m_b\rangle &= j_b(j_b + 1)|j_a, m_a; j_b, m_b\rangle, \\
B_3|j_a, m_a; j_b, m_b\rangle &= m_b|j_a, m_a; j_b, m_b\rangle,
\end{align*}
\]

(A.3.6)

In this thesis we are particularly interested in the symmetric representations of \(SO(2,2)\) characterized by

\[
C_2^{so(2,1)} = C_2^{so(2,1)} = \frac{1}{4}C_2^{so(2,2)},
\]

(A.3.7)

or

\[
j_a = j_b = j,
\]

(A.3.7')

where the \(SO(2,2)\) Casimir invariant is given by

\[
C_2^{so(2,2)} = J_3^2 + K_3^2 - J_1^2 - J_2^2 - K_1^2 - K_2^2.
\]

(A.3.8)

Two convenient bases, \(|\omega, m_1, m_2\rangle\) and \(|j, m_a, m_b\rangle\), which are defined by

\[
\begin{align*}
C_2^{so(2,2)}|\omega, m_1, m_2\rangle &= \omega(\omega + 2)|\omega, m_1, m_2\rangle, \\
J_3|\omega, m_1, m_2\rangle &= m_1|\omega, m_1, m_2\rangle, \\
K_3|\omega, m_1, m_2\rangle &= m_2|\omega, m_1, m_2\rangle,
\end{align*}
\]

(A.3.9)

and

\[
\begin{align*}
C_2^{so(2,1)}|j, m_a, m_b\rangle &= j(j + 1)|j, m_a, m_b\rangle, \\
A_3|j, m_a, m_b\rangle &= m_a|j, m_a, m_b\rangle, \\
B_3|j, m_a, m_b\rangle &= m_b|j, m_a, m_b\rangle,
\end{align*}
\]

(A.3.10)

respectively, where \(C_2^{so(2,1)} = \frac{1}{4}C_2^{so(2,2)}\), are often used for the symmetric representations of \(SO(2,2)\). Notice that \(\omega, m_1, m_2\) are related to \(j, m_a, m_b\) through

\[
\begin{align*}
\omega &= 2j, \\
m_1 &= m_a + m_b, \\
m_2 &= m_a - m_b.
\end{align*}
\]

(A.3.11)
In Chapter 3 we introduced a symmetric realization of $SO(2,2)$ on the $(2,2)$ hyperboloid. The unitary symmetric representations of $SO(2,2)$ can be grouped into three classes according to the spectrum of $C_{2}^{\text{so}}(2,1)$. They will follow the same pattern of the classification of the unitary representations of $SO(2,1)$. We will not repeat it here.

It is known that there exist discrete series representations for $SO(2,n)$, where the quantum number for the two-space rotation of the polar angle $\phi$, i.e.

$$m = (-i \frac{\partial}{\partial \phi}), \quad (A.3.12)$$

has a lower bound (or an upper bound). In the following we will give a theorem from reference [24] for the discrete series of the $SO(2,2)$ representations where the quantum number $m_1$ has a lower bound (or an upper bound).

Theorem A.3.1. Let $\rho$ be an algebraically irreducible representation of $SO(2,2)$ on

$$V = \bigoplus_{m_1,m_2} |m_1, m_2\rangle.$$

If $m_1$ is bounded from below (or from above), then

1. $m_a, m_b$ are bounded from below (or from above);
2. $m, m_a, m_b$ are positive (or negative) definite;
3. given $(m_1, m_2)$ for the fixed $m_1$,

$$|m_2| < |m_1|.$$

For the symmetric representations of $SO(2,2)$ in the discrete series we have further consequences, for example:

4. for the fixed $m_2$, $m_1$ takes the possible values

$$m_1 = -2j + |m_2| + 2n, \quad (A.3.13)$$

where the integer $n = 0, 1, 2, \ldots$.

Notice that there exists no generator of $SO(2,2)$ which shifts $m_1$ by $\pm 1$ and preserves $m_2$. Starting from certain value of $m_2$ we have to apply shifting generators an even number
of times in order to return to the same value of $m_2$. This is the reason why the consecutive values of $m_1$ for the fixed $m_2$ in (4.3.13) differ by 2.

The weight diagram of a symmetric representation of $SO(2,2)$ in the discrete series is shown in Fig.(4.3.1).
Fig. (A.3.1). The weight diagram of a symmetric representation of $SO(2,2)$ in the discrete series, where $m_1$ has a lower bound, is shown. The horizontal line show the possible values of $m_1$ for a fixed value of $m_2$, i.e., $m_1 = -\omega + |m_2| + 2n$, $n = 0, 1, 2, \ldots$. 
Appendix D.

Representations of SO(3,2)

The $SO(3,2)$ algebra consists of a set of ten linearly independent operators, denoted by $L_{ab}$, $(a, b = 1, 2, 3, 4, 5)$, where

$$L_{ab} = -L_{ba}.$$  \hfill (A.4.1)

They satisfy the commutation relations

$$[L_{ab}, L_{cd}] = -i(g_{ac}L_{bd} + g_{ad}L_{cb} + g_{bc}L_{da} + g_{bd}L_{ac}),$$  \hfill (A.4.2)

where

$$g_{ab} = (-1, -1, -1, +1, +1).$$  \hfill (A.4.3)

The center of the universal enveloping algebra of $SO(3,2)$ is generated by the Casimir invariants $C_2$ and $P_4$ defined thus:

$$C_2 = \frac{1}{2} L_{ab} L^{ab},$$  \hfill (A.4.4)

$$P_4 = -W_a W^a,$$

where

$$W^a = \frac{1}{8} \epsilon^{abcd} L_{bc} L_{de}.$$  \hfill (A.4.5)

Here $\epsilon^{12345} = +1$ and $\epsilon^{abcde}$ is anti-symmetric in all indices. The summation convention of tensor analysis is understood for repeating indices and indices are raised and lowered by means of the tensor $g^{ab}$ and $g_{ab}$ respectively.

Define the operators $L_i, A_i, B_i, (i = 1, 2, 3)$ and $J_3$ by

$$\begin{cases}
L_k = \frac{1}{2} \epsilon_{ijk} L_{ij}, \\
B_i = L_{i4}, \\
A_i = L_{i5}, \\
J_3 = L_{45},
\end{cases}$$  \hfill (A.4.6)

\((i, j, k = 1, 2, 3).$$
Then, the commutation relations (A.4.2) are written explicitly as

\[
\begin{align*}
[L_i, L_j] &= i\epsilon_{ijk}L_k, & [L_i, J_3] &= 0, \\
[L_i, B_j] &= i\epsilon_{ijk}B_k, & [A_i, J_3] &= -iB_i, \\
[L_i, A_j] &= i\epsilon_{ijk}A_k, & [B_i, J_3] &= iA_i, \\
[A_i, A_j] &= -i\epsilon_{ijk}L_k, & [A_i, B_j] &= -i\delta_{ij}J_3, \\
[B_i, B_j] &= -i\epsilon_{ijk}L_k, & (i, j, k = 1, 2, 3).
\end{align*}
\] (A.4.7)

Generally speaking, the $SO(3,2)$ representations are labeled by a pair of numbers $(x, y)$, which are related to the two $SO(3,2)$ Casimir invariants $C_2$ and $P_4$ through

\[
C_2 = \frac{5}{2} - x^2 - y^2, \\
P_4 = (x^2 - \frac{1}{4})(y^2 - \frac{1}{4}).
\] (A.4.8)

However, the general representation theory of $SO(3,2)$ is rich, and we will only present the part of it concerned in our discussion. For details readers may refer to reference [23] and [24].

The maximum compact subgroup of $SO(3,2)$ is $O(3) \otimes O(2)$. The vector space $V$ for an $SO(3,2)$ representation can be written symbolically as

\[
V = \oplus \sum_{l, m, m'} |x, y; l, m, m'),
\] (A.4.9)

where the basis $|x, y; l, m, m')$ is characterized by

\[
\begin{align*}
C_2|x, y; l, m, m') &= (\frac{5}{2} - x^2 - y^2)|x, y; l, m, m'), \\
P_4|x, y; l, m, m') &= (x^2 - \frac{1}{4})(y^2 - \frac{1}{4})|x, y; l, m, m'), \\
L^2|x, y; l, m, m') &= l(l + 1)|x, y; l, m, m'), \\
L_3|x, y; l, m, m') &= m|x, y; l, m, m'), \\
J_3|x, y; l, m, m') &= m'|x, y; l, m, m').
\end{align*}
\] (A.4.10)

Note that the multicipies of $(l, m')$ may be greater than one. If it happens we need additional quantum numbers to label the states, which are called “hidden variables”. Fortunately, in
this thesis we are only concerned with those representations with no \((l, m')\) multiplicities greater than one, called "singleton representations".

There exists a subgroup chain

\[ SO(3,2) \supset SO(2,2), \]

where the \(SO(2,2)\) algebra consists of \(L_{ab}, (a, b = 1, 2, 4, 5)\). In this \(SO(2,2)\) subalgebra the operators often used are

\[
M_3 = \frac{1}{2}(J_3 + L_3), \\
N_3 = \frac{1}{2}(J_3 - L_3).
\]

(A.4.11)

Since they are commuting with \(J_3, L_3\), they have definite values in the basis (A.4.10). Their eigenvalues are denoted by

\[
\lambda = \langle M_3 \rangle = \frac{1}{2}(m' + m), \\
\lambda' = \langle N_3 \rangle = \frac{1}{2}(m' - m).
\]

(A.4.11')

For the discrete series representations of \(SO(3,2)\) where \(m'\) is bounded from below (or above), we have the following consequences from Theorem A.3.1:

1. \(\lambda\) and \(\lambda'\) are bounded from below (or above).
2. \(m', \lambda\) and \(\lambda'\) are positive-definite (or negative-definite).
3. given \((l, m')\) for fixed \(m'\),

\[
l < |m'|.
\]

Therefore, for the discrete series representations of \(SO(3,2)\), both \(l\) and \(|m'|\) have lower bounds, denoted by \(l_0\) and \(m'_0\) respectively, i.e.

\[
l = l_0, l_0 + 1, l_0 + 2, \ldots,
\]

\[
m = -l, -l + 1, \ldots, l,
\]

\[
m' = m'_0, m'_0 + 1, m'_0 + 2, \ldots,
\]

(A.4.12)

\((or - m'_0, -m'_0 - 1, \ldots).\)

it can be shown that the two Casimir invariants are related to them through

\[
C_2 = l_0(l_0 + 1) + m'_0(m'_0 - 3),
\]

\[
P_1 = l_0(l_0 + 1)(m'_0 - 1)(m'_0 - 2).
\]

(A.4.13)
When \( l_0 = 0 \), the discrete series representations are singleton representations which usually are labeled by a quantum number \( \omega < 0 \) through

\[
C_2 = \omega(\omega + 3). \tag{A.4.14}
\]

We can easily identify

\[
m_0' = -\omega. \tag{A.4.14'}
\]

For this singleton representation, given fixed \( l \),

\[
m' - (m_0' + l) = 0, 2, 4, \ldots,
\]

or

\[
m' = 2n - \omega + l, (n = 0, 1, 2, \ldots). \tag{A.4.15}
\]

When \( l_0 = 0 \) and \( P_4 = 0 \), there exist the continuous series representations which are labeled by

\[
\omega = -\frac{3}{2} + ik, (k > 0) \tag{A.4.16}
\]

through (A.4.14). They are also singleton representations. For the differential realization of \( SO(3, 2) \) on the \((3, 2)\) hyperboloid the Casimir invariant \( P_1 = 0 \) and \( C_2 \) has a mixed spectrum. In this realization we have \( l_0 = 0 \) and both discrete and continuous series are singleton representations.
Appendix E.

The Representation Theory of \(SO(3,1)\)

In reference [27], Naimark presents a complete discussion of the representation theory of the Lorentz group. We will only summarize some of the major points in this appendix. Readers may refer to Naimark's book for details.

The \(SO(3,1)\) algebra consists of a set of linearly independent operators, denoted by \(L, K\), which satisfy the following commutation relations:

\[
\begin{align*}
[L_i, L_j] &= i\epsilon_{ijk}L_k, \\
[L_i, K_j] &= i\epsilon_{ijk}K_k, \\
[K_i, K_j] &= -i\epsilon_{ijk}L_k, \quad (i, j, k = 1, 2, 3).
\end{align*}
\]

The two \(SO(3,1)\) Casimir invariants are

\[
\begin{align*}
C_2 &= L^2 - K^2, \\
C'_2 &= K \cdot L.
\end{align*}
\]

For the sake of convenience we introduce

\[
L_\pm = L_1 \pm iL_2, \\
K_\pm = K_1 \pm iK_2.
\]

Since the maximum compact group of \(SO(3,1)\) is \(O(3)\), the vector space for each \(SO(3,1)\) representation can be written as

\[
H_{[SO(3,1)]} = \bigoplus_l \sum_{m=-l}^{l} |l, m\rangle,
\]

where \(|l, m\rangle\) is characterized by

\[
\begin{align*}
L_+|l, m\rangle &= (l + 1/2)|l, m\rangle, \\
L_0|l, m\rangle &= m|l, m\rangle, \\
L_-|l, m\rangle &= (l - 1/2)|l, m\rangle.
\end{align*}
\]

Theorem A.5.1. Each irreducible representation of \(SO(3,1)\) is characterized by a pair of numbers \((l_0, c)\), where \(l_0\) is a positive integer or half-integer and \(c\) is a complex number.

For each representation \((l_0, c)\) there exists a standard basis \(|l, m\rangle\), such that

\[
\begin{align*}
L_+|l, m\rangle &= [(l + 1/2)^2 - (m + 1/2)^2]^{1/2}|l, m + 1\rangle, \\
L_-|l, m\rangle &= [(l + 1/2)^2 - (m - 1/2)^2]^{1/2}|l, m - 1\rangle, \\
L_0|l, m\rangle &= m|l, m\rangle.
\end{align*}
\]
and

\[
K_\pm |l, m\rangle = [(l - m)(l - m - 1)]^{1/2}C_l |l - 1, m + 1\rangle \\
- [l(l + m)(l + m + 1)]^{1/2}A_l |l, m + 1\rangle \\
+ [(l + m + 1)(l + m + 2)]^{1/2}C_{l+1} |l + 1, m + 1\rangle,
\]

\[
K_+ |l, m\rangle = -[(l + m)(l + m - 1)]^{1/2}C_l |l - 1, m - 1\rangle \\
- [l(l + m)(l - m + 1)]^{1/2}A_l |l, m - 1\rangle \\
- [(l - m + 1)(l - m + 2)]^{1/2}C_{l+1} |l - 1, m + 1\rangle,
\]

\[
K_3 |l, m\rangle = [(l - m)(l + m)]^{1/2}C_l |l - 1, m\rangle \\
- mA_l |l, m\rangle - [(l + m + 1)(l - m + 1)]^{1/2}C_{l+1} |l, m - 1\rangle,
\]

where

\[
A_l = \frac{i l_0 c}{l(l + 1)}, \\
C_l = \frac{i(\nu^2 - v^2)(\nu^2 - c^2)}{4l^2 - 1}^{1/2}.
\]

If \(c^2 = (l_0 + n)^2\) for some positive integer \(n\), then the representation is finite-dimensional and the possible values of the indices \(l\) and \(m\) are

\[
m = -l, -l + 1, \ldots, l; \\
l = l_0, l_0 + 1, \ldots, l_1.
\]

If \(c^2 \neq (l_0 + n)^2\) for any positive integer, then the representation is infinite-dimensional and the possible values of the indices \(l\) and \(m\) are

\[
m = -l, -l + 1, \ldots, l; \\
l = l_0, l_0 + 1, \ldots.
\]

In the representation \((l_0, c)\) the two Casimir invariants are related to the pair of numbers through

\[
C_2 = L^2 - K^2 = l_0^2 + c^2 - 1,
\]

\[
C_2' = K \cdot L = il_0 c.
\]

**Theorem A.5.2.** If the irreducible representation \((l_0, c)\) is unitary, then the pair of numbers \((l_0, c)\) determining it satisfies one of the following conditions:
(1) $c$ is purely imaginary,
(2) $c$ is real and $0 \leq c \leq 1$, and $l_0 = 0$.

There exist an $l$-raising operator and an $l$-lowering operator in the representation given by (A.5.6) and (A.5.7), i.e.

$$X(l, +) = K_+ J_+ + K_3 J_3 + (l + 1) K_3 + \frac{il_0 c}{l + 1} J_3 + il_0 c I,$$

$$X(l, -) = K_+ J_- + K_3 J_3 + l K_3 + \frac{il_0 c}{l} J_3 + il_0 c I,$$

where $I$ is the unit operator. The $l$-shifting operators are characterized by

$$X(l, +)|l, m\rangle = \left(\frac{(2l + 1)((l + 1)^2 - m^2)((l + 1)^2 - l_0^2)((l + 1)^2 - c^2)}{2l + 3}\right)^{1/2} \frac{-i}{l + 1} |l + 1, m\rangle,$$

$$X(l, -)|l, m\rangle = \left(\frac{(2l + 1)(l^2 - m^2)(l^2 - l_0^2)(l^2 - c^2)}{2l - 1}\right)^{1/2} \frac{i}{l} |l - 1, m\rangle,$$
Appendix F.

Representations of $E(3)$

So far as the scattering problem is concerned we are not interested in the general representation theory of $E(3)$ but rather restrict ourselves to the representations of $E(3)$ which are obtained through the contraction from the unitary representations of $SO(3,1)$ given in Appendix E.

A set of linearly independent operators, denoted by $P_i, L_i, (i = 1, 2, 3)$, which satisfy the commutation relations

\[
\begin{align*}
\{P_i, P_j\} &= 0, \\
\{L_i, P_j\} &= i\epsilon_{ijk} P_k, \\
\{L_i, L_j\} &= i\epsilon_{ijk} L_k,
\end{align*}
\]

form an $E(3)$ algebra. Define the linear operators $P_{\pm}, L_{\pm}$ by means of

\[
P_{\pm} = P_1 \pm iP_2, \\
L_{\pm} = L_1 \pm iL_2.
\]

Then we have

\[
\begin{align*}
\{L_3, P_{\pm}\} &= \{P_3, L_{\pm}\} = \pm P_{\pm}, \\
\{L_+ , P_+ \} &= \{L_-, P_- \} = \{L_3, P_3 \} = 0, \\
\{L_+ , P_- \} &= \{P_+, L_- \} = 2P_3, \\
\{L_+ , L_- \} &= 2L_3, \\
\{L_3, L_{\pm}\} &= \pm L_{\pm},
\end{align*}
\]

The two $E(3)$ Casimir invariants are

\[
P^2 = P_1^2 + P_2^2 + P_3^2, \\
P \cdot L = P_1 L_1 + P_2 L_2 + P_3 L_3.
\]

The unitary representations we are concerned with are labeled by a pair of numbers $(k_0, k)$, which are related to the Casimir invariants through

\[
\begin{align*}
P^2 &= k^2, \\
P \cdot L &= -k_0 k.
\end{align*}
\]
The contraction process from the $SO(3, 1)$ representation $(l_0, c)$ to the $E(3)$ representation $(k_0, k)$ is performed through setting

$$P'_l = \epsilon K_i,$$  \hspace{1cm} (A.6.6)

and then taking the limit $\epsilon \to 0$ while keeping

$$\epsilon c = ik,$$  \hspace{1cm} (A.6.6')

and

$$l_0 = k_0.$$  \hspace{1cm} (A.6.6'')

The unitary irreducible representation of $E(3)$ obtained from the contraction of the $SO(3, 1)$ representation given by (A.5.6) and (A.5.7) is characterized by

$$L^+|l, m\rangle = \left(\frac{l + m + 1}{2}\right)^{1/2}|l+1, m\rangle,$$

$$L_0|l, m\rangle = \left(\frac{l + m}{2}\right)^{1/2}|l, m - 1\rangle,$$

$$L^-|l, m\rangle = \left(\frac{l - m + 1}{2}\right)^{1/2}|l - 1, m + 1\rangle,$$

and

$$P_+|l, m\rangle = [(l + m)(l + m - 1)]^{1/2}C_l|l - 1, m - 1\rangle,$$

$$- [(l - m)(l + m - 1)]^{1/2}A_l|l, m + 1\rangle,$$

$$+ [(l + m + 1)(l + m + 2)]^{1/2}C_{l+1}|l + 1, m + 1\rangle,$$

$$P_0|l, m\rangle = [(l + m)(l + m - 1)]^{1/2}C_l|l - 1, m - 1\rangle,$$

$$- [(l + m)(l - m + 1)]^{1/2}A_l|l, m - 1\rangle,$$

$$- [(l - m + 1)(l - m + 2)]^{1/2}C_{l+1}|l + 1, m - 1\rangle,$$

$$P_3|l, m\rangle = [(l - m)(l + m)]^{1/2}C_l|l - 1, m + 1\rangle,$$

$$- mA_l|l, m\rangle - [(l + m + 1)(l - m + 1)]^{1/2}C_{l+1}|l + 1, m\rangle,$$

where

$$A_l = \frac{kk_0}{l(l + 1)},$$  \hspace{1cm} (A.6.9)

$$C_l = \frac{ikl^2 - l_0^2}{l\sqrt{4l^2 - 1}}^{1/2}.$$
In this representation the quantum numbers $l$ and $m$ take the possible values

$$l = k_0, k_0 + 1, k_0 + 2, \ldots$$

$$m = -l, -l + 1, -l + 2, \ldots, l.$$  \hspace{1cm} (4.6.10)

Notice that there exist $E(3)$ representations $\pm k$ corresponding to the same value of $P^2 = k^2, (k^2 > 0)$.

In this representation the $l$-shifting operators of $E(3)$, denoted by $X(l, \pm)$, are

$$X(l, +) = P_+ L_+ + P_3 L_3 + (l + 1)P_3 - \frac{kk_0}{l+1} L_3 - kk_0 I,$$

$$X(l, -) = P_+ L_- + P_3 L_3 + lP_3 - \frac{kk_0}{l} L_3 - kk_0 I,$$  \hspace{1cm} (4.6.11)

where $I$ is the unit operator. Their shifting operations are given by

$$X(l, +)|l, m\rangle = \left[ \frac{(2l + 1)(l + 1)^2 - m^2)((l + 1)^2 - l_0^2)}{2l + 3} \right]^{1/2} \frac{-ik}{l + 1} |l + 1, m\rangle,$$

$$X(l, -)|l, m\rangle = \left[ \frac{(2l + 1)(l^2 - m^2)(l^2 - l_0^2)}{2l - 1} \right]^{1/2} \frac{ik}{l} |l, m - 1\rangle.$$  \hspace{1cm} (4.6.12)
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