MICROSCOPIC AND MACROSCOPIC THEORIES
OF
NUCLEAR DYNAMICS

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The dynamical problem of nuclear many-body systems has been studied from two different, but complementary, aspects of microscopic and macroscopic theories. The exact quantum many-body theory is formulated in terms of a hierarchy of generalized Green's functions from which a truncation at the second level has resulted in a new extended time-dependent Hartree-Fock (ETDHF) approximation. This microscopic theory, as a generalization of the mean-field theory of time-dependent Hartree-Fock (TDHF) approximation, includes specifically particle collisions due to the nucleon-nucleon residual interaction and contains the TDHF approximation as a collisionless limit. The formal structure of this theory, mathematically simple, but physically transparent, consists of a set of modified time-dependent Hartree-Fock equations, coupled to a master equation for the time-dependent occupation probability amplitudes of the single-particle states. The latter, as an analogue of the quantum Boltzmann equation in configuration space, provides an appropriate framework from which to study the dissipative behavior of the nuclear system from a microscopic standpoint. Much statistical dynamical information can be extracted from this ETDHF scheme. The concepts of entropy and local and global thermal equilibrium have been introduced quantitatively and an H-theorem has been obtained. The salient features of the approach of a finite system to thermal equilibrium can be illustrated by means of analytic solutions to some simple cases of the master equation. Estimates have been made of the thermal relaxation time scale for excited heavy ions within the framework of ETDHF. Macroscopic equations in a hydrodynamical form and the associated conservation theorems have been obtained.
Nuclear hydrodynamics, a macroscopic approach to nuclear dynamics, can be obtained as a different approximation to the many-body theory. Extensive calculations have been performed to simulate the dynamical processes of nuclear collisions. In this theoretical framework, a nuclear system is ascribed to be a compressible, viscous and thermally conducting (probability) fluid which is endowed with the realistic features of known nuclear bulk properties, such as the binding energy and surface diffuseness. In a two-dimensional model in which the time-dependent problem has been solved for systems colliding with arbitrary impact parameters and bombarding energies, systematic studies have been made over a large energy range. In collisions of a few MeV up to tens of MeV per nucleon, the calculations have reproduced the qualitative features of low-energy phenomena, such as fusion, fission and deep-inelastic scatterings. At higher energies, features of Mach shock phenomena emerge and become increasingly prominent with larger energies. Dissipation has also been shown to play an important role in both low- and high-energy collisions in terms of the intermediate dynamical states and final reaction products. Many of the hydrodynamical features found in the two-dimensional model have been confirmed by realistic calculations on the three-dimensional head-on collisions, for projectiles and targets of equal and unequal masses, and at energies from 50 MeV up to 400 MeV per nucleon. The angular and energy distributions of the outgoing nucleons from these reactions, thus far calculated, indicate prominent features, which can be confronted with experiments. The role played by dissipation has been found to be significant. These studies have led to definite conclusions concerning the validity of nuclear hydrodynamics applied to low-, medium- and high-energy regions, which can be tested by experiments in the near future in an unambiguous manner.
DEDICATION

To the memories of my parents:

to my father who had, since my early childhood, cultivated in me a disinterested curiosity for knowledge;

to my mother who believed that the end of learning should be directed towards the betterment of mankind.
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CHAPTER I

INTRODUCTION
In the realm of physics, attempts at understanding the dynamics of a many-body system from a microscopic viewpoint remains one of the greatest intellectual challenges of research, both theoretical and experimental. Such problems are found in liquid and solid state physics, atomic physics, nuclear physics and even elementary particle physics. But nowhere is this challenge more succinctly met than in the study of the dynamics of atomic nuclei, where the number of particles can still be small enough to allow a completely microscopic description of the dynamics, but large enough to exhibit collective behavior. Solutions of such a problem in nuclear physics are not only significant for understanding nuclear properties, but they will also be of great help in related many-body problems in other branches of physics. It is important to search for a microscopic theory which is computationally manageable and contains the correct and physically important principles. Such a search constitutes a substantial part of the present investigation.

It is worthwhile to review the current status of such outlook. Almost five decades have passed since Dirac first discovered a rigorous derivation of the Hartree-Fock (HF) equations (Di 32). In the HF approximation, the particles in a system are assumed to move independently of each other, except that each particle is subject to a self-consistent potential generated by all particles in the system. Its being derivable from a variational principle and its relation to the formal many-body theory as the lowest order approximation are among the most appealing aspects of the HF theory, from a theoretical standpoint. However, the application of this approximation to finite nuclei is but a recent event. One of the basic problems lies with the
uncertainties in the interaction to be used in the theory.

The fundamentally new element in the nuclear HF theory is the introduction of Skyrme interactions (Sk 56, 59). In the works of Vautherin and Brink (Va 72), the original Skyrme parametrization was simplified by the use of effective, density-dependent nucleon-nucleon interactions. Such interactions were found to account well for the ground states and low-lying states of closed-shell nuclei throughout the periodic table.

A natural extension of the static HF theory is the time-dependent Hartree-Fock (TDHF) approximation. Its application to nuclear collisions was initiated by the one-dimensional calculations of Bonche, Koonin and Negele (Bo-b 76), which reproduced many qualitative features of nuclear dynamics. Since then, many studies had been stimulated. The results obtained have been encouraging enough to maintain the TDHF investigations one of the most active areas of nuclear physics in recent years.

Perhaps it would represent too great a digression to even attempt to summarize the results of all workers in this field. But it is important to understand the essential features which lead to the successes and failures of TDHF. The necessity for such understanding has been emphasized by Negele and co-workers (Ne 77, 78, Ho 78, 79). As the TDHF approximation is basically a one-body operator theory, it can be expected to reproduce well the expectation values of one-body operators, but fails to describe accurately quantities which depend sensitively on higher-order correlations. This is, for example, a source of difficulties in the development of a fully quantum mechanical scattering theory from TDHF.
It is our objective to extend the conventional TDHF formalism so that the internucleon collision effects as well as the mean-field effects can be included in a more general framework. To the extent that we are successful, the generalized theory provides a step forward towards a better understanding of particle-particle correlations, and hopefully, more realistic reproduction of the rapidly increasing body of experimental data on complex nuclear collisions. Moreover, it is our hope that such a microscopic theory will permit us to isolate in greater detail the underlying physical phenomena that determine the observable characteristics of such collisions.

The TDHF approximation is expected to be best applied for low-energy phenomena where the Pauli effect plays an important role. At higher energies (tens of MeV above the Coulomb barrier), the particle-particle collisions and the opening up of the phase space allow the possibility of describing the dynamics by macroscopic theories such as nuclear hydrodynamics. In recent years, there has been considerable interest in the theoretical foundation of hydrodynamics obtained from the many-body theory (Fr 73, 74, Be 74, 77, Wo 74, 75, 76, 77i-iii, 78i-iii). However, the ultimate test of nuclear hydrodynamics lies in a confrontation of its predictions with experiments.

The advantages of a hydrodynamical treatment of the reaction process, if successful, are manyfold. As different energetic heavy ions lead to different compressed densities, the equation of state can, in principle, be traced by using projectiles of different energies. Many exotic
phenomena such as pion condensation and density isomers manifest themselves in peculiar shapes of the equation of state (Go 78). Hence over a wide energy range (from tens of MeV up to at least hundreds of MeV per nucleon), it is of importance to ascertain to what degree the heavy-ion collision can be described by nuclear hydrodynamics. The second part of this dissertation concentrates on the hydrodynamical calculations of heavy-ion reactions, their results and physical implications.

Of the two parts of this dissertation, the first one which is made up of two chapters concerns the microscopic theory. In Chapter II, the many-body theory is formulated by means of a hierarchy of generalized Green's functions. These Green's functions are more general than the conventional ones in that the expectation values are taken with respect to arbitrary states. The TDHF approximation is shown to be a special collisionless limit within this hierarchy. By a truncation at the second level, an extended time-dependent Hartree-Fock (ETDHF) approximation is obtained. It includes particle-particle collisions due to the residual interaction of the nucleons which is neglected in a pure TDHF theory. The final equations, mathematically simple in structure, but physically transparent, consist of a modified time-dependent Hartree-Fock equations, coupled to a master equation for the time-dependent single-particle occupation numbers. The implications of this ETDHF approximation are discussed in Chapter III. The master equation, an analogue of the quantum Boltzmann equation in configuration space, provides an appropriate framework within which one can study the dissipative behavior of a finite many-
fermion system. The concepts of entropy and global thermal equilibrium are introduced quantitatively. An h-theorem that entropy never decreases is recovered. The salient features of the approach of a finite system to thermal equilibrium will be illustrated by means of analytic solutions to some simple cases of the master equation. Estimates can be made of the thermal relaxation time for excited heavy ions within the framework of ETDHF. Macroscopic equations in the hydrodynamical form and the corresponding conservation theorems are obtained and discussed.

The second part of the dissertation concerns the macroscopic approach, nuclear hydrodynamics. Indeed, recent experimental work of Gutbrod and co-workers on intermediate- and high-energy heavy-ion reactions strongly indicates hydrodynamical features (particularly for collisions with small impact parameters) and gives additional impetus to the study of nuclear hydrodynamics (Gu 78, Na 79). In Chapter IV, the genealogy of nuclear hydrodynamics and its relations to the many-body theory are discussed. The basic equations of motion and the parametrization of the equation of state of the nuclear fluid, the transport coefficients and the effective nucleon-nucleon interaction used are discussed. Chapter V concerns a two-dimensional model of hydrodynamics. Here, the two-dimensional equations are solved explicitly for arbitrary scatterings. Extensive calculations have been done for energies which cover a wide range of cases: from a few MeV up to hundreds of MeV per projectile nucleon. Many qualitative and semi-quantitative results are obtained. In Chapter VI, three three-dimensional calculations are done for the limited case of the central
collisions. In particular, the systems $^{20}\text{Ne} + ^{197}\text{Au}$ and $^{208}\text{Pb} + ^{208}\text{Pb}$ are investigated. The results obtained confirm many features found in the two-dimensional model and they are sufficiently quantitative to be confronted with experiments in the near future.

The calculations discussed in this dissertation demonstrate the feasibility of such an approach to heavy-ion collisions. From a technical point of view, it is worthy to emphasize that as a contrast to some earlier work on this subject (Am-b 75, 77ii), nuclear hydrodynamics is solved by a different, and as far as we are able to judge, better method, in this investigation. The two-dimensional model is a step beyond the limited case of the one-dimensional problem. The low dimensionality, however, permits fast and economic computations which are not performed at the expense of accuracy. The two-dimensional results have helped to gain insight into the characteristic features of nuclear hydrodynamics. In going from two- to three-dimensional calculations, the main difficulty is one of practice, rather than in principle. The problem one encounters is the computer storage problem, which is formidable if not impossible. The axial symmetry in the exactly central collisions allows one to circumvent this obstacle. The numerical techniques from the two-dimensional model require only slight modifications for the three-dimensional head-on collisions. From some general arguments (Chapters IV, V), one expects hydrodynamics to be best applied to cases of small impact parameters. Recently, theoretical proposals have been made to identify central collisions (Wo 79iii) and
experimental data which isolate these events have already been obtained (Gu 78). Hence, it is reasonable to expect that the calculations of the central collisions (Chapter VI) are representative of cases of non-zero, but small impact parameters.
CHAPTER II

FORMULATION

OF

EXTENDED TIME-DEPENDENT HARTREE-FOCK APPROXIMATION
A Unified View From the Quantum Many-Body Theory

To the extent that Galilean relativity provides an adequate representation of the space-time geometry, the time-dependent many-particle Schrödinger equation, with an appropriate choice of interaction, can be regarded as the starting point of the formulation of the microscopic, dynamical laws of many-body systems. In the case of atomic nuclei, the advantage of such a fundamental view, if valid, is manyfold. As the understanding of the strong interaction is one of the most important motivations for the study of nuclear phenomena, one may hope to represent the underlying dynamics, to a good approximation at least, by means of an essentially parameter-free dynamical theory in which the only input is some suitably chosen, realistic nucleon-nucleon interaction. The relevant degrees of freedom, as manifested by the various prominent features in a given situation, will then be dictated by this basic theory, rather than presumed in any ad hoc manner. Moreover, even though such a theory cannot be solved to the last details, a successful formulation will present a general framework which allows one to make meaningful connections with the other more phenomenological approaches, whereby one gains a basic understanding of how and when a set of mathematical approximations and physical assumptions may be validly applied.

It is well known that the Hartree-Fock (HF) theory arises naturally as a lowest order approximation to the formal many-body problem. But because of the strongly interacting nature of nucleons at short ranges, a naive perturbative approach is clearly inappropriate. However, with an introduction of density-dependent, effective interactions (Va 72, Ne 71, 77, 78), for
which much of the higher-order particle-particle correlations have been, in principle, incorporated, the time-dependent Hartree-Fock theory has been found to describe well the static properties of the ground states and low-lying states of closed-shell nuclei.

Recent interest in the application of the time-dependent Hartree-Fock (TDHF) approximation for the microscopic description of nuclear reactions was pioneered by the one-dimensional studies of Bonche, Koonin and Negele (Bo-b 76). This work stimulated many subsequent investigations, in both theoretical studies (e.g. Li 76, 78, 79, Bl 76, Wo 78ii, 79i, Ka-b 78) and realistic calculations (e.g. Ko-a 76, 77, Cu 76i, 76ii, Fl 77). The encouraging results strongly suggest the fruitfulness of a microscopic treatment for a great variety of nuclear problems.

With this in mind, the microscopic studies which constitute the first part of the present dissertation starts with the assumption that there exists dynamical conditions under which the TDHF theory gives a reasonable first approximation to the nuclear dynamics. However, the nucleons in a pure TDHF description are assumed to interact only through a mean potential generated self-consistently by all particles in the system; the correlations due to the residual interaction are capable of altering the energy and momentum states of the colliding particles and are responsible for the dissipation of energy. Hence the pattern of behavior of nuclear systems can be that of hydrodynamics in one extreme limit (Wo 77i-iii), or of elastic response in the other (Be-a 74, 77,78i, Wo 77ii), depending on the importance of particle
collisions.

Accordingly, it is of great interest to generalize the usual TDHF approximation by including particle collisions due to the residual interaction. On one hand, such an extension can provide a better understanding of when and where corrections to the mean-field theory may be important. It is thus significant not only in enlarging the range of applicability of the TDHF theory, but also in gaining insight into the problems of further generalizations by which the present approximations can be systematically improved. On the other hand, it also provides a clear picture of the connections between the microscopic and macroscopic degrees of freedom in the dynamics and hence a better evaluation of the validity of macroscopic approaches such as nuclear hydrodynamics.

The rest of this chapter is organized as follows. The exact (non-relativistic) many-body theory is formulated in terms of generalized Green's functions. The mathematical language, though applied in the context of nuclear physics, is in fact much more general, being valid for the dynamics of many other fermion systems. In section B, after the introduction of the basic concepts of Green's functions and time-path ordering of field operators, a hierarchy of equations for the Green's functions is derived. They constitute the basic equations of motion in the present microscopic approach. In section C, the first level of this hierarchy is studied. With a decomposition of the two-body Green's function into a mean-field part and a correlated part, the TDHF approximation is shown to be a collisionless limit. In section D, the second level of the hierarchy is examined. A truncation of the higher-order terms results in an approximate integral representation of the correlated two-body
Green's function in terms of only one-body Green's functions. In section E, the separation of the effective and residual interactions and their formal relations with the bare interaction, as appropriate for nuclear problems, are discussed. In the subsequent discussions of this dissertation, the terminology "mean-field approximation" will be used in a similar (but not necessarily the same) sense as that in the current literature (e.g. Ne 77, 78). The essential feature is the use of density-dependent effective interactions. But the primary objective here is to formulate an extended time-dependent Hartree-Fock (ETDHF) approximation and to study its physical consequences. The fundamental problems on what classes of diagrams to be included in the effective interaction and how to derive the corresponding residual interaction wherefrom will not be treated in this dissertation. In section F, the basic equation of the ETDHF approximation is derived and expressed in a non-Markovian Green's function representation. The question of a simplification by an appropriate choice of single-particle basis is discussed in section G. In section H, after making an assumption of slow temporal variation of the mean field and a diagonal approximation, the final simple forms of ETDHF equations are obtained. A brief summary of the present formalism is given at the end of this section.
B Many-Body Theory in Terms of Green's Functions

1. Basic Operators and Their Equations

In the second-quantized form of the dynamical theory of a many-fermion system, the basic operators are the creation and annihilation field operators \( \hat{\psi}^\dagger \) and \( \hat{\psi} \), respectively. In the discussions of this dissertation, they are taken to be in the Heisenberg representation, unless stated otherwise.

The canonical quantization condition of the theory are expressed by the usual equal-time anti-commutation relations:

\[
\left[ \hat{\psi}(x_t), \hat{\psi}^\dagger(x'_t) \right]_+ = \hat{\psi}(x_t) \hat{\psi}^\dagger(x'_t) + \hat{\psi}^\dagger(x'_t) \hat{\psi}(x_t) = \delta(x, x') \quad (\text{II-1a})
\]

and

\[
\left[ \hat{\psi}(x_t), \hat{\psi}(x'_t) \right]_+ = \left[ \hat{\psi}^\dagger(x_t), \hat{\psi}^\dagger(x'_t) \right]_+ = 0 \quad (\text{II-1b})
\]

where \( x \) represents the collection of the spatial coordinate \( \vec{r} \) and those of the other relevant degrees of freedom, such as the spin and isospin \( \xi \). In the case with spin and isospin, the delta-function in eq. (II-1a) is the compact form of

\[
\delta(x, x') = \delta(\vec{r}-\vec{r}') \delta_\xi \xi' \quad (\text{II-2})
\]

The dynamics of a many-body system of fermions of mass \( m \), interacting through an instantaneous two-body potential \( v(x, x') \), is completely determined by the full Hamiltonian operator:

\[
\hat{H} = -\frac{\hbar^2}{2m} \int dx \nabla \hat{\phi}^\dagger(x_t) \cdot \nabla \hat{\psi}(x_t)
\]
\[ + \frac{i\hbar}{2} \int \int dx \, dx' \, \hat{\psi}^\dagger(xt) \, \hat{\psi}^\dagger(x't) \, v(x, x') \, \hat{\psi}(x't) \, \hat{\psi}(xt) \]  

(II-3)

where

\[ dx = \sum_f \int d^3 r \]  

(II-3a)

From the Heisenberg equation and the anti-commutation relations (II-1a) and (II-1b), the equations of motion of the basic operators \( \hat{\psi} \) and \( \hat{\psi}^\dagger \) can be readily obtained:

\[
-\frac{i\hbar}{\partial t} \hat{\psi}(xt) = -\frac{\hbar^2}{2m} \psi^2(\psi)(xt) + \int dx' \, v(x, x') \, \hat{\psi}(x't) \, \hat{\psi}(x't), \hat{\psi}(xt) \]  

(II-4a)

and

\[
-\frac{i\hbar}{\partial t} \hat{\psi}^\dagger(xt) = -\frac{\hbar^2}{2m} \psi^2(\psi)(xt) + \int dx' \, v(x, x') \, \hat{\psi}^\dagger(xt) \, \hat{\psi}(x't) \, \hat{\psi}(x't) \]  

(II-4b)

2. Time-Path Ordering of Field Operators

In recent years, there has been considerable interest in the study of Green's functions of very general forms and their applications to non-equilibrium many-body problems. To facilitate systematic analyses using this language, such as the development of appropriate perturbation expansions, Mills (Mi 69), Craig (Cr 68), Hall (Ha-a 74, 75) and others (e.g. Ke-a 65, Sa 70) have introduced the concept of time-path ordering of the fermion field operators.
In fact, the notion of time-path also arises in the other more restricted cases. The Green's function in the zero-temperature formalism is the expectation value of a time-ordered product of field operators, taken with respect to the exact interacting ground state of the system in question. By a theorem of Gell-Mann and Low (Ge 51, Fe 71), the exact ground state is connected to a non-interacting or model state via an adiabatic procedure, by which the non-interacting state is effectively propagated in time from the remote past to the remote future. In the case of the non-zero temperature formalism, the thermodynamic Green's function is the expectation value taken with respect to a grand canonical ensemble. The resultant quasi-periodic boundary condition of Martin and Schwinger (Ma-c 59, see also Ab 75, Fe 71, Ka-a 76) allows elegant treatments of the problem in the imaginary time domain. The real time integrals in the final equations of motion can be obtained by analytic continuation.

The basic concept of path-ordering can be illustrated by the following simple example. We can consider the expectation value of an operator $\hat{O}$ taken with respect to an arbitrary N-particle state $\Psi$. In particular, $\hat{O}$ can be taken to be a function involving an equal number of creation and annihilation operators:

$$< \hat{O} > = <\Psi | \hat{O} | \Psi> = \sum_{m n} c_{mn} <n | \hat{O} | m> \quad (\Pi-5)$$

where $c_{mn}$ are the matrix elements of the exact initial density matrix operator $|\Psi><\Psi|$, relative to a suitable, but arbitrary basis:
The expansion in eq. (II-5) is completely general. It can be applied to a great variety of physical situations. In the usual studies of many-body systems, since the eigen-states of the full Hamiltonian can at the best be constructed by successive approximations, one can envisage (as a gedanken experiment) the preparation of the system in some appropriate model state at a convenient initial time $t_i$. For example, in the theoretical study of heavy-ion collisions, one can consider two approaching nuclei, each initially in some Hartree-Fock-like state. With the residual interaction turned on, the effects of two-body correlations can be analyzed and compared with the dynamics of the pure mean-field theory. A typical matrix element of eq. (II-5) in the time-dependent problem then appears as:

$$\langle n | \hat{O}(t) | m \rangle = \langle n, t_i | \hat{U}(t_1, t) \hat{O}(t) \hat{U}(t, t_i) | m, t_i \rangle$$ (II-6)

where $\hat{U}$ is the evolution operator (in the appropriate representation). The physical interpretation of the matrix element is clear. The initial state $|m\rangle$ is propagated from an initial time $t_i$ to the present time $t$, at which the interaction of $\hat{O}$ operates on the system. The resultant state is then propagated back to the initial time, at which it is projected onto the state $|n\rangle$.

By the group property of the evolution operator, there can be different, but equivalent, decompositions of the time-path. However, the main point of this example is that, in the formal expansion of the operator $\hat{U}(t_1, t_2)$ (in the interaction picture), the time-ordering of each term is causal when
and anti-causal when \( t_1 < t_2 \). This is the origin of the concept of the time-path ordering, as pointed out by Craig (Cr 68). The derivation of such a general expression is outlined in Appendix I which will demonstrate how the notion of time-path ordering arises naturally in a perturbation theory for generalized Green's functions.

The prescribed time-path \( P \) in the present investigation constitutes two branches along the real time axis (figure 2.1):

\[
P = P_+ + P_-, \quad (\text{II-7})
\]

where \( P_+ \) is the path along the forward direction and \( P_- \), that along the backward direction, both between times \( t_i \) and \( t_f \). These limits will be specified later. By way of book-keeping, one can imagine the introduction of infinitesimally small external fields \( \varphi_+ \) and \( \varphi_- \) in the operator equations (II-4a) and (II-4b), so that the time evolution of the fermion field operators are slightly different in the forward and backward directions. This artifice, though unphysical, formally permits one to index the time coordinate of the operators as being on the forward or backward branch and makes possible an elegant and compact formulation of the present problem. It is to be understood that at the end of a calculation, one takes the limit \( \varphi_\pm \to 0 \).

The essential notations associated with the time-path ordering are now introduced. A generalized, path-dependent, step-function is defined for a pair of time coordinates in terms of the usual step-function \( \Theta \):
\[ \theta_P(t, t') = \begin{cases} \theta(t-t') & \text{if } t, t' \in P_+ \\ \theta(t'-t) & \text{if } t, t' \in P_- \\ 1 & \text{if } t \in P_- \text{ and } t' \in P_+ \\ 0 & \text{if } t \in P_+ \text{ and } t' \in P_- \\ \end{cases} \] (II-8)

Furthermore, a generalized, path-dependent, delta-function can be defined for a pair of time coordinates in terms of the usual Dirac delta-function \( \delta \):

\[ \delta_P(t, t') = \begin{cases} \delta(t-t') & \text{if } t, t' \in P_+ \\ -\delta(t'-t) & \text{if } t, t' \in P_- \\ 0 & \text{if } t \in P_- \text{ and } t' \in P_+ \\ 0 & \text{if } t \in P_+ \text{ and } t' \in P_- \end{cases} \] (II-9)

It can be readily seen that the generalized step-function and the generalized delta-function are related as follows:

\[ \delta_P(t, t') = \frac{\partial}{\partial t} \theta_P(t, t') \] (II-10a)

and

\[ \delta_P(t, t') = -\frac{\partial}{\partial t'} \theta_P(t, t') \] (II-10b)

with the symmetry property for \( \delta_P \):

\[ \delta_P(t, t') = \delta_P(t', t) \] (II-11)

It is clear that the functions \( \theta_P \) and \( \delta_P \) are geometric extensions of the conventional step-function and delta-function with regard to the prescribed time-path.
The time-path ordering operator $T_P$ is defined for a product of $n$ fermion field operators as a straightforward generalization of the usual time-ordering operation:

$$T_P[\hat{O}_1(l)\hat{O}_2(2)\hat{O}_3(3)\ldots\hat{O}_n(n)]$$

$$= \sum_P (-)^P \theta_P(t^l, t^2) \theta_P(t^3, \ldots t^n) \theta_P(t^{\alpha_{n-1}}, t^{\alpha_n})$$

$$\times \hat{\alpha}_1(\alpha_1) \hat{\alpha}_2(\alpha_2) \ldots \hat{\alpha}_n(\alpha_n), \quad (\Pi - 12)$$

where the numerical index $k$ in the argument of the fermion operator $\hat{O}_k$ represents the collection of spatial, spin-isospin and time coordinates, $P$ symbolizes the permutation from $(12\ldots n)$ to $(\alpha_1\alpha_2\ldots\alpha_n)$, $(-)^P$ is +1 or -1 depending on whether $P$ is an even or odd permutation respectively and the summation exhausts all $n!$ permutations of the $n$-tuple $(12\ldots n)$.

A generalized $n$-body Green's function is defined as:

$$g_n(12\ldots n; 1'2'\ldots n')$$

$$= (\frac{1}{i})^n < T_P [\hat{\psi}(1) \hat{\psi}(2) \ldots \hat{\psi}(n) \hat{\psi}^+(n') \ldots \hat{\psi}^+(2') \hat{\psi}^+(1') ] > \quad (\Pi - 13)$$

where the expectation value is taken with respect to any appropriate initial wavepacket-like, normalized $N$-particle state. From the definition given in eq. (\Pi - 12), it is easy to prove the following symmetry properties for any Green's function:

$$g_n(12\ldots n; 1'2'\ldots n')$$

$$= (-)^P g_n(\alpha_1\alpha_2\ldots\alpha_n; 1'2'\ldots n')$$

$$= (-)^{P'} g_n(12\ldots n; \alpha_1'\alpha_2'\ldots\alpha_n'), \quad (\Pi - 14)$$
where $P$ is the permutation from $(12 \ldots n)$ to $(\alpha_1 \alpha_2 \ldots \alpha_n)$ and $P'$ is the permutation from $(1'2' \ldots n')$ to $(\alpha_1' \alpha_2' \ldots \alpha_n')$. The factors $(-)^P$ and $(-)^{P'}$ are the same notation as that used in eq. (II-12). As special cases of eq. (II-14), one has

$$g_n(1 \ldots j \ldots k \ldots n; 1'2' \ldots n') = - g_n(1 \ldots k \ldots j \ldots n; 1'2' \ldots n') \quad (\text{II-15a})$$

and

$$g_n(12 \ldots n; 1' \ldots j' \ldots k' \ldots n') = - g_n(12 \ldots n; 1' \ldots k' \ldots j' \ldots n') \quad (\text{II-15b})$$

Eqs. (II-15a) and (II-15b) directly reflect the effect of the fermion statistics.

Because of the different possibilities of the time ordering of field operators, the class of all possible Green's functions is, in principle, infinite in number, containing not only the properties of one fixed Fock space, but also the relations among various Fock spaces of different particle numbers. Hence the Green's functions form a much more general set than that of the reduced density matrices (a set of $N$ functions for an $N$-body problem, Wo 77iii) which are special equal-time limits. Moreover, the Green's functions in eq. (II-13) are defined with respect to an arbitrary initial state. Thus, such Green's functions contain the most general dynamical information of the theory and are well adapted for the study of non-equilibrium problems under very general dynamical conditions.
Figure 2.1 Contour of the time-path for the generalized Green's functions. The part $P_+$ is called the forward branch in the text and $P_-$, the backward branch.
TIME-PATH FOR GREEN'S FUNCTIONS

Fig. 2.1
3. Equations of Green's Functions

The one-body Green's function contains information of the interaction between a particle and the rest of the many-body system. The structure of the two-body Green's function reflects the interaction between two particles and also that between a particle pair and the many-body background and so on. Therefore, a knowledge of particle-particle correlations of higher orders can be obtained as one goes to Green's functions at higher levels.

A direct application of the definition of time-path ordered operator product leads to the following operator equations:

\[
\left( i\hbar \frac{\partial}{\partial t_1} + \frac{\hbar^2}{2m} \nabla_1^2 \right) T_P \left[ \hat{\psi}(1) \ldots \hat{\psi}(n) \hat{\psi}^\dagger(n') \ldots \hat{\psi}^\dagger(1') \right] 
\]

\[
= i\hbar \sum_{k' = 1}^{n'} (-)^{k'-1} \delta_P (1, k') 
T_P \left[ \hat{\psi}(2) \ldots \hat{\psi}(n) \hat{\psi}^\dagger(n') \ldots \hat{\psi}^\dagger((k+1)') \ldots \hat{\psi}^\dagger((k-1)') \ldots \hat{\psi}^\dagger(1') \right] 
- i \int_P d(n+1) V(1, n+1) \ T_P \left[ \hat{\psi}(1) \ldots \hat{\psi}(n+1) \hat{\psi}^\dagger((n+1)') \hat{\psi}^\dagger(n') \ldots \hat{\psi}^\dagger(1') \right] 
\]

\[ \text{(II-16a)} \]

and

\[
\left( -i\hbar \frac{\partial}{\partial t_{1'}} + \frac{\hbar^2}{2m} \nabla_{1'}^2 \right) T_P \left[ \hat{\psi}(1) \ldots \hat{\psi}(n) \hat{\psi}^\dagger(n') \ldots \hat{\psi}^\dagger(1') \right] 
\]

\[
= i\hbar \sum_{k = 1}^{n} (-)^{k-1} \delta_P (k, 1') 
T_P \left[ \hat{\psi}(1) \ldots \hat{\psi}(k-1) \ldots \hat{\psi}(k+1) \ldots \hat{\psi}(n) \hat{\psi}^\dagger(n') \ldots \hat{\psi}^\dagger(2') \right] 
- i \int_P d(n+1) V(n+1, 1') \ T_P \left[ \hat{\psi}(1) \ldots \hat{\psi}((n+1)') \hat{\psi}^\dagger(n+1) \ldots \hat{\psi}^\dagger(1') \right] 
\]

\[ \text{(II-16b)} \]
where the superscript + attached to the coordinate in eq. (II-16a) indicates a time "slightly later" than a corresponding time in the sense of the time-path prescribed. Similarly, the superscript - in eq. (II-16b) indicates a time "slightly before" a corresponding time. For the economy of expressions, the following (path-dependent) notations have been introduced in eqs. (II-16a) and (II-16b):

$$\delta p_{(j,k')} = \delta(x_j, x_{k'}) \delta p_{(t_j, t_{k'})}, \quad (II-17)$$

$$V(j,k') = V(x_j, x_{k'}) \delta p_{(t_j, t_{k'})}, \quad (II-18)$$

and

$$\int p \, dk \equiv \int p \, dt_k \int dx_k. \quad (II-19)$$

In the time integration of eq. (II-19), all the time orderings of operators in the integrand are understood to be relative to the time-path.

Taking expectation values of eqs. (II-16a) and (II-16b) leads to the adjoint equations for the n-body Green's function:

$$\left( i \hbar \frac{\partial}{\partial t_1} + \frac{\hbar^2}{2m} \nabla_1^2 \right) g_n(12 \ldots n; 1'2' \ldots n')$$

$$= \hbar \sum_{k''}^{n'} (-)^{k''-1} \delta p(1, k') g_{n-1}(2 \ldots n; 1' \ldots (k-1)' (k+1)' \ldots n')$$

$$- i \int d(n+1) V(1, n+1) g_{n+1}(1 \ldots (n+1); 1' \ldots n' (n+1)) \quad (II-20a)$$

and
The hierarchy of equations given by (II-20a) and (II-20b) are very
general. They are the same form as that obtained by Martin and Schwinger
(Ma-c 59), except for the differences in the definitions of the Green's
functions used in the two cases. The functional similarity arises from the
general validity of the operator equations (II-14a) and (II-14b). But it
should be emphasized again that the generalized Green's functions used here
do not assume any equilibrium as an initial condition of the system.
C First Level of the Green's Function Hierarchy

The time evolution of the one-body Green's function is governed by the following adjoint equations:

\[
\begin{align*}
\left[ \frac{i\hbar}{\partial t_1} + \frac{\hbar^2}{2m} \nabla^2_1 \right] g(1;1') &= \hbar \delta_{p}(1,1') + (-i) \int_{p} d^2 V(1, 2) \ g_2(12;1'2^+) \\
\left[ -\frac{i\hbar}{\partial t_1} + \frac{\hbar^2}{2m} \nabla^2_1 \right] g(1;1') &= \hbar \delta_{p}(1,1') + (-i) \int_{p} d^2 V(2, 1') \ g_2(12^-;1'2) 
\end{align*}
\]

(II-21a)

and

\[
\begin{align*}
\left[ \frac{i\hbar}{\partial t_1} + \frac{\hbar^2}{2m} \nabla^2_1 \right] g(1;1') &= \hbar \delta_{p}(1,1') + (-i) \int_{p} d^2 V(1, 2) \ g_2(12;1'2^+) \\
\left[ -\frac{i\hbar}{\partial t_1} + \frac{\hbar^2}{2m} \nabla^2_1 \right] g(1;1') &= \hbar \delta_{p}(1,1') + (-i) \int_{p} d^2 V(2, 1') \ g_2(12^-;1'2) 
\end{align*}
\]

(II-21b)

For convenience of later discussions, the following auxiliary Green's functions are introduced:

\[
g^< (1;1') = i <\hat{\psi}(1') \hat{\psi}(1)> 
\]

(II-22a)

and

\[
g^> (1;1') = -i <\hat{\psi}(1) \hat{\psi}^+(1')> 
\]

(II-22b)

which are defined for all times \( t_1 \) and \( t_1' \) and satisfy the equal-time relation:

\[
i \lim_{t_1' \to t_1} [g^>(1;1') - g^<(1;1')] = \delta(x_{1'}, x_{1'}) 
\]

(II-23)

These functions are related to the original one-body Green's function as:
As the exact equation for the one-body Green's function is coupled to the two-body Green's function, it can only be closed by approximating the two-body Green's function in terms of one-body Green's functions. A simple way is to decompose the two-body Green's function into an uncorrelated (mean-field) part $g_{20}$ and a correlated part $g_{2c}$ due to the particle collisions, where

$$g_{20}(12;1'2') = g(1;1')g(2;2') - g(1;2')g(2;1')$$  \hspace{1cm} (II-25)$$

and

$$g_{2c}(12;1'2') = g_2(12;1'2') - g_{20}(12;1'2')$$  \hspace{1cm} (II-26)$$

The decomposition of n-body Green's functions (n ≥ 2) into products of lower-order Green's functions and correlation functions is a general problem which is best studied by systematic diagrammatic techniques. However, the simpler problem involving only the lowest few levels can be handled by following the algebraic and functional methods developed by Martin and Schwinger (Ma-c 59, Ka-a 76), from which, the decomposition given by eqs. (II-25) and (II-26) arises in a natural fashion.

From eqs. (II-25) and (II-26), the one-body equations (II-21a) and (II-21b) can be re-written as follows:

$$\left[ i\hbar \frac{\partial}{\partial t_1} + \frac{\hbar^2}{2m} \nabla_{1}^2 - U_{MHF}(1) \right] g(1;1')$$

$$= \hbar \delta_{p}(1,1') - i \int_{P} d^{2} V(1,2) \ g_{2c}(12;1'2^{+})$$  \hspace{1cm} (II-27a)$$
where a modified Hartree-Fock potential is introduced by means of the operator $U_{\text{MHF}}$ on $g$:

$$U_{\text{MHF}}(1) \ g(1;1') = -i \int \ d^2 \ V(1,2) \ g_{20}(12^{-};1'2)$$

and

$$U_{\text{MHF}}(1') \ g(1;1') = -i \int \ d^2 \ V(2,1') \ g_{20}(12^{-};1'2)$$

In the absence of $g_{2c}$, eqs. (II-27a) and (II-27b) are reduced to equations of the TDHF approximation. The term "modified" is used for $U_{\text{MHF}}$ to indicate that in the presence of particle collisions, the one-body density matrix (which is an equal-time limit of the one-body Green's function), as distinct from the TDHF case, is not derivable from a single determinantal wavefunction. The adjoint equations (II-27a) and (II-27b) can now be combined in a symmetric form:

$$\left\{ i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_1'} \right) + \frac{\hbar^2}{2m} \left( v_{11}^2 - v_{11}'^2 \right) - [U_{\text{MHF}}(1) - U_{\text{MHF}}(1')] \right\} g(1;1') = I(1;1')$$

where the effects of particle collisions are expressed by a collision matrix.
I(1;1') = -i \int P \, d^2 \left[ V(1,2) g_{2c}^{(12;1'2')} - V(2,1') g_{2c}^{(12';1'2)} \right] . \tag{II-30}

Apparently, a successful evaluation of the collision matrix I(1;1') will lead to a generalization of the TDHF approximation.

The correspondence between the TDHF approximation and eq. (II-29) obtained from the Green's function hierarchy is now very transparent. When the collision matrix is neglected, eq. (II-29) is reduced to that of TDHF. In particular, the one-body reduced density matrix is related to the one-body Green's function $g^<$ as:

\[
N(x;x';t) = -i \lim_{t' \to t} g^< (xt;x't') . \tag{II-31}
\]

The usual TDHF equation in the density matrix form can be directly obtained from eq. (II-29) when I(1;1') is neglected:

\[
i \hbar \frac{\partial}{\partial t} N(x;x';t) = -\frac{\hbar^2}{2m} (v^2 - v'^2) N(x;x';t) + \int d^2 x_2 \lim_{x_2' \to x_2} \left[ (v(x,x_2) - v(x',x_2')) \right. \\
\left. \left[ N(x;x,t) N(x_2';x_2';t) - N(x;x_2';t) N(x_2;x_2';t) \right] \right] . \tag{II-32}
\]

An exploitation of the idempotent property, symmetry and finite normalization conditions reduces the TDHF equation in the density matrix form to that in the single-particle form. Details of this equivalence will be proven in Appendix II.

The recognition of the TDHF approximation as the simplest truncation
of the Green's function hierarchy is useful not only as a way to understand how the TDHF equation can be related to the formal many-body theory, but it also suggests how to construct corrections to the TDHF approximation. To obtain explicitly an approximation to the collision matrix, it is necessary to examine the next level of the Green's function hierarchy.
D Second Level of the Green's Function Hierarchy

To obtain explicit forms of the particle-particle correlations of very high orders, it is necessary to develop general rules for diagrammatic methods. In the present case, however, the inclusion of the collision term as a lowest order correction to the TDHF approximation can be accomplished by analyzing the structure of the Green's functions at the first two levels.

To construct an approximate representation of the correlated two-body Green's function \( g_{2c} \), it is instructive to obtain its equation of motion. It is also helpful to recall the exact equation of the full two-body Green's function:

\[
\begin{align*}
\left[ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right] g_2(12; 1'2') &= \hbar \left[ \delta_p(1, 1') g(2; 2') - \delta_p(1, 2') g(2; 1') \right] \\
&+ (-i) \int d^3 V(1, 3) g_3(123; 1'2'3') 
\end{align*}
\]  

(II-33)

In general, for an \( n \)-body Green's function \( g_n \), different types of correlations are reflected by various cluster decompositions. The simplest form of an uncorrelated \( n \)-body Green's function is a determinant of one-body Green's (see, for example, Am-62):

\[
g_{n0}(12 \ldots n; 1'2' \ldots n') = \begin{vmatrix}
g(1;1') & g(1;2') & \ldots & g(1;n') \\
\vdots & \vdots & & \vdots \\
g(n;1') & g(n;2') & \ldots & g(n;n') 
\end{vmatrix}, \quad (II-34)
\]

with the associated correlated function defined in an obvious way:
\[ g_{nc}(12 \ldots n; 1'2' \ldots n') = g_n(1 \ldots n; 1' \ldots n') - g_{n0}(1 \ldots n; 1' \ldots n'). \] (II-35)

The mean-field part of the two-body Green's function defined in eq. (II-25) is a special case of this scheme. In addition, it is easy to verify that when all the correlated functions \( g_{nc} \) are neglected, the corresponding \( g_{n0} \) satisfy the equations of the free field problem.

Motivated by such observations, one can construct a potential operator \( U^{(2)}_{MF} \) for \( g_2 \) and \( g_{20} \) in a way analogous to the mean-field potential for \( g \).

For any c-number function \( f(12;1'2') \), the operational definition of \( U^{(2)}_{MF} \) is given as follows:

\[ U^{(2)}_{MF}(1) f(12;1'2') = -i \int d^3 V(1,3) \left[ + g(1;1') f(23;2'3') - g(1;2') f(23;1'3') + g(1;3') f(23;1'2') \right] \] (II-36a)

and

\[ U^{(2)}_{MF}(2) f(12;1'2') = -i \int d^3 V(2,3) \left[ - g(2;1') f(13;2'3') + g(2;2') f(13;1'3') - g(2;3') f(13;1'2') \right]. \] (II-36b)

Now, applying eq. (II-27a) and the definition for \( U^{(2)}_{MF} \), the following equation for \( g_{2c} \) can be obtained, after some tedious, but straightforward algebraic steps:

\[ \left[ i\hbar \frac{\partial}{\partial t_1} + \frac{\hbar^2}{2m} v^2_{11} - U^{(2)}_{MF}(1) \right] \left[ -i\hbar \frac{\partial}{\partial t_1'} + \frac{\hbar^2}{2m} v^2_{1'} - U^{(2)}_{MF}(1') \right] g_{2c}(12;1'2') \]

\[ = i V(1,2) g_2(12;1'2') + \text{higher-order terms} \] (II-37)
The higher-order terms in eq. (II-37) involve expressions containing $g_3$. When they are neglected, then eq. (II-37) can be inverted to solve for $g_{2c}$ in an approximate manner:

$$
g_{2c}(12;1'2') = \frac{i}{\hbar} \int d1 d2 \ V(\bar{1},\bar{2}) \ g(1;\bar{1}) \ g(2;\bar{2}) \ g_{20}(\bar{12};1'2') \tag{II-38}
$$

which can also be verified by a direct substitution back into eq. (II-37). It is worthy to note that the approximate expression for $g_{2c}$ in eq. (II-38) satisfies the correct symmetry conditions:

$$
g_{2c}(12;1'2') = -g_{2c}(21;1'2') \tag{II-39}
$$

Except for the time-path ordering of the operators in the time integration, eq. (II-38) is the same form as that of the Born collision term in the conventional perturbation expansion (Ka-a 76). In passing, it should also be noted that eq. (II-38) can be derived by following the method of Martin and Schwinger (Ma-c 59, Ka-a 76). By means of a functional technique, generalized to the case of path ordering, one can prove the existence of a mass operator by explicit construction. Then the correlated two-body Green's function emerges as a result of successive approximations to the one-body equation.

With the higher-order terms neglected, eq. (II-38) can henceforth be taken as the definition of the correlated two-body function in the present truncation scheme. The remainder of this chapter and also the next one are devoted to the consequences of a theory in which this correlation is included.
E  Effective and Residual Interactions

Without specification of the interaction used and the many-fermion system in question, the arguments given in the previous section of this chapter are completely general. However, it is well known that for the nuclear field, it is not possible to consider only the first few terms of the perturbation series for the correction of the Green's functions, if one starts with the bare nucleon-nucleon interaction $v^0(x,x')$. Consequently, it is necessary to carry out partial summations so that some appropriate infinite sequence of terms in the perturbation series involving the bare interaction makes a first-order contribution involving an effective interaction to the quantities under consideration. In the application to nuclear dynamics, the expansion in eq. (II-38) requires some amendment in order to be useful. For this purpose, one would like to examine how the effective interaction and residual interaction are related to the bare interaction, at least in a formal way.

It is convenient to introduce a vertex function for the bare interaction. (Since much of the following arguments is schematic, it is sufficient to take the un-antisymmetrized form):

$$\Gamma^{(0)}(12;34) = v_b(x_1,x_2) \delta(t_1-t_2) \delta(1,3) \delta(2,4) \qquad (II-40)$$

It is easy to show that the exact equation of motion involves the vertex function $\Gamma(12;34)$ in the following form (see, for example, Ab 75):

\[
(\gamma \frac{\delta}{\delta t_1} + \hbar \frac{\gamma^2}{2m} \gamma^2_1) \chi_{11'} \\
= \hbar \delta(1;1') + (-i) \int d2 \, d3 \, d4 \, \Gamma(12;34) \, g_{20}(34;1'2^+) \quad (II-41)
\]
where the exact vertex function is given by the integral equation:

\[
\Gamma(12;34) = \Gamma^{(0)}(12;34) + \frac{i}{\hbar} \int d\bar{1} d\bar{2} d\bar{3} d\bar{4} \Gamma^{(0)}(12;\bar{3}\bar{4}) g(\bar{3};1) g(\bar{4};2) \Gamma(I\bar{2};34).
\]  

(II-42)

In general, the vertex function in eq. (II-42) cannot be solved exactly. However, an approximate solution can be obtained, for example, by a partial sum involving compact diagrams to very high orders. Once this is accomplished, the diagonal part of the vertex function can be identified as the effective interaction \(v\) for the mean field.

\[
\Gamma(12;34) = v(1,2) \delta(1,3) \delta(2,4)
\]

(II-43)

for \((x_3, t_3) = (x_1, t_1)\) and \((x_4, t_4) = (x_2, t_2)\). For the off-diagonal elements of the vertex function, since different space-time points are involved at the vertex, the simplest parametrization in terms of a residual interaction \(v'\) can be given in the following form:

\[
\Gamma(12;34) = v'(1,2) g(1;3) g(2;3) v'(3;4)
\]

(II-44)

for \((x_3, t_3) \neq (x_1, t_1)\) and \((x_4, t_4) \neq (x_2, t_2)\), involving a second order in the interaction \(v'\). This is chosen to correspond to the result of eq. (II-38) involving, however, the residual interaction \(v'\) instead of \(v\).

With these identifications and the truncation scheme described above, one obtains closed equations involving only one-body Green's functions:
\[
(i\hbar \frac{\partial}{\partial t_1} + \frac{\hbar^2}{2m} \nabla_1^2) g(1;1') - (-i) \int \frac{d2}{p} V(1,2) \ g_{20}(12;1'2^+) \\
= \hbar \delta_p(1;1') - (-i) \int \frac{d2}{p} V'(1,2) \ g_{2c}(12;1'2^+) 
\]

(II-45a)

and

\[
(-i\hbar \frac{\partial}{\partial t_1'} + \frac{\hbar^2}{2m} \nabla_1'^2) g(1;1') - (-i) \int \frac{d2}{p} V(2,1') \ g_{20}(12^-;1'2) \\
= \hbar \delta_p(1;1') - (-i) \int \frac{d2}{p} V'(2,1') \ g_{2c}(12^-;1'2) 
\]

(II-45b)

where the correlated two-body function is now given as:

\[
g_{2c}(12;1'2') = \frac{i}{\hbar} \int \frac{d1 \ d2}{p} V'(1,2) \ g(1;1') \ g(2;2') \ g_{20}(12;1'2') 
\]

(II-46)

and

\[
V'(1;2) = V'(x_1, x_2) \ \delta_p(t_1, t_2) 
\]

(II-47)

The adjoint equations (II-45a) and (II-45b) can be combined in a more symmetric form:

\[
\left\{ i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_1'} \right) + \frac{\hbar^2}{2m} \left( \nabla_1^2 - \nabla_1'^2 \right) - \left[ U_{\text{MHF}}(1) - U_{\text{MHF}}(1') \right] \right\} g(1;1') \\
= I(1;1') 
\]

(II-48)

where the modified Hartree-Fock potential \( U_{\text{MHF}} \) is defined in the same way as eqs. (II-28a) and (II-28b), except that the effective interaction is used here. The collision matrix is now entirely due to the residual interaction:

\[
I(1;1') = -i \int \frac{d2}{p} \left[ V'(1,2) \ g_{2c}(12;1'2^+) - V'(2,1') \ g_{2c}(12^-;1'2) \right] 
\]

(II-49)
One can now envisage that the mean-field theory of TDHF approximation, obtained by using an effective interaction $v$ in eqs. (II-45a) and (II-45b) and neglecting the residual interaction $v'$, is a good first approximation to the nuclear dynamics. In using an effective interaction with density-dependence such as the Skyrme type (Bo-b 76, Ne 77, 78) and containing the effect of rearrangement terms because of such a dependence, many of the higher-order correlations are, in principle, incorporated. It now becomes necessary to take into account the last terms in eqs. (II-45a) and (II-45b) as a correction due to the residual interaction. The effective and residual interactions need not be the same and henceforth $v$ and $v'$ will be used for the mean-field part and the correlated part of the Green's function for a strongly interacting nuclear system.
Extended Time-Dependent Hartree-Fock Approximation in Green's Function Representation

The origin of the time-path ordering of field operators has been discussed in section B. However, the limits of the path $t_i$ and $t_f$, so far, have not been specified. The choice of these limits is ultimately related to the questions of the exact initial correlations of the system and its decay in time. These questions have been addressed, directly and indirectly, by different authors. In the works of Fujita (Fu 66, 69, 71), the initial correlations are included, but the final equations contain correction terms of considerable complexity. Such treatments may be very useful when it is necessary to include correlations of high orders. In the studies of Craig (Cr 68), the time-path is taken to run from the remote past to the present time and then from the present time back to the remote past in a simple manner. Though this choice of time-path does not completely solve the difficult problem of initial correlations (Ko-b 69), it has the merit of being simple and for many physical situations, is considered adequate (Ha-a 74, 75).

For the present purposes, one can follow a prescription similar to Craig's. The residual interaction is considered to be operative since the remote past and it is consistent to take $t_i$ to be $-\infty$. The time integral for the correlated two-body function $g_{2c}^{\tau}$, so far written in a very general way, can be converted into a simpler form. In the intermediate steps of the following derivations one can take, for convenience, $t_f = \max(t_1, t_{1'})$. In the last analysis, it is sufficient to limit one's attention to the case of $t_{1'} = t_1^+$ as the final form of the result.
With this in mind, the time integrals in eq. (II-46) can be explicitly evaluated. The mathematical procedure is rather straightforward. The time integral along each branch of the contour is replaced by a definite integral over the interval \((-\infty, t_f)\), while the field operators in the integrand are properly ordered relative to the branch. Thus, for example, in the double time integral in eq. (II-46), each of the integration variables \(\tilde{t}_1\) and \(\tilde{t}_2\) runs over the entire contour, leading to a final replacement of the general Green's functions \(g\) by the auxiliary functions \(g^<\) and \(g^>\) (defined in eqs. (II-22a) and (II-22b)). In particular, the integrals in eq. (II-46) can be rewritten as follows:

\[
\iint d\tilde{t}_1 d\tilde{t}_2 = \int_{P+} d\tilde{t}_1 \int_{P+} d\tilde{t}_2 + \int_{P-} d\tilde{t}_1 \int_{P-} d\tilde{t}_2 + 
\]

\[
+ \int_{P+} d\tilde{t}_1 \int_{P-} d\tilde{t}_2 + \int_{P-} d\tilde{t}_1 \int_{P+} d\tilde{t}_2.
\]

(II-50)

When \(t_1\) and \(t'_1\) are taken to be on the forward branch \(P_+\), the last two components on the right side of eq. (II-50) do not make any contribution to the final result (because of \(\delta_{PP}\) associated with the interaction term) and eq. (II-46) can now be written in the following form:

\[
g_{2c}^{(12; 1'2')} = \frac{i}{\hbar} \int \int d\tilde{x}_1 d\tilde{x}_2 \ v'(\tilde{x}_1, \tilde{x}_2) \ \int_{-\infty}^{\max(t_1, t'_1)} dt_1 
\]

\[
\left\{ g^>(1; \tilde{1}) g^>(2; \tilde{2}) \ G[g^<(\tilde{1}; t_1') g^<(\tilde{2}; t_1') \] 

\[- g^<(1; \tilde{1}) g^<(2; \tilde{2}) \ G[g^>(\tilde{1}; t_1') g^>(\tilde{2}; t_1') \] 

\right\}_{t_2 = \tilde{t}_1} + \Delta_{2c}^{(12; 1'2')} 
\]

(II-51)
where \( G \) is the anti-symmetrization operator defined in the usual way:

\[
G \left[ f(1;1') f(2;2') \right] \equiv f(1;1') f(2;2') - f(1;2') f(2;1') .
\]

For the sake of precision, a correction term \( \Delta_{2c} \) is included in eq. (II-51).

In the general case of an arbitrary ordering of the times \( t_1, t_1', t_2 \) and \( t_2' \), \( \Delta_{2c} \) contains small contributions due to the time-paths between pairs of these time coordinates. However, this residual correction term \( \Delta_{2c} \) vanishes identically when the four time coordinates collapse to a common limit \( (t_1', t_2', t_2' \rightarrow t_1) \). For this reason, it is not necessary to write out \( \Delta_{2c} \) explicitly.

The ETDHF equation for the one-body Green's function \( g^\ast \) can now be explicitly written:

\[
\left[ i\hbar \frac{\partial}{\partial t_1} + \frac{\hbar^2}{2m} \nabla^2_{1} - U_{\text{MHF}}(1) \right] g^\ast (1;1') = \hbar \delta (1;1')
\]

\[
+ \frac{i}{\hbar} \int \int \int dx_2 d\bar{x}_1 d\bar{x}_2 \ v'(x_1', x_2') \ v'(\bar{x}_1', \bar{x}_2')
\]

\[
\int_{-\infty}^{\max(t_1', t_{11}')} dt_1
\]

\[
\left\{ \begin{array}{l}
g^\ast (1;\bar{1}) g^\ast (2;\bar{2}) \ G \left[ g^\ast (\bar{1};1') g^\ast (\bar{2};2') \right] \\
-g^\ast (1;\bar{1}) g^\ast (2;\bar{2}) \ G \left[ g^\ast (\bar{1};1') g^\ast (\bar{2};2') \right]
\end{array} \right\}_{t_2=t_1', \ \bar{t}_2=\bar{t}_1'}
\]

\[
(\text{II-53})
\]

where, for simplicity, \( \Delta_{2c} \) has been dropped since it makes no contribution to the final equal-time results. Here, the mean-field potential \( U_{\text{MHF}} \) is constructed in terms of only \( g^\ast \)s.
The adjoint equation of eq. (II-53) can be written in an analogous way and combined with eq. (II-53) in the same manner as described before. The collision matrix now reads as:

\[
I(1;1') = \frac{i}{\hbar} \int \int \int dx_2 \, dx_1 \, dx_2' \left[ v'(x_1, x_2) - v'(x_1', x_2') \right] v'(x_1', x_2') \\
\int_{-\infty}^{\max(t_1', t_2')} dt_1 \left\{ g^>(1;1') g^>(2;2') \mathbb{G} \left[ g^<(1;1') g^<(2;2') \right] \\
+ g^<(1;1') g^<(2;2') \mathbb{G} \left[ g^>(1;1') g^>(2;2') \right] \right\}
\]

(II-54)

Eq. (II-53) and its adjoint equation may be called the extended time-dependent Hartree-Fock approximation in the Green’s function representation. The additional collision term is second order in the residual interaction, much as the collision integral in the classical Boltzmann equation. Furthermore, because of the integration over the "past collision history" extends from the remote past to the present time only, the time-reversal invariance is broken, much as in the case of the Boltzmann equation. Eq. (II-53) is non-Markovian in the sense that the equation of motion depends not only on the present configuration of the system, but also on the past collision history.
G Single-Particle Representation

Despite the mathematical equivalence of the density matrix and single-particle forms, the practical application of TDHF is simpler when one deals with the single-particle equations than the density matrix equation (Bo-b 76, 77, Ko-a 76, 77, Ne 77, 78). Similarly, in the ETDHF approximation, the introduction of a single-particle basis will simplify its application. A proper choice of the basis functions will also allow a clean separation of the effects due to the particle collisions and those due to the temporal variation of the mean field and accordingly, makes the connections between the single-particle equations of TDHF and those of ETDHF very transparent.

In the present investigation, one is mainly interested in the time evolution of the one-body Green's function $g^<$. In general, this Green's function and its complementary function $g^>$, considered as continuous matrices of $x_t$ and $x_t'$, can be expanded in terms of any complete orthogonal set of single-particle states $\psi_\lambda$, with the introduction of occupation number matrices $n^<$ and $n^>$:

$$-i \ g^< (x_t; x_t') = \sum_{\lambda, \lambda' = 1}^\infty n^<_{\lambda \lambda', (t; t')} \ \psi_{\lambda}(x_t) \ \psi^*_{\lambda'}(x_t') \quad \text{(II-55a)}$$

and

$$i \ g^> (x_t; x_t') = \sum_{\lambda, \lambda' = 1}^\infty n^>_{\lambda \lambda', (t; t')} \ \psi_{\lambda}(x_t) \ \psi^*_{\lambda'}(x_t') \quad \text{(II-55b)}$$

with

$$\langle \psi_{\lambda'}(t) \mid \psi_{\lambda}(t) \rangle = \delta_{\lambda', \lambda} \quad \text{(II-56)}$$
The expansions given in eqs. (II-55a) and (II-55b) are completely general. When the single-particle basis is chosen to be time-independent, the burden of the dynamical description of the system will then be concentrated on the occupation number matrices. However, if a judicial prescription can be discovered for a time evolution of the single-particle basis, the resultant equations will be much more physically transparent as well as aesthetically pleasant.

When \( g^< \) and \( g^> \) are the exact Green's functions, it is easy to see from their definitions (eqs. (II-22a) and (II-22b)) and eq. (II-23) that:

\[
\lim_{t' \to t} \left[ n^<_{\lambda \lambda'}(t;t') - n^<_{\lambda \lambda'}(t;t') \right] = \delta_{\lambda \lambda'} . \quad (\text{II-57})
\]

The hermiticity of the occupation number matrices can also be readily established:

\[
n^<_{\lambda \lambda'}(t;t') = n^<_{\lambda \lambda'}(t';t) \quad \quad (\text{II-58a})
\]

and

\[
n^>_{\lambda \lambda'}(t;t') = n^>_{\lambda \lambda'}(t';t) * \quad \quad (\text{II-58b})
\]

It is clear from eqs. (II-58a) and (II-58b) that the diagonal matrix elements in the equal-time limits must be real, positive numbers, hence the term "occupation number matrix".

The properties given in eqs. (II-57), (II-58a) and (II-58b) are valid for any single-particle representation for exact one-body Green's functions. In an approximation scheme such as the ETDHF, however, they cannot be
proven in general. Instead, they can be implemented as the physically meaningful constraints imposed on the initial conditions of the approximate one-body Green's functions.

As in the subsequent discussions one is mainly interested in the equal-time limits of the Green's functions, it is convenient to introduce the following abbreviations:

\[ \langle \lambda \lambda' | n_{X}^{<} \rangle (t) = \langle \lambda \lambda' | n_{X}^{<} (t; t^+) \rangle \quad (II-59a) \]

and

\[ \langle \lambda \lambda' | n_{X}^{>} \rangle (t) = \langle \lambda \lambda' | n_{X}^{>} (t; t^-) \rangle \quad (II-59b) \]

1. Conservation of Hermiticity and Normalization

When the equations of motion (II-45a) and (II-45b) with eq. (II-46) are taken as the starting point for the ETDHF approximation, the following initial conditions are preserved in time within the approximation scheme:

(a) Hermiticity:

\[ \langle \lambda \lambda' | n_{X}^{<} (t=0) \rangle = \langle \lambda \lambda' | n_{X}^{<} (t=0)^* \rangle \quad (II-60a) \]

and

\[ \langle \lambda \lambda' | n_{X}^{>} (t=0) \rangle = \langle \lambda \lambda' | n_{X}^{>} (t=0)^* \rangle \quad (II-60b) \]

(b) Normalization:

\[ \sum_{\lambda=1}^{\infty} \langle \lambda \lambda' | n_{X}^{<} (t=0) \rangle = A \quad (II-61) \]

where \( A \) can be chosen to be the total particle number of the system.
A formal proof of these conditions is given in Appendix III.

The conditions of hermiticity and normalization of the occupation matrix are related to the questions of probability amplitudes and particle conservation and are important physical conditions in any practical implementation of the ETDHF approximation. In the general non-Markovian form of ETDHF, their conservation in time can be established in a representation-independent manner (Appendix III). But in principle, they should be checked again whenever further approximations have been used to simplify the equations of motion.

2. Single-Particle Equations of ETDHF

We can now specialize to the equation for the one-body Green's function \( g^< \) in the equal-time limit. A direct substitution of the single-particle expansion of eq. (II-55a) into eq. (II-48) results in the following:

\[
\begin{align*}
&i\hbar \sum_{\lambda\lambda'} n_{\lambda\lambda'}(t) \psi_{\lambda}(x;t) \psi^*_{\lambda'}(x';t) \\
&\quad - \frac{\hbar^2}{2m} \sum_{\lambda\lambda'} n_{\lambda\lambda'}(t) \left[ \nabla^2 \psi_{\lambda}(x;t) \psi^*_{\lambda'}(x';t) - \psi_{\lambda}(x;t) \nabla^2 \psi^*_{\lambda'}(x';t) \right] \\
&\quad - \sum_{\lambda_1\lambda_1'\lambda_2\lambda_2'} n_{\lambda_1\lambda_1'}(t) n_{\lambda_2\lambda_2'}(t) \\
&\quad \quad \int dx'' \left( v(x,x'') - v(x',x'') \right) \\
&\quad \quad \left[ \psi_{\lambda_1}(x;t) \psi^*_{\lambda_1'}(x'';t) \psi_{\lambda_2}(x'';t) \psi^*_{\lambda_2'}(x'';t) \\
&\quad \quad \quad - \psi_{\lambda_1}(x;t) \psi^*_{\lambda_1'}(x'';t) \psi_{\lambda_2}(x'';t) \psi^*_{\lambda_2'}(x';t) \right] = -I(x;t; x'^+) \quad . \quad (II-62)
\end{align*}
\]
In eq. (11-62), the superscript $<_{\lambda\lambda'}$ has been dropped from the coefficients $n^<_{\lambda\lambda'}$. The apparently complicated expression is in fact simple in structure. The left side of eq. (11-62) can be considerably simplified as follows.

One can define a time-dependent basis by the introduction of a single-particle (modified Hartree-Fock) Hamiltonian operator $h_{\text{MHF}}$, which, in configuration space, is defined as:

$$i\hbar \frac{\partial}{\partial t} \psi(\lambda; t) = [h_{\text{MHF}} \psi(\lambda)](t) \quad (\text{11-63})$$

where

$$[h_{\text{MHF}} \psi(\lambda)](t) = \left(-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{MHF}}(x(t))\right) \psi(\lambda;x(t)) \quad (\text{11-64})$$

and the modified Hartree-Fock potential operator $V_{\text{MHF}}$ is defined for a single-particle state as:

$$[V_{\text{MHF}} \psi(\lambda)](t) = \int dx' v(x,x') N(x';x,t) \psi(\lambda;x(t))$$

$$- \int dx' v(x,x') N(x;x';t) \psi(\lambda;x';t) \quad (\text{11-65})$$

where $N$ is the one-body reduced density matrix defined in eq. (11-31). The interaction $v(x,x')$ can be density-dependent in which case it must include the rearrangement terms (Wo 77iii). The mean-field potential $V_{\text{MHF}}$ in eq. (11-65) is defined in the same form as the TDHF potential, except that in the presence of particles, $N$ must include all possible single-particle states. This can be written in terms of the basis functions:
In the collisionless limit, with no loss of generality (Appendix III), \( n_{\alpha\alpha'} \) can be constructed to be diagonal for all time and the summations are over a finite set of A single-particle states. In this limit, eq. (II-66) recovers the usual expression of the TDHF mean potential.

Since the Hamiltonian \( h_{\text{MHF}} \) can be defined in a representation-independent manner in terms of only \( \mathcal{N} \) (or \( g^< \)), it can be easily shown that \( h_{\text{MHF}} \) is hermitian in the usual sense (though it is a non-linear operator):

\[
<h_{\text{MHF}}(t) \psi_\lambda(t)|\psi_\lambda'(t)> = <h_{\text{MHF}}(t)\psi_\lambda'(t)|\psi_\lambda(t)>
\] (II-67)

The significance of this is that the initial orthogonality and normalization of the basis set defined by eq. (II-63) are preserved in time.

With the choice of the single-particle basis defined by eqs. (II-63)-(II-65), eq. (II-62) can now be significantly simplified:

\[
\sum_{\lambda'\lambda} n_{\lambda\lambda'}(t) \psi_\lambda(x^t) \psi_{\lambda'}^*(x'^t) = -\frac{1}{\hbar} I(x^t;x'^t)
\] (II-68)

whence the equation of motion for the occupation number matrix can be projected out:
\[ \frac{d n_{\lambda \lambda'}(t)}{dt} = -\frac{1}{\hbar} < \psi_{\lambda}(t) \mid 1 \mid \psi_{\lambda'}(t) > \] 

(Hence, with this single-particle representation, the main effect of the particle collisions due to the residual interaction is to change the occupation probabilities of the single-particle states in time. In the limit of vanishing residual interaction, one recovers the TDHF equations with frozen single-particle probabilities.

The equation of the occupation number matrix (\( \Pi-65 \)) can be expressed in terms of the single-particle wavefunctions. The resultant set of ETDHF equations is closed in itself, but is non-Markovian as the time coordinate in the wavefunctions and the occupation numbers runs from the remote past to the present time. Such equations are very difficult to solve. For many practical applications, it is necessary to reduce the non-Markovian collision matrix to a Markovian form. Such simplification will be discussed in the next section.)
Slow Mean-Field and Diagonal Approximations

A reduction of the non-Markovian equations of ETDHF into a simpler Markovian form necessitates an analytic integration over the collision history. Obviously, this cannot be achieved in general, but is possible in the special case where the mean field is varying slowly enough so that simplifying assumptions can be made concerning the temporal behavior of the Green's functions. Since these assumptions are made only in connection with the explicit evaluation of the collision matrix, which itself is a correction term, a refinement of these simplifying assumptions may lead only to higher-order corrections and may be less important.

In order to obtain the correlated two-body Green's function $g_{2c}$ by integrating over the past collision history, one can consider a substitution of the expansions (II-55a) and (II-55b) into the expression of the collision matrix given in eq. (II-54) and the integration over $t_1$. In the case in which the mean-field is sufficiently slowly varying function of time, the Green's function is expected to peak sharply about the relative time coordinate ($K_a - a_76$, $Ha - a_74$) so that only a typical small region around $t_1$ with a width of some characteristic correlation time is important in the time integral of eq. (II-54).

It is reasonable to assume that the time-dependence of the single-particle wavefunctions satisfy the following approximate relation:

$$
\psi_\lambda(x, T+\tau/2) \approx e^{-i \epsilon_\lambda(T) \tau/2\hbar} \psi_\lambda(x, T) \quad (\text{II-70})
$$
for small $\tau$ and $\epsilon_\lambda$ is the time-dependent single-particle energy:

$$\epsilon_\lambda(t) \equiv <\psi_\lambda^{(t)} | h_{M_{HF}}^{(t)} | \psi_\lambda^{(t)}>.$$  \hspace{1cm} (II-71)

In fact, such an approximation has been found to account well for the predominant time-dependence of TDHF single-particle wavefunctions for small time intervals $\tau$ (Da 77). Hence, to be consistent with such considerations, one should further limit the single-particle basis to be one which is very close to a TDHF basis, at the initial time. Before a calculation, the occupation number matrix may be essentially diagonal, with at the most very small off-diagonal elements. It is reasonable to assume the following temporal behaviors for small $\tau$:

$$\epsilon_\lambda\left(\frac{1}{2}(t_0 + \tilde{t}_1)\right) \approx \epsilon_\lambda(t_1)$$  \hspace{1cm} (II-72)

and

$$n_{\lambda\lambda'}\left(\frac{1}{2}(t_0 + \tilde{t}_1)\right) \approx n_{\lambda\lambda'}(t_1).$$  \hspace{1cm} (II-73)

With these approximations, an integration over time in the correlated two-body Green's function leads to the following closed form, after some tedious algebraic manipulations:

$$g_{2c}(x_{t}, x'^{+}_{t}; x'_{t}, x''^{+}_{t})$$

$$\approx \sum_{\lambda_1'\lambda_2' \lambda_3' \lambda_4'} \frac{[\left(1-n_{\lambda_1'\lambda_1}\right)\left(1-n_{\lambda_2'\lambda_2}\right) n_{\lambda_3'\lambda_3} n_{\lambda_4'\lambda_4} - n_{\lambda_1'\lambda_1} n_{\lambda_2'\lambda_2} \left(1-n_{\lambda_3'\lambda_3}\right)\left(1-n_{\lambda_4'\lambda_4}\right)\left(\frac{\epsilon_{\lambda_1} + \epsilon_{\lambda_1'} + \epsilon_{\lambda_2} + \epsilon_{\lambda_2'} - \epsilon_{\lambda_3} - \epsilon_{\lambda_3'} - \epsilon_{\lambda_4} - \epsilon_{\lambda_4'}}{2 - i\eta}\right)^{-1}\left[\left(\psi_{\lambda_1'}^{*}\psi_{\lambda_2'}^{*}(x'^{+}_{t}) \psi_{\lambda_3'}^{*}(x'^{+}_{t}) \psi_{\lambda_4'}^{*}(x'^{+}_{t}) \psi_{\lambda_1}^{*}\psi_{\lambda_2}^{*}(x'^{+}_{t}) \psi_{\lambda_3}^{*} \psi_{\lambda_4}^{*}(x'^{+}_{t})\right) - \frac{\lambda_{1'}^{*}\lambda_{2'}^{*}\lambda_{1} \lambda_{2}}{\lambda_{3} \lambda_{4}}\right]v' | \lambda_{3} \lambda_{4}>}{\lambda_{1} \lambda_{2}} \hspace{1cm} (II-74)
where the anti-symmetrized matrix element of the residual interaction is defined in the usual way:

\[
\begin{aligned}
&\langle \lambda_1 \lambda_2 | v' | \lambda_4 \lambda_3 - \lambda_3 \lambda_4 \rangle \\
= & \int \int dx \, dx' \, \psi_{\lambda_1}^*(xt) \, \psi_{\lambda_2}^*(x't) \, v'(x, x') \\
& \begin{bmatrix}
\psi_{\lambda_4}(xt) \, \psi_{\lambda_3}(x't) - \psi_{\lambda_4}(xt) \, \psi_{\lambda_3}(x't)
\end{bmatrix}.
\end{aligned}
\]  

Finally, the equation of motion of the occupation number matrix is given by the following, after some lengthy calculations:

\[
\frac{dn_{\lambda_1 \lambda_2}(t)}{dt} = -\frac{1}{\hbar} \sum_{\lambda_3 \lambda_4} \left[ \delta_{\lambda_1 \lambda_3} \delta_{\lambda_2 \lambda_4} \langle \lambda_3 | v' | \lambda_1 \lambda_2 \rangle - \delta_{\lambda_3 \lambda_1} \langle \lambda_4 | v' | \lambda_2 \lambda_3 \rangle \right]
\]

\[
\left[ (1-n_{\lambda_1 \lambda_1}) (1-n_{\lambda_2 \lambda_2}) n_{\lambda_3 \lambda_3} n_{\lambda_4 \lambda_4}
- \eta \lambda_1 \lambda_1 \lambda_2 \lambda_2 (1-n_{\lambda_3 \lambda_3}) (1-n_{\lambda_4 \lambda_4}) \right]
\]

\[
\left[ \bar{\epsilon}_{\lambda_1} + \bar{\epsilon}_{\lambda_2} - \bar{\epsilon}_{\lambda_3} - \bar{\epsilon}_{\lambda_4} - i\eta \right]^{-1} \eta \rightarrow 0 \langle \lambda_1 \lambda_2 - \lambda_2 \lambda_1 | v' | \lambda_4 \lambda_3 \rangle
\]

(II-76)

where

\[
\bar{\epsilon}_{\lambda_i} = \frac{1}{2} (\epsilon_{\lambda_i} + \epsilon{_{\lambda_i}}).
\]

(II-76a)

This is a generalized master equation, involving an eight-fold summation over the single-particle indices, which, however, can be reduced to a seven-fold summation because of the Kronecker delta-functions.
The formulation of the ETDHF approximation is completed once the modified TDHF equations (II-63)-(II-65) and the generalized master equation (II-76) are obtained. However, eq. (II-76) presents a formidable task if one attempts to solve the ETDHF equations in this form.

Consistent with the notion that TDHF is a good first approximation, it is reasonable to assume that the diagonal parts of the occupation number matrix can adequately represent most of the dynamics, the off-diagonal parts being higher-order corrections. Henceforth, one can further consider the diagonal approximation:

\[ n_{\lambda \lambda'}^{<} (t) = \delta_{\lambda \lambda'} n_{\lambda \lambda'}^{<} (t) \]  

(II-77)

and hereafter, use the abbreviation for the diagonal elements:

\[ n_{\lambda} (t) = n_{\lambda}^{<} (t) . \]  

(II-78)

With the diagonal approximation invoked, the final ETDHF equations now appear as a set of modified TDHF equations:

\[ i \hbar \frac{\partial}{\partial t} \psi_{\lambda} (xt) = - \frac{\hbar^2}{2m} \nabla^2 + V_{\text{MHF}} \psi_{\lambda} (xt) \]  

(II-79)

which is coupled to a simplified master equation:

\[ \frac{d n_{\lambda} (t)}{dt} = \frac{\pi}{\hbar} \sum_{\lambda \lambda' \lambda_4} \delta (\epsilon_\lambda + \epsilon_\lambda_2 - \epsilon_\lambda_3 - \epsilon_\lambda_4) \left[ (1-n_{\lambda}) (1-n_{\lambda_2}) n_{\lambda_3} n_{\lambda_4} - n_{\lambda} n_{\lambda_2} (1-n_{\lambda}) (1-n_{\lambda}) \right] \\
 \frac{d}{dt} |<\lambda \lambda_2 | v' | \lambda_4 \lambda_3 - \lambda_3 \lambda_4> |^2 \]  

(II-80)
The modified TDHF potential is now simplified to

\[ [V_{\text{MHF}} \psi_{\lambda}] = \int dx' \rho(x',t) v(x, x') \psi_{\lambda}(x,t) \]
\[- \sum_{\lambda'} n_{\lambda'}(t) \int dx' v(x, x') \psi_{\lambda'}(x,t) \psi^*_{\lambda'}(x',t) \psi_{\lambda}(x',t) \]

(II-81)

with the number density defined as:

\[ \rho(x,t) = \sum_{\lambda} n_{\lambda}(t) \psi_{\lambda}(x,t) \psi^*_{\lambda}(x,t) \]

(II-82)
I Summary of ETDHF

As summary of the present formulation, the ETDHF approximation can be expressed at different levels of exactness. The crucial step in this formulation is the evaluation of the collision matrix which gives the correction to the TDHF approximation. With a truncation of the higher-order terms, it can be approximated in terms of one-body Green's functions (second order in the residual interaction). The most general expression of the collision matrix in this truncation scheme is a non-Markovian integral in time. Indeed, if one takes such a form in an axiomatic manner, much of the analytic discussions on the structure of ETDHF can be carried out in a representation-independent manner. To cast the equation into a more practical form, one goes to a single-particle representation. By an appropriate choice of the basis, it is possible to separate the effects due to the mean field and the residual interaction. A slow mean field approximation reduces the collision matrix from a non-Markovian to a Markovian form. In addition, a diagonal approximation leads to a simple, final result, which consists of a set of modified TDHF equations, which is coupled to a master equation equation of the time-dependent single-particle occupation numbers.

The physical interpretation of the result is now very clear. The single-particle equation reflects the physical assumption that TDHF provides a reasonable first approximation. The new feature in ETDHF is the additional degree of freedom due to the time-dependent occupation numbers. Its equation of motion, the master equation, describes properly the physics of particle collisions by taking into account energy conservation, Pauli principle and the
transition probabilities due to the residual interaction.
CHAPTER III

PHYSICAL CONSEQUENCES

OF

EXTENDED TIME-DEPENDENT HARTREE-FOCK APPROXIMATION
A principal result of the preceding chapter is the derivation of the extended time-dependent Hartree-Fock approximation from the exact hierarchy of generalized Green's functions. This is achieved by a truncation of the hierarchy at the second level. The resultant equations of motion include the Born collision term due to the residual interaction, but also contain the mean-field theory of the time-dependent Hartree-Fock approximation as a collisionless limit. In addition, practical results obtained include the reduction of the general ETDHF equations to a simple single-particle form and, finally, of the collision term from a non-Markovian form to a Markovian representation.

In the case of nuclear dynamics, with the appropriate use of a density-dependent effective interaction, this ETDHF is useful in providing corrections to the usual TDHF approximation and thus may enlarge the range of validity of this mean-field theory.

The apparent advantage of this approach is that one can easily relate ETDHF to the exact many-body theory. The various mathematical approximations and physical assumptions invoked in obtaining the final simple results are well-defined, so that refinements and generalizations may be readily made if and when they are called for in the future.

However, the ETDHF results can also be obtained from a less rigorous, nevertheless intuitive, approach. This is based on the analogue of the classical Boltzmann equation which usually appears as follows:
\[
\frac{\partial}{\partial t} f(r, v, t) + \mathbf{v} \cdot \frac{\partial}{\partial r} f(r, v, t) + \frac{\mathbf{F}}{m} \cdot \mathbf{v} = \int d\Omega \int d^3v_2 \sigma(\Omega) |\mathbf{v} - \mathbf{v}_2| \left[ f(r, v_1', t) f(r, v_2', t) - f(r, v_1, t) f(r, v_2, t) \right]
\]

(III-1)

where \( f \) is the classical distribution function in space, momentum and time and \( \mathbf{F} \) is the force due to an external field. The right side of eq. (III-1) is the collision integral which contains \( \sigma(\Omega) \), the cross-section for the elastic scattering of a pair of particles with initial velocities \( \mathbf{v}' \) and \( \mathbf{v}_2' \) and final velocities \( \mathbf{v}_1' \) and \( \mathbf{v}_2' \).

Historically, the classical Boltzmann equation, which opened up new horizons in the study of statistical mechanics, was first obtained for dilute gases. The original approach was founded on several intuitive, though plausible, assumptions (Hu 63, Ha-d 71):

(a) The validity of the concepts of the molecular structure and the distribution function of a gas are assumed. (This was a non-trivial hypothesis in an era when there was no consensus regarding the microscopic structure of matter.)

(b) The gas is taken to be sufficiently dilute and the inter-particle forces so weak that the motion of one particle is essentially independent of all the others.

(c) The change of particle trajectories is assumed to be mainly the result of binary collisions because of the short-range forces. But it is assumed that the "memory" of the initial configuration is "lost" after each collision.
The left-hand side of eq. (III-1) can be obtained from kinematical arguments. But it is the collision integral on the right-hand side, with the assumptions built in, that violates the microscopic time-reversal invariance.

The Boltzmann equation thus obtained is not without limitations. In general, it is rather difficult to generalize it in this form to situations in which the basic assumptions (a) – (c) are not valid. This shortcoming, however, can be remedied by the more fundamental and systematic hierarchy of reduced distribution functions due to Bogoliubov and Gurov (Bo-a 47), Born (Bo-e 49), Kirkwood (Ki 47) and Yvon (Yv 35), the so-called BBGKY hierarchy, which provides the classical analogue of the quantum Green's function hierarchy. From this BBGKY hierarchy, the Boltzmann equation can be obtained as a limiting case at the lowest level of the system in a natural manner. Moreover, systematic generalizations and corrections to the Boltzmann equation can be readily constructed by analyzing the higher levels in this hierarchy.

The quantum TDHF case presents an independent-particle description of the dynamics in which the particle-particle collisions are neglected. The underlying dynamics is dominated by the common potential generated by all particles in the system. This feature is reflected in many TDHF calculations for low-energy, head-on collisions of atomic nuclei, which indicate very little transfer of the longitudinal momentum to the transverse directions. In addition, the particles in the composite system during reaction generally take very long times (compared with the typical reaction time) to
equilibrate, if indeed they do so at all.

In the dynamical evolution of a system described by the TDHF approximation, the self-consistent field appears as an "external" potential for each particle. It is easy to see that the situation is, in some sense, analogous to the collisionless Boltzmann equation. In fact, this intuitive picture can be more firmly established when one transforms the TDHF equation onto the Wigner space (Wi 32, Be-a 77, Wo 77iii). With the residual interaction turned on, however, the energy states of the particles can change whenever collisions occur. Hence the inclusion of the residual interaction in the dynamics can be regarded as the analogue of the collision integral in the Boltzmann equation.

With this comparison in mind, if the dynamical process does not prohibit the establishment of a mean field in the system, it is reasonable to postulate that one can start with a set of TDHF-like single-particle equations. The only modification is that many more states must be included in the scheme and each must now be weighted by a time-dependent occupation number.

To complete this analysis, one must also find a prescription for the equation of motion of the occupation numbers. Since the residual interaction induces transitions among the single-particle states, to the lowest order approximation, one can apply the Fermi Golden Rule which relates the transition probability $W$ to the matrix element of the interaction:

$$W(\lambda_1 \lambda_2; \lambda_3 \lambda_4) = \frac{2\pi}{\hbar} |\langle \lambda_1 \lambda_2 | v' | \lambda_3 \lambda_4 \rangle|^2.$$  \hspace{1cm} (III-2)

A consideration of the energy conservation and the Pauli principle for the
fermions leads to the following equation for the occupation number $n_{\lambda}$:

$$\frac{dn_{\lambda}(t)}{dt} = \frac{1}{2} \sum_{\lambda_2, \lambda_3, \lambda_4} W(\lambda_2, \lambda_3, \lambda_4) \delta(\epsilon_{\lambda} + \epsilon_{\lambda_2} - \epsilon_{\lambda_3} - \epsilon_{\lambda_4}) \times \left[ (1-n_{\lambda})(1-n_{\lambda_2})n_{\lambda_3}n_{\lambda_4} - n_{\lambda}n_{\lambda_2}(1-n_{\lambda_3})(1-n_{\lambda_4}) \right].$$

(III-3)

Eq. (III-3) is in fact the same as the master equation (II-80). Hence one can consider the master equation as the quantum Boltzmann equation for a finite system in configuration.

Such an intuitive approach to the ETDHF equations is based largely on plausibility arguments which can claim no rigor. However, it is gratifying that a systematic but abstract treatment of the exact equations and a heuristic yet simple analysis should yield the same result. As will be seen again later, other comparisons of the quantum mechanical problem with the classical Boltzmann equation are illuminating and helpful in deepening our understanding of the underlying physics.
B  Entropy, H-Theorem and Thermal Equilibrium

The ETDHF equations (II-79) and (II-80), as the configuration space analogue of the quantum Boltzmann equation, are applicable to both finite as well as infinite systems. In the case of an infinite system, the energy levels are no longer discrete, but the momentum will be a more suitable quantum label for the states. Here, the master equation is reduced to the usual Uehling and Uhlenbeck equation for an infinite system (Ue 33). From the ETDHF equations of a finite system, just as in the case of the Boltzmann equation of Uehling and Uhlenbeck, much information can be extracted concerning the macroscopic variables, conservation laws and the approach to thermal equilibrium.

1. Entropy

The mathematical structure of the ETDHF equations strongly suggests the establishment of an H-theorem. One can first introduce a field quantity which can be identified with the entropy density:

$$ \sigma(x,t) = -k_B \sum_{\lambda} \psi_\lambda(x,t) \psi_\lambda^*(x,t) \left[ n_\lambda(t) \ln n_\lambda(t) + (1-n_\lambda(t)) \ln(1-n_\lambda(t)) \right] $$

(III-4)

where $k_B$ can be any constant, but for the sake of analogy, may be taken to be the Boltzmann constant to give the relevant equations the correct dimension.

The total entropy of the system then given by:

$$ S(t) \equiv \int dx \sigma(x,t) $$
Then an H-theorem can be established.

2. H-Theorem

The theorem states that the total entropy of the system never decreases

\[ \frac{dS(t)}{dt} \geq 0. \]  

(III-6)

The equality sign in (III-6) holds if and only if for any combination of four levels \( \lambda_1, \lambda_2, \lambda_3 \), and \( \lambda_4 \),

\[ (1-n_\lambda_1)(1-n_\lambda_2)n_\lambda_3n_\lambda_4 = n_\lambda_1n_\lambda_2(1-n_\lambda_3)(1-n_\lambda_4) \]  

(III-7)

which, in turn, implies that for any \( \lambda \),

\[ \frac{dn_\lambda(t)}{dt} = 0. \]  

(III-8)

The formal proof of this H-theorem is given in Appendix IV. As will be seen in the proof, the above statements of the H-theorem remain valid even in the more general case where the delta-function in the master equation is replaced by any other appropriate distribution \( D(\lambda_1, \lambda_2; \lambda_3, \lambda_4) \) of single-particle energies, which is symmetric under the exchanges of indices \( \lambda_1 \leftrightarrow \lambda_2 \), \( \lambda_3 \leftrightarrow \lambda_4 \), and \( (\lambda_1 \leftrightarrow \lambda_3, \lambda_2 \leftrightarrow \lambda_4) \).

3. Global Thermal Equilibrium and Relaxation Time

With the H-theorem, global equilibrium for a finite many-fermion
system can be quantitatively defined as the configuration for which the distribution of the single-particle occupation numbers is stationary. Moreover, one now has a way to measure how far a non-equilibrium system, such as a nuclear system during collision, is at any time from thermal equilibrium. This can be done simply by calculating the rate of change of the entropy at that time.

In later sections, the questions of the characteristics of thermal equilibrium will be analyzed. But it is of great interest to make an estimate of the time scale for thermal relaxation of a nuclear system. For cases not far from equilibrium, \( \lambda \sim 0 \) for states \( \lambda \) much above or below the Fermi surface (which is now not sharp). One can focus the attention to the occupation numbers of states near the Fermi surface for which the factor containing the occupation numbers in the master equation assumes a value of the order of \( \frac{1}{2} \). For such a term in eq. (II-80), one can restrict to the case where \( \lambda_1 \) and \( \lambda_2 \) (and also \( \lambda_3 \) and \( \lambda_4 \)) are time-reversal doublets. When a value of 2 MeV is assigned to be the average value of the matrix element (Sc-c 72) following can be obtained:

\[
\frac{dn}{dt} \sim \frac{\rho(\epsilon_F)}{2} \cdot \frac{\pi}{\hbar} \cdot (2\text{MeV})^2 \cdot \frac{1}{2}
\]

where \( \rho(\epsilon_F) \) is the density of single-particle states at the Fermi surface and is given on the average by (Br-a 72):

\[
\rho(\epsilon_F) \sim \frac{3A}{80} \text{MeV}^{-1}
\]

This leads to the approximate rate of change of the occupation probability
A time scale can be obtained by taking the inverse of eq. (III-11). This is approximately the time for the occupation probability of the rate on the top of the Fermi surface to change by unity. As in the classical case where a few units of the time between collisions are sufficient to lead to thermal equilibrium (Ba-a 75), so here, the relaxation time $\tau_{\text{relax}}$ leading to global thermal equilibrium is expected to be a few units (say 5) of the time for a state on the top of the Fermi surface to change its occupation number by unity:

$$\tau_{\text{relax}} \sim \frac{4000}{A} \text{ fm/c} \quad (\text{III}-12)$$

A dynamical motion is therefore considered to be slow or fast depending upon whether the time scale involved is much greater or much less than this unit. Recent model studies of the quantum Boltzmann equation by Toeffer and Wong (To 79, Wo 79ii) gave preliminary results which are consistent with this estimated time scale.
C Static Solution of the Master Equation

In the previous section, the concept of thermal equilibrium has been introduced for a nuclear many-body system in terms of stationary occupation numbers. This is related to the H-theorem which plays a central role in the study of irreversible dissipative phenomena. It is desirable to discover the features characteristic of the solutions of the master equation at thermal equilibrium.

From the H-theorem, the necessary and sufficient condition for (global) thermal equilibrium is that for any combination of four single-particle levels \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \)

\[
D(\lambda_1, \lambda_2, \lambda_3, \lambda_4) [(1 - n_{\lambda_1})(1 - n_{\lambda_2})n_{\lambda_3}n_{\lambda_4} - n_{\lambda_1}n_{\lambda_2}(1 - n_{\lambda_3})(1 - n_{\lambda_4})] = 0 \quad (III-13)
\]

where in general, the constraint of energy conservation is expressed by \( D \), which may be any appropriate distribution function peaked at

\[
\epsilon_\lambda_1 + \epsilon_\lambda_2 - \epsilon_\lambda_3 - \epsilon_\lambda_4 = 0 \quad \text{and vanishingly small outside a certain neighborhood. In particular, it can be a delta-function or a Lorentzian distribution function as the case may be.}
\]

1. Fermi-Dirac Distribution as the Static Solution of an Idealized Case

Consider the case where a delta-function is used for \( D \) in eq. (III-13). This is the idealized situation in which the single-particle states are taken to have infinite lifetimes. Eq. (III-13) is then equivalent to the following functional relation:
under the constraint
\[ \epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4 = 0, \tag{III-15} \]
where the abbreviated subscripts 1, 2, 3 and 4 have been used to stand for \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \), respectively.

It is easy to see that a Fermi-Dirac distribution
\[
n(\epsilon) = \left\{ 1 + \exp\left[\frac{(\epsilon - \epsilon_F)}{T}\right] \right\}^{-1} \tag{III-16}
\]
characterized by a temperature \( T \) and a Fermi energy \( \epsilon_F \), will satisfy eqs. (III-14) and (III-15). However, it is also of interest to study the converse in the hope of establishing the uniqueness of this type of solution to the number equation.

Apparently, when the single-particle levels are well separated from each other, it is possible to have local deviations from the Fermi-Dirac distribution while eqs. (III-14) and (III-15) are still satisfied. But there are also cases of interest such as the level crossing during a heavy-ion reaction, in which it can be assumed that the spacing of the single-particle levels is sufficiently small and regular that the discrete set \( \{n(\epsilon_{\lambda})\} \) can be replaced by a smooth distribution function \( n(\epsilon) \) \( (0 < n(\epsilon) < 1) \). Under this condition, an auxiliary distribution function \( y(\epsilon) \) can be introduced, so that
\[
n(\epsilon) = \left[ 1 + y(\epsilon) \right]^{-1} \tag{III-17}
\]
It is clear that $0 < y(\epsilon) < \infty$. Then eqs. (III-14) and (III-15) can be re-expressed by a single functional equation

$$y(\epsilon_1) y(\epsilon_2) = y(\epsilon_3) y(\epsilon_1 + \epsilon_2 - \epsilon_3)$$

which is satisfied for independent values of $\epsilon_1$, $\epsilon_2$, and $\epsilon_3$, each over a domain of non-zero measure. For any arbitrary, but fixed value of $\epsilon_3$, partial differentiations of eq. (III-18) with respect to $\epsilon_1$ and $\epsilon_2$ lead to the following:

$$\frac{y'(\epsilon_1)}{y(\epsilon_1)} = \frac{y'(\epsilon_2)}{y(\epsilon_2)}$$

Since eq. (III-19) holds for any $\epsilon_1$ and $\epsilon_2$, it follows that it must be a constant, which, in turn, implies that $y$ must be an exponential function. Accordingly, the original function $n(\epsilon)$ must have the functional form:

$$n(\epsilon) = \left\{ 1 + \exp[\beta(\epsilon - \mu)] \right\}^{-1}$$

for some constants $\beta$ and $\mu$. One notes that if $\mu$ is associated with the Fermi energy, then $\beta$, the inverse temperature, can only be positive in order to have a reasonable shape of the distribution. Hence, the uniqueness of the Fermi-Dirac distribution as the static solution for a system at thermal equilibrium is established for this case.

2. Static Solution for a Discrete System and Non-Zero Single-Particle Widths

The arguments in the previous paragraphs leading to the uniqueness of the Fermi-Dirac distribution as the static solution are based on two
assumptions, namely, the replacement of the discrete distribution function by a continuous function and infinite lifetimes (zero widths) of the single-particle states. With regard to the first assumption, the restriction should be removed whenever possible (though justifiable for heavy systems). The second assumption, on the other hand, needs to be modified if one wants to be self-consistent.

The finite lifetimes of particles and holes are a result of particle collisions. The master equation can be rewritten in the following form:

\[
\frac{dn_\lambda}{dt} = - P^{(-)}_\lambda n_\lambda + P^{(+)}_\lambda (1 - n_\lambda), \quad (III-21)
\]

where \( P^{(-)}_\lambda \) stands for the rate of outgoing probability and \( P^{(+)}_\lambda \), for the rate of incoming probability. They are given by

\[
P^{(-)}_\lambda = \frac{\pi}{\hbar} \sum_{\lambda_2, \lambda_3, \lambda_4} \delta(\epsilon_\lambda + \epsilon_{\lambda_2} - \epsilon_{\lambda_3} - \epsilon_{\lambda_4}) n_{\lambda_2} (1 - n_{\lambda_3}) (1 - n_{\lambda_4}) \left| \langle \lambda \lambda_2 | v' | \lambda_4 \lambda_3 - \lambda_3 \lambda_4 \rangle \right|^2 \quad (III-22a)
\]

and

\[
P^{(+)}_\lambda = \frac{\pi}{\hbar} \sum_{\lambda_2, \lambda_3, \lambda_4} \delta(\epsilon_\lambda + \epsilon_{\lambda_2} - \epsilon_{\lambda_3} - \epsilon_{\lambda_4}) (1 - n_{\lambda_2}) n_{\lambda_3} n_{\lambda_4} \left| \langle \lambda \lambda_2 | v' | \lambda_4 \lambda_3 - \lambda_3 \lambda_4 \rangle \right|^2 \quad (III-22b)
\]

To obtain an approximate time dependence of the occupation numbers, one assumes that the rates \( P^{(-)}_\lambda \) and \( P^{(+)}_\lambda \) are independent of time. In that case, the occupation number of a single-particle state is approximately given
by:

\[ n_\lambda(t) = n_\lambda(t_0) e^{-\frac{\Gamma_\lambda(t-t_0)}{\hbar}} + \frac{\Gamma_\lambda^{(+)}}{\Gamma_\lambda} \left[ 1 - e^{-\frac{\Gamma_\lambda(t-t_0)}{\hbar}} \right] \]  

(III-23)

where

\[ \Gamma_\lambda = \Gamma_\lambda^{(+)} + \Gamma_\lambda^{(-)} \]  

(III-23a)

and

\[ \Gamma_\lambda^{(\pm)} = \hbar p_\lambda^{(\pm)} \]  

(III-23b)

Accordingly, for a short enough duration for which the second term on the right side of eq. (III-23) is very small, the occupation number decreases exponentially with a decay constant of \( \hbar/\Gamma_\lambda \). Similarly, the degree of emptiness of a single-particle state \((1-n_\lambda)\), which appears in the Green's function \( g^> \), depends on time according to

\[ 1 - n_\lambda(t) = \left[ 1 - n_\lambda(t_0) \right] e^{-\frac{\Gamma_\lambda(t-t_0)}{\hbar}} + \frac{\Gamma_\lambda^{(-)}}{\Gamma_\lambda} \left[ 1 - e^{-\frac{\Gamma_\lambda(t-t_0)}{\hbar}} \right] \]  

(III-24)

Again, for a sufficiently short duration, the degree of emptiness decreases in time with a decay constant \( \hbar/\Gamma_\lambda \).

To provide a correction to the master equation, one is well advised to modify the assumption of eq. (II-73) by including the time dependence of \( n_\lambda \) given in eqs. (III-23) and (III-25). But an inclusion of all terms will turn the problem intractable. Since for intervals where \((t-t_0)\) is small, the second terms on the right side of eqs. (III-23) and (III-25) are small. Hence
the Green's function for the holes is approximately

\[- i g < ( x_t; x_{t'} ) \approx \sum_{\lambda} n_{\lambda}(t) e^{-\Gamma_{\lambda}(t'-t)/2\hbar} \psi_{\lambda}(x_t) \psi^*_{\lambda}(x'_{t'})\]

(III-25a)

and the complementary Green's function for the particles is

\[i g > ( x_t; x_{t'} ) \sum_{\lambda} \sum_{\lambda'} [1- n_{\lambda}(t)] e^{-\Gamma_{\lambda}(t'-t)/2\hbar} \psi_{\lambda}(x_t) \psi^*_{\lambda'(x'_{t'})}\]

(III-25b)

When eqs. (III-25a) and (III-25b) are substituted, one obtains a master equation which is the same as eq. (II-80), except that the delta-function is now modified and replaced by a Lorentzian distribution.

One can now return to the problem of the static equilibrium of a discrete system and \( \Gamma_{\lambda} \approx 0 \). In this case, the time-dependent occupation number is

\[n_{\lambda} = \frac{\Gamma^{(+)}_{\lambda}}{\Gamma^{(+)}_{\lambda} + \Gamma^{(-)}_{\lambda}}\]

(III-26)

where

\[\Gamma^{(-)}_{\lambda} = \pi \sum_{\lambda' \lambda_2 \lambda_3 \lambda_4} D(\epsilon + \epsilon_{\lambda_2} - \epsilon_{\lambda_3} - \epsilon_{\lambda_4}) n_{\lambda_2} (1-n_{\lambda_3}) (1-n_{\lambda_4}) \left| \langle \lambda \lambda_2 | v' | \lambda_3 \lambda_4 \rangle \right|^2\]

(III-26a)

and

\[\Gamma^{(+)}_{\lambda} = \pi \sum_{\lambda' \lambda_2 \lambda_3 \lambda_4} D(\epsilon + \epsilon_{\lambda_2} - \epsilon_{\lambda_3} - \epsilon_{\lambda_4}) (1-n_{\lambda_2}) n_{\lambda_3} n_{\lambda_4} \left| \langle \lambda \lambda_2 | v' | \lambda_3 \lambda_4 \rangle \right|^2\]

(III-26b)

with
Thus, knowing the energies of the single-particle states and the matrix elements, one can iterate eqs. (III-26) to obtain the occupation numbers, using a Lagrangian multiplier to conserve the total particle number during the iteration. As the widths depend on the occupation numbers in a non-linear way, the resultant solution should depend on the initial distribution assumed. The solutions can therefore be classified by the parameters which specify the initial distribution. For example, one can use the parameter of a Fermi-Dirac temperature to characterize both the initial distribution and the resultant distribution after iterations. The case of zero temperature ground state occupation number distribution has been studied by Orland and Schaeffer (Or 79). Other more general cases, however, have not been investigated.

Before closing this section, one should note in passing that the widths as one has discussed in this static thermal equilibrium are due to particle collisions. In the general case, when there is also dynamical motion of the mean field. The latter width, which can be called the dynamical width $\Gamma^{(d)}_\lambda$ is approximately given by

$$\Gamma^{(d)}_\lambda \approx \langle \psi_\lambda | (\hbar m_{\text{HF}} - \epsilon_\lambda)^2 \psi_\lambda \rangle^{\frac{1}{2}}.$$  

(III-27)

Thus, in the general case, the total width $\Gamma_\lambda$ is

$$\Gamma_\lambda = \Gamma^{(d)}_\lambda + \Gamma^{(+)}_\lambda + \Gamma^{(-)}_\lambda.$$  

(III-28)
In the case of using a Lorentzian form of distribution, it in principle is defined with respect to this total width.
The usefulness of the concept of entropy is illustrated by the H-theorem which supplies the necessary and sufficient condition under which a finite many-fermion system achieves global thermal equilibrium. In the previous section, the Fermi-Dirac distribution has been shown to be an asymptotic solution to the master equation (for large times). However, the important questions of non-equilibrium situations and the approach to equilibrium configurations are dynamical in nature and can only be settled by the time-dependent solutions of the ETDHF equations.

The set of occupation number equations are coupled to the modified TDHF equations through the two-body matrix elements. They in principle run over the entire set of occupation numbers of all single-particle states which span the Hilbert space. Apparently, the construction of rigorous solutions to the full dynamical problem is a highly non-trivial task. Nevertheless, many important features of the underlying dynamics can be well illustrated by means of simple models which yield analytic solutions.

In this section, the case of four isolated states is considered. The evolution is followed for some time interval by means of an explicit construction of the analytic solution to the master equation.

1. Generalized Occupation Numbers

One now considers a coupled set of equations for $n_\lambda$, where $\lambda = 1, 2, 3$ and 4. Because of the symmetry of the factor containing the occupation numbers in the master equation, it is convenient to introduce
generalized occupation numbers $S$ (sum), $F$, $D_1$ (difference) and $D_2$ (difference), related to the following transformation:

$$
\begin{bmatrix}
S \\
F \\
D_1 \\
D_2
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
n_1 \\
n_2 \\
n_3 \\
n_4
\end{bmatrix}
$$

(III-29)

from which, it is easy to verify that

$$
\frac{dS}{dt} = \frac{dD_1}{dt} = \frac{dD_2}{dt} = 0
$$

(III-30)

Hence, the generalized occupation numbers $S$, $D_1$, and $D_2$ are constants and the original problem is reduced to that of solving the following:

$$
\frac{dF(t)}{dt} = -\frac{\pi}{2\hbar} |q_1| |q_2| |q_3| |q_4|^2 D(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) w(F, S, D_1, D_2)
$$

(III-31)

where $w$ is a polynomial, cubic in $F$,

$$
w(F, S, D_1, D_2) = F \left[ F^2 + 4S \left(1 - \frac{S}{4}\right) - 2(D_1^2 + D_2^2) \right] - 4 \left(1 - \frac{S}{2}\right) (D_1^2 - D_2^2).
$$

(III-31a)

For simplicity, a constant $a$ (which may be complex) and a real constant $b$ are introduced, so that:

$$
w(F) = F(F + a^2) + b
$$

(III-32)

where

$$
a^2 = 4S \left(1 - \frac{S}{4}\right) - 2(D_1^2 + D_2^2)
$$

(III-32a)
and

\[ b = -4 \left( 1 - S/2 \right) \left( D_1^2 - D_2^2 \right) \quad \text{(III-32b)} \]

To convert the relation (III-31) into a proper equation of motion for \( F \), one can consider the time \( t = t_0 \) when the "levels" \( E_A = \epsilon_1 + \epsilon_2 \) and \( E_B = \epsilon_3 + \epsilon_4 \) cross in the sense

\[ \epsilon_1(t_0) + \epsilon_2(t_0) - \epsilon_3(t_0) - \epsilon_4(t_0) = 0 \quad \text{(III-33)} \]

and some interval around \( t_0 \). If a delta-function is used for \( D \) in eq. (III-31), then it is necessary only to evaluate the remainder of the expression at \( t_0 \). When \( D \) is some general distribution peaked at \( t_0 \), one makes the assumption that \( E_A \) and \( E_B \) are linear functions of time near \( t_0 \) and that the norm of the residual interaction matrix element varies slowly enough so that it can be taken to be a constant. As long as the distribution is sharply peaked at \( t_0 \), this assumption is actually very mild, since it can always be realized for a small enough interval around \( t_0 \). The equation for \( F \) becomes:

\[ \frac{dF(t)}{dt} = -\frac{\pi}{4} G^2 \left[ F \left( F^2 + a^2 \right) + b \right] \delta(t - t_0) \quad \text{(III-34)} \]

where the delta-function of energy has been converted into that of time and the constant \( G^2 \) has been introduced:

\[ G^2 = \frac{2}{\hbar} \left| \langle 12 | v' | 43-34 \rangle \right|^2 / \left| \frac{d}{dt} (E_A - E_B) \right| \quad \text{(III-35)} \]
where all terms on the right-hand side of the equation is evaluated at \( t_0 \).

With \( G^2 \) independent of time and \( w \) a cubic polynomial of \( F \), eq. (III-34) can be solved analytically. One can discuss the types of solutions for level crossing of three different kinds characterized by the different (but constant) values of \( D_1 \) and \( D_2 \). A level crossing is said to be of the first kind when the differences \( D_1 \) and \( D_2 \) are zero. This is the case when \( n_1 - n_2 = n_3 - n_4 = 0 \). The solution for this case is particularly simple, but the physical situation it represents is of interest. A level crossing is said to be the second kind when either the differences obey \( D_1 = \pm D_2 \neq 0 \), or the total occupation number \( S = 2 \). A level crossing is said to be of the third kind when the initial occupation numbers are completely arbitrary. This is the most general case for a four level system and the solution is slightly more complicated than the other two cases.

2. Level Crossing of the First Kind (\( D_1 = D_2 = 0 \))

The level crossing of the first kind is the special case in which initially \( n_1 = n_2 \) and \( n_3 = n_4 \), from which, eq. (III-34) is simplified to become

\[
\frac{dF(t)}{dt} = -\frac{\pi}{4} G^2 F (F^2 + a^2) \xi(t - t_0),
\]

(III-36)

where \( a^2 \) assumes the special value

\[
a^2 = 4S(1 - S/4),
\]

(III-36a)

and \( a^2 \geq 0 \). The solution to eq. (III-36) is readily obtained by direct integration:
\[ n_1(t) = n_2(t) = \frac{1}{2} \left[ (n_{10} + n_{30}) + (n_{10} - n_{30}) f(t) \right] \]  
(III-37a)

and

\[ n_3(t) = n_4(t) = \frac{1}{2} \left[ (n_{10} + n_{30}) - (n_{10} - n_{30}) f(t) \right] \]  
(III-37b)

where \( n_{10} \) are the initial values of \( n_i \) before the levels cross. The time dependence is governed by the function \( f(t) \),

\[ f(t) = \left\{ \left[ 1 + \left( \frac{F_o}{a} \right)^2 \right] \exp \left[ \frac{\pi}{4} a^2 G^2 \theta(t-t_0) \right] - \left( \frac{F_o}{a} \right)^2 \right\}^{-\frac{1}{2}} \]  
(III-38)

where

\[ F_o = n_{10} + n_{20} - n_{30} - n_{40} \]  
(III-39)

and \( \theta \) is the usual step-function.

Solutions given by eqs. (III-37a)-(III-39) represent a physically simple but interesting situation. The jump-discontinuity of the step-function leads to redistributions of the occupation numbers after the levels cross at \( t = t_0 \). This kind of level crossing has distinct features. If the rate of change of the energy difference \( |d(E_A - E_B)/dt| \) is large, or if the matrix element of \( v' \) is small, then \( G^2 \) is small. Hence the occupation numbers are essentially unchanged for \( t > t_0 \). This situation is depicted in figure 3.1a. On the other hand, if the reverse is true, that is, if \( |d(E_A - E_B)/dt| \) is small, or if the matrix element of \( v' \) is large, then \( G^2 \) will be large. In this limit, the solutions (III-37a)-(III-37b) give \( n_1 \sim n_3 \sim \frac{1}{2} (n_{10} + n_{30}) \), as illustrated schematically by figure 3.1b. Such a redistribution can be understood. When \( G^2 \)}
is large, the master equation at $t = t^+$ becomes

$$n_1 n_2 (1 - n_3) (1 - n_4) - (1 - n_1) (1 - n_2) n_3 n_4 \to 0$$

which admits $n_1 = n_2 = n_3 = n_4$ as the solution when there are the initial constraints $n_1 = n_2$ and $n_3 = n_4$.

It is of interest to compare and contrast the present result with the Landau-Zener level crossing formula. There, one can also consider two-particle states represented by $E_A$ and $E_B$ and a residual interaction such that $G^2$ defined in eq. (III-35) is the same as the parameter $G^2$ in the Landau-Zener formula (for example, see figure 34 of Hi 53). The present result is the same as that of the Landau-Zener formula for small values of $G^2$ (when either the collective motion is slow or when the matrix element is large), only the lowest level is occupied in a Landau-Zener level crossing in contradistinction with the present case where the occupation probabilities are evenly distributed between the upper and lower levels after the levels cross. Such a difference arises from the special ways of including correlations in the present problem. Specifically, to arrive at the Markovian form of equations of motion, one evaluates, in essence, the two-body correlation anew at every instant of time by propagating backward in time, allowing a collision to take place and then propagating forward in time. Once such a correlation is obtained, at the present time, say, the past collision history is "erased" and the correlation in the future time has to be reworked again. In such a picture, time-reversal invariance is broken since the microscopic information concerning higher-order correlations is neglected and in the Markovian
approximation details of the past history are not kept. In contrast, in the Landau-Zener case, all the correlations from the past to the present and to the future are kept intact. There is no "erasing of memory" so that a coherent propagation of correlation is maintained. It is clear that for a system with only a few particles for which the coherent phase information can be kept throughout, the Landau-Zener formula is more appropriate. However, for a system with a large number of particles, such as a heavy nucleus, a statistical treatment of the present kind will often be more suitable and manageable. In fact, this provides an example of how, irrespective of the initial distribution of the occupation numbers, this kind of redistribution eventually leads to a Fermi-Dirac distribution characterized by a temperature.

3. Level Crossing of the Second Kind

The simplification of the solution in the previous section can be traced back to the fact that in the polynomial \( w \) given in eq. (III-32), one has specialized to the case where

\[
b = -4 \left( 1 - S/2 \right) \left( D_1^2 - D_2^2 \right) = 0.
\]  

(III-40)

For a level crossing of the second type, either \( n_1 - n_2 = \pm (n_3 - n_4) = 0 \), or \( S = 2 \). Eq. (III-40) holds again. Thus eq. (III-31) leads to the same time-dependent function \( f(t) \). When the initial values of the occupation numbers are taken into account, one obtains the general solution for a level crossing of the second kind:
where \( f(t) \) is given by eq. (III-38) except that \( \alpha^2 \) is now given by the general expression of eq. (III-32a).

Similar to the previous kind of level crossing, in the limiting case of very small \( G^2 \), the occupation numbers remain practically unchanged after level crossing. In the other extreme of large \( G^2 \), however, they are redistributed according to the following approximate way:

\[
\begin{align*}
n_1 & \sim \frac{1}{4} (n_{10} + n_{20} + n_{30} + n_{40}) + \frac{1}{3} (n_{10} - n_{20}), \\
n_2 & \sim \frac{1}{3} (n_{10} + n_{20} + n_{30} + n_{40}) - \frac{1}{3} (n_{10} - n_{20}), \\
n_3 & \sim \frac{1}{2} (n_{10} + n_{20} + n_{30} + n_{40}) + \frac{1}{2} (n_{30} - n_{40})
\end{align*}
\]

and

\[
\begin{align*}
n_4 & \sim \frac{1}{4} (n_{10} + n_{20} + n_{30} + n_{40}) - \frac{1}{3} (n_{30} - n_{40})
\end{align*}
\]

Thus, when the collective motion is slow or when the interaction matrix element is large, the occupation numbers are redistributed in such a way that the sum and difference of \( n_1 \) and \( n_2 \) become equal to the corresponding sum and difference of \( n_3 \) and \( n_4 \).
4. Level Crossing of the Third Kind

The level crossing of the third kind is the general case for which the initial values of \( n_1, n_2, n_3 \), and \( n_4 \) are completely arbitrary. Even in this case, it is remarkable that eq. (III-31) admits an analytic solution in a simple form which contains much of the structure of the previous cases.

To obtain this general solution, one notes that the polynomial \( w \) can be written in the factorized form:

\[
w(x) = (x - y) \left[ x^2 + yx + (a^2 + y^2) \right], \tag{III-43}
\]

where \( y \), without loss of generality, is a real root of \( w(x) = 0 \) and satisfies the equation

\[
w(y) = y(\gamma^2 + a^2) + b = 0. \tag{III-44}
\]

Applying the method of partial fraction, one arrives at the following solution after some lengthy, but otherwise straightforward, algebraic manipulations:

\[
n_1(t) = \bar{n} + \frac{1}{2} D_1 + \frac{1}{2} F_0 \alpha(t) - \frac{i}{2} \gamma \beta(t), \tag{III-45a}
\]

\[
n_2(t) = \bar{n} - \frac{1}{2} D_1 + \frac{1}{2} F_0 \alpha(t) - \frac{i}{2} \gamma \beta(t), \tag{III-45b}
\]

\[
n_3(t) = \bar{n} + \frac{1}{2} D_1 - \frac{1}{2} F_0 \alpha(t) + \frac{i}{2} \gamma \beta(t), \tag{III-45c}
\]

and

\[
n_4(t) = \bar{n} - \frac{1}{2} D_2 - \frac{1}{2} F_0 \alpha(t) + \frac{i}{2} \gamma \beta(t), \tag{III-45d}
\]

where

\[
\bar{n} = \frac{1}{2} \left( n_{10} + n_{20} + n_{30} + n_{40} \right). \tag{III-46}
\]
\[\alpha(t) = \left\{ f^2 \left[ 1 + \frac{3\gamma^2}{a^2} \left( 2 + 3 \left( \frac{M - \gamma}{a} \right)^2 \right) \right] \right\}^{\frac{1}{2}} \text{ (III-47)}\]

and

\[\beta(t) = \alpha(t) + \frac{3}{2} \left[ \frac{M - \gamma}{a} \right]^2 f^2(t) - 1 \text{ (III-48)}\]

with

\[F = f(t) \text{ and } F = \gamma \left[ \frac{M - \gamma}{a} \right]^2 \text{ (III-49)}\]

\[f(t) = \sqrt{\exp \left[ \frac{\pi}{2} G^2 (a^2 + 3\gamma^2) \theta(t - t_o) - \gamma(\eta(F) - \eta(F_o)) \right] - \left( \frac{M - \gamma}{a} \right)^2} \text{ (III-50)}\]

and

\[\eta(x) = \frac{3}{\sqrt{4a^2 + 3\gamma^2}} \tan^{-1} \left[ \frac{\sqrt{4a^2 + 3\gamma^2}}{2x + \gamma} \right] \text{ (III-51)}\]

Thus, functions \(\alpha(t)\) and \(\beta(t)\) which appear in the solutions (III-45a)-(III-45d) can be written analytically in terms of the function \(f(t)\). The latter quantity, however, contains a function of the unknown

\[F(t) = n_1(t) + n_2(t) - n_3(t) - n_4(t) \text{ .}\]

The final result needs to be obtained in an iterative manner when one uses eqs. (III-45)-(III-51).
As a limiting case, when $\gamma = 0$ (i.e. when $b = 0$), $f$ is simplified to the same expression as that given by eq. (III-38) and $\alpha = f$, thus recovering the solutions (III-37a), (III-37b) and (III-41) for the level crossing of the first and second kinds.

The step-function $\theta(t - t_o)$ in all these solutions results from the integration of the delta-function in time. When the finite lifetimes of the single-particle states are taken into account, then the energies $\epsilon_i$ will then take complex values and the delta-function is replaced by a Lorentzian distribution function. In that case, the solutions to the master equation are only modified by the following replacement:

$$
\theta(t - t_o) \rightarrow \frac{1}{\pi} \tan^{-1} \left\{ \frac{2 |d(E_A - E_B)/dt| (t - t_o)/\Gamma_{1234}}{(t - t_o)/\Gamma_{1234}} \right\} + \frac{1}{2}
$$

(III-52)

where $\Gamma_{1234}$ is the sum of the widths of four single-particle states labelled by 1, 2, 3 and 4.
Figure 3.1  Level crossing of the first kind.
(a) The case of small $G^2$. (b) The case of large $G^2$. 
Fig. 3.1  LEVEL CROSSING OF THE FIRST KIND

(a)  \[ n_{30} = n_{40} \]
\[ n_{10} = n_{20} \]
\[ \varepsilon_1 + \varepsilon_2 \]
\[ n_1 \approx n_{10} \]
\[ n_3 \approx n_{30} \]
\[ \varepsilon_3 + \varepsilon_4 \]
\[ t_0 \]

(b)  \[ n_{30} = n_{40} \]
\[ n_{10} = n_{20} \]
\[ \varepsilon_1 + \varepsilon_2 \]
\[ n_1 \approx \frac{1}{2} (n_{10} + n_{30}) \]
\[ n_3 \approx \frac{1}{2} (n_{10} + n_{30}) \]
\[ \varepsilon_3 + \varepsilon_4 \]
\[ t_0 \]
The analytic solutions discussed in the previous section illustrate, from a detailed microscopic standpoint, how a non-equilibrium system approaches equilibrium. The key point to this dynamical process lies in the (two-body) particle-particle collisions. Though a collision is subject to the restriction of energy conservation, the condition of the exact equality of \( \epsilon_{\lambda_1} + \epsilon_{\lambda_2} \) and \( \epsilon_{\lambda_3} + \epsilon_{\lambda_4} \) may be relaxed slightly to allow for single-particle widths. In order to have the occurrence of a "collision", a near equality of \( \epsilon_{\lambda_1} + \epsilon_{\lambda_2} \) and \( \epsilon_{\lambda_3} + \epsilon_{\lambda_4} \) is achieved by the dynamical motion of the system, thereby a change in the mean field and the kinetic energy of the particles bring about a change in the values of \( \epsilon_{\lambda} \). The change of \( \epsilon_{\lambda} \) as a function of time then gives rise to an accidental near equality of the levels \( E_A \) (\( \epsilon_{\lambda_1} + \epsilon_{\lambda_2} \)) and \( E_B \) (\( \epsilon_{\lambda_3} + \epsilon_{\lambda_4} \)), an event which we call level crossing, and a "collision" then takes place to rearrange the occupation numbers of the four single-particle states. It is in this context that level crossings, particularly from those involving states near the Fermi surface, are very important in the discussion of the approach to thermal equilibrium.

It is instructive to devise a scenario for the occurrence or non-occurrence of thermalization, based on the insight gained from the analytic solutions. For simplicity, we can consider level crossings of the first kind and take states \( \lambda_1 \) and \( \lambda_2 \) (and similarly \( \lambda_3 \) and \( \lambda_4 \)) to belong to the same single-particle level and the same time-reversal doublet. In this case, levels \( E_A \) and \( E_B \) will cross when the states \( \lambda_1 \) and \( \lambda_3 \) have a level crossing. The matrix element in eq. (III-35) has the magnitude of the pairing
matrix element. This matrix element is so much larger than the average matrix elements of the other kinds that one may ignore the level crossing of the third kind in a qualitative description of the thermalization process. Level crossings of the second kind have qualitative features much like those of the first kind and need not be discussed separately.

First, one has the following scenario for a system which does not acquire thermal equilibrium. This is depicted in figures 3.2. One can envisage that at t = 0 the single-particle levels $\epsilon_\lambda$ of a system are arranged as in figure 3.2a with the occupation numbers indicated by the lengths of the solid lines. As time proceeds, the change of the mean field, as well as the dynamics, will result in a shift in the positions of the levels. Such a shift is indicated in figures 3.2 by lines joining the levels for a few of those near the Fermi surface. For example, there can be a level crossing on top of the Fermi sea, as illustrated in figure 3.2b. We can envisage a situation in which the dynamics proceeds in such a rapid manner that the parameter $G^2$ in eq. (III-35) is very small so that after level crossing (of the first kind) has taken place, the occupation numbers remain unchanged. The system will reach the configuration at figure 3.2d at a later time. As unoccupied states now lie below occupied states, this is clearly not a thermalized system.

Thermalization takes place when the dynamical process is sufficiently slow or when the occurrence of collisions is frequent. A full description of how the dynamics is affected by particle collisions can be best obtained from realistic ETDHF calculations. But one can also understand thermalization from simple examples. The scenario of a system in a slow motion is depicted
in figures 3.3. Here, when a level crossing on top of the Fermi sea occurs as shown in figure 3.3b and the motion is slow, a redistribution of the occupation probabilities takes place to divide them evenly between the two levels (figure 3.3c). If further level crossings take place as in figure 3.3d and the motion remains slow, then the occupation probabilities are further redistributed to give a smooth tail for the occupation probability distribution. It is this kind of redistribution which eventually leads to a Fermi-Dirac distribution of the single-particle occupation numbers characterized by a temperature.

It is important to emphasize again that all these features of thermal equilibrium are consequences of the ETDHF equations, in particular, the master equation.
Figure 3.2  A non-equilibrium system which has not reached thermal equilibrium. The lengths of the solid lines are proportional to the single-particle occupation numbers. The dotted lines represent the unoccupied states which are initially above a sharp Fermi surface. The positions of a few levels (of both occupied and unoccupied states) near the Fermi surface are followed for several time steps.

Figure 3.3  A non-equilibrium system approaching thermal equilibrium. The states near the Fermi surface share occupation numbers among themselves as level crossings take place. This, after a few collisions, leads to a "tail" in the occupation number distribution.
Fig. 3.2  A NONEQUILIBRIUM SYSTEM WHICH HAS NOT ACHIEVED THERMAL EQUILIBRIUM

(a)  (b)  (c)  (d)

\[ \varepsilon_\lambda \]

\[ n_\lambda \]

\[ t_a > t_b > t_c > t_d \]
Fig. 3.3  APPROACH TO THERMAL EQUILIBRIUM

(a)  (b)  (c)  (d)  (e)

\[ t_Q > t_c > t_d > t_e \]
Macroscopic Equations and Conservation Theorems

It is well known that macroscopic equations can be systematically constructed from the microscopic equations of Green's functions (Ma-c 59, Ka-a 76) by taking appropriate moments and limits. These equations describe the time evolution of macroscopic variables and are useful for the formulation of macroscopic dynamical theories.

In the ETDHF approximation, the associated macroscopic equations - the equation of continuity, the Euler-like equation for momentum flux density and the equation of energy density - contain terms which are derived from the collision matrix and hence reflect how the microscopic and macroscopic degrees of freedom in the system are coupled via the mechanism of particle-particle collisions.

This macroscopic point view can be taken at two levels. On one hand, one can take the formal, non-Markovian equations (II-48)-(II-49) in terms of Green's functions as the starting point, from which the macroscopic equations and conservation theorems can be arrived at. Such procedure is very general and can be carried out in an analogous way to study the macroscopic equations of other Green's functions at higher levels. On the other hand, one can also consider the modified TDHF equations (II-79) and the master equation (II-80) which have been obtained from a Markovian reduction and the slow mean-field and diagonal approximations. Following the method of Madelung (Ma-a 26), one arrives at macroscopic equations in a hydrodynamical form. This explicitly demonstrates how one can make a transition from a microscopic to a macroscopic description of the dynamics.
In the remainder of this section, both of these aspects will be explored. The formal aspect is examined so that the problem can be viewed from a very general level. But the Madelung analysis is also carried out to pave way for a theoretical foundation for the second part of this thesis.

1. Continuity Equation and Total Particle Number Conservation

The (number) density $\rho$ and the (number) current density field $\mathbf{j}$ of a many-fermion system can be defined in a representation-independent manner in terms of the one-body Green's function $g^<$:

$$p(x, t) = -i \lim_{x' \to x} g^<(xt; x't^+)$$  \hspace{1cm} (III-53)

and

$$\mathbf{j}(x, t) = -\frac{\hbar}{2m} \lim_{x' \to x} \left[ \nabla - \nabla' \right] g^<(xt; x't^+) .$$  \hspace{1cm} (III-54)

The equation of motion for the density field is obtained in the form of a continuity equation by taking the limit $x' \to x$ in eq. (II-48). From eq. (II-49),

$$\lim_{x' \to x} I(xt; x't^+) = 0 \ ,$$  \hspace{1cm} (III-55)

it follows that

$$\frac{\partial \rho(x, t)}{\partial t} + \nabla \cdot \mathbf{j}(x, t) = 0 .$$  \hspace{1cm} (III-55)

It should be noted that in arriving at eq. (III-55), simple properties of the interaction are assumed.

The total particle number is given as
and the conservation of the total particle number is readily obtained by an integration of eq. (III-55) over all x-space and applying the usual boundary condition that all field variables vanish at infinity:

\[
\frac{dN_{\text{tot}}(t)}{dt} = 0 \quad (\text{III-57})
\]

Eqs. (III-55)-(III-57) are very general, but it should be noted again that they should be checked whenever other approximations have been invoked in an implementation of the ETDHF scheme.

In the Markovian, slow mean-field and diagonal approximations, one starts with eqs. (II-79)-(II-82). The density field in this version of ETDHF can be expressed in terms of the single-particle states and the occupation numbers:

\[
\rho(x, t) = \sum_{\lambda} n_{\lambda} (t) \psi_{\lambda} (xt) \psi_{\lambda}^* (xt) \quad (\text{III-58})
\]

which is clearly a generalization of the usual TDHF density field (Wo 77i, 77ii). The current density field is given as:

\[
\mathbf{J}(x, t) = \sum_{\lambda} n_{\lambda} (t) \mathbf{J}_{\lambda} (xt) \quad (\text{III-59})
\]

where \( \mathbf{J}_{\lambda} \) is the local current density contributed from the state \( \lambda \) and is defined in the usual way:

\[
\mathbf{J}_{\lambda}(x, t) = \frac{\hbar}{2m_i} \left[ \psi_{\lambda}^* (xt) \nabla \psi_{\lambda} (xt) - \nabla \psi_{\lambda}^* (xt) \psi_{\lambda} (xt) \right] \quad (\text{III-60})
\]
From this, one can introduce a derived quantity, the velocity field:

\[ \mathbf{u}(x, t) = \frac{\mathbf{j}(xt)}{\rho(xt)} \]  

(III-61)

The total particle number is now given in an obvious way:

\[ N_{\text{tot}}(t) = \sum_{\lambda} n_{\lambda}(t) \]  

(III-62)

In the presence of only zero-range exchange interactions for the density-dependent part of the effective interaction (Wo 77i), the continuity equation (III-55) remains valid. In fact, it still holds even when the delta-function in the master equation is replaced by any distribution function \( D(\epsilon_1 \epsilon_2; \epsilon_3 \epsilon_4) \) which is symmetrical with respect to the exchanges \( \epsilon_1 \leftrightarrow \epsilon_4 \) and \( \epsilon_2 \leftrightarrow \epsilon_3 \), such as a Lorentzian form. The conservation of the total particle number can be readily recovered after a simple application of the master equation.

For completeness, it should be noted that when there is a non-zero exchange component in the effective interaction, there will be an additional source term on the right side of the continuity equation. But the particle conservation is preserved in this case. Hence the presence of the additional source term associated with the exchange of particles at different locations does not affect the global conservation of the total particle number.

2. Euler Equation and Total Momentum Conservation

The vector equation of the momentum flux density is obtained by taking the limit \( x' \rightarrow x \) after operating \( (\nabla - \nabla') \) onto eq. (II-48). After an application of the general definition (III-54) and some manipulations, the resultant equation appears in the cartesian form as:
\[
\frac{\partial J_1(x, t)}{\partial t} = -\left(\frac{\hbar}{2m}\right)^2 \frac{1}{i} \sum_{j=1}^{3} \nabla_j \lim_{r' \to r} \left[ (\nabla_i - \nabla_i') (\nabla_j - \nabla_j') g^{<}(xt;x't^+) \right]
\]

where the second divergence term on the left side of eq. (III-63) comes from the kinetic energy part and can be directly verified from the bi-orthogonal expansion of eq. (II-55a).

With the usual assumption about the dependence of \(v\) and \(v'\) on \(|\mathbf{r} - \mathbf{r}'|\), the conservation of the total linear momentum can be obtained by an integration of eq. (III-63) over all x-space:

\[
\frac{\partial}{\partial t} \int dx \ J_1(x, t) = 0 , \quad (\text{III-64})
\]

Now, one can consider the version of ETDHF given by eqs. (II-79)-(II-82). One follows Madelung by writing a single-particle wavefunction in a polar form:

\[
\psi_\lambda(x, t) = \varphi_\alpha(\lambda)(\mathbf{r}, t) \exp\left[ \frac{i m}{\hbar} S_\alpha(\lambda)(\mathbf{r}, t) \right] \Xi_\sigma(\lambda)(\xi)
\]

where \(\varphi_\alpha\) and \(\Xi_\sigma\) are the spatial and spin-isospin wavefunctions, respectively. With no loss of generality, the amplitude functions \(\varphi_\alpha\) and \(S_\alpha\) can be chosen to be real.

With this polar form, after some tedious algebraic steps, eq. (III-63) can be expressed in a more transparent way:
\[
\frac{\partial (\rho u_{ij})}{\partial t} + \sum_{j=1}^{3} \nabla_j \left[ \rho u_{ij} + \frac{1}{m} (p^{(q)}_{ij} + p^{(t)}_{ij}) \right] = - \frac{(-i)^2}{m} \int \mathrm{d}x' \nabla_i \nu(x, x') \left[ g^<(x_t; x_t^+) g^<(x' t; x't^+) \right.

\left. - g^<(x_t; x't^+) g^<(x' t; x_t^+) \right]

+ \tilde{F}^{(c)}(x, t) \tag{III-66}
\]

where \( p^{(q)}_{ij} \) and \( p^{(t)}_{ij} \) are the generalized quantum and thermal stress tensors, respectively,

\[
p^{(q)}_{ij}(x, t) = \sum_{\lambda} n_\lambda(t) p^{(q)}_{ij; \lambda}(x_t) \tag{III-67}
\]

and

\[
p^{(t)}_{ij}(x, t) = \sum_{\lambda} n_\lambda(t) p^{(t)}_{ij; \lambda}(x_t) \tag{III-68}
\]

These stress terms are obtained from summations over components due to all single-particle states:

\[
p^{(q)}_{ij; \lambda}(x, t) = \left[ - \frac{\hbar^2}{4m} \nabla_i \nabla_j \phi^2(\vec{r}, t) + \frac{\hbar^2}{m} \nabla_i \phi(\vec{r}, t) \nabla_j \phi(\vec{r}, t) \right] x \Xi_\sigma(\xi) \Xi_\sigma(\xi^*) \tag{III-67a}
\]

and

\[
p^{(t)}_{ij; \lambda}(x, t) = m \phi^2 \left( \nabla_i S_{\alpha} - u_i \right) \left( \nabla_j - u_j \right) \Xi_\sigma \Xi_\sigma^* \tag{III-67b}
\]

When the summations are taken over a finite set of single-particle states, one recovers the Euler equation associated with the TDHF theory (Wo 77i).
The first term on the right side of the Euler equation (III-66) represents the force density arising from the mean field. It has the form of a folding of the two-body force with the uncorrelated part of the two-body distribution function, \( i^2 g_{20} \). As the two-body distribution function gives the probability for finding a particle at position \( x' \) when another particle is at \( x \), such that a force density term has a simple physical interpretation.

The additional force term \( \vec{F}^{(c)} \) is due to the presence of particle collisions and is a new feature absent in the pure TDHF theory. It is related to the collision matrix according to:

\[
\vec{F}(x, t) = \frac{i}{2} \lim_{x' \to x} \left[ (\nabla - \nabla') I(x; x'^+) \right] \tag{III-69}
\]

From the diagonal approximation of the collision matrix, \( \vec{F}^{(c)} \) can be expressed in terms of single-particle wavefunctions

\[
\vec{F}^{(c)}(x, t) = \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \left[ \epsilon_{\lambda_1} + \epsilon_{\lambda_2} - \epsilon_{\lambda_3} - \epsilon_{\lambda_4} \right]^{-1} \eta_{n=0} \left[ \left( 1 - n_{\lambda_1} \right) \left( 1 - n_{\lambda_2} \right) n_{\lambda_3} n_{\lambda_4} \right. \\
\left. - n_{\lambda_1} n_{\lambda_2} \left( 1 - n_{\lambda_3} \right) \left( 1 - n_{\lambda_4} \right) \right] \langle 12 | v' | 43-34 \rangle \\
\int dx'' \ n v'(x, x'') \ \psi_{\lambda_1}(x) \ \psi_{\lambda_2}(x'') \ \psi^*_{\lambda_3}(x''') \ \psi^*_{\lambda_4}(x) \tag{III-70}
\]

This force density term is in fact a folding of the two-body residual force with the correlated part of the two-body Green's function \( i^2 g_{2c} \), with an obvious physical interpretation.
With a conservative two-body force which depends on the relative spatial coordinates, the contributions from the force densities arising from the mean field and the residual interaction vanish after an integration of the Euler equation over all space. This remains true in the Markovian and diagonal approximations for which the conservation of total linear momentum is ensured. The result is valid even when the delta-function in the master equation is replaced by a general distribution \( D(\epsilon_1, \epsilon_2; \epsilon_3, \epsilon_4) \) which is symmetric with respect to the exchanges \( \epsilon_1 \leftrightarrow \epsilon_4 \) and \( \epsilon_2 \leftrightarrow \epsilon_3 \).

3. Energy Density Equation and Total Energy Conservation

The energy density equation of the general, non-Markovian form of ETDHF can be obtained by taking the limit \( x' \rightarrow x \) after operating \( \nabla \cdot \nabla' \) onto eq. (II-48). After some applications of the ETDHF equation and lengthy calculations, one arrives at the following:

\[
\frac{\partial E_{\text{tot}}(x, t)}{\partial t} = -\frac{\hbar}{2m} \nabla \cdot \int dx'' \, v(x, x'') \lim_{x' \rightarrow x} \left[ (\nabla - \nabla') g_{20}(x_t, x''t'; x' t^+, x'' t^+)_{t' = t^+} \right]
\]

\[-v'(x, x'') \lim_{x' \rightarrow x} \left[ (\nabla - \nabla') g_{2c}(x_t, x''t'; x' t^+, x'' t^+)_{t' = t^+} \right]
\]

\[ (\text{III-71}) \]

where the total energy density \( E_{\text{tot}} \) associated with ETDHF is defined as the sum of a kinetic energy density \( T(x, t) \) and an internal energy density \( E_{\text{int}}(x, t) \):

\[ E_{\text{tot}}(x, t) = T(x, t) + E_{\text{int}}(x, t) \]  \[ (\text{III-72}) \]
In terms of the Green's function, the kinetic energy density is defined as

\[ T(x, t) = \frac{\hbar^2}{2m} \lim_{{x' \to x}} \left[ (\nabla - \nabla') g^{<(xt; xt')^+)} \right] . \]  

(III-73)

The internal energy density is due to the potential energy of the mean-field part and a correlated part from the residual interaction:

\[ E_{\text{int}}(x, t) = \frac{1}{2} \int dx' \left[ \psi(x, x') \right] \frac{2}{\pi} g_{20}(xt, x't'; xt^+, x't'^+) \]

\[ + \psi'(x, x') \frac{2}{\pi} g_{2c}(xt, x't'; xt^+, x't'^+) \]  

(III-74)

where the factor \( \frac{1}{2} \) is due to the double counting in performing the folding of the interactions with the corresponding distribution functions.

An integration of eq. (III-71) over all space leads to the conservation law for total energy:

\[ \frac{\partial}{\partial t} \int dx \ E_{\text{tot}}(x, t) = 0 . \]  

(III-75)

If one now applies the polar form of the single-particle wavefunctions to the ETDHF equation in the Markovian and diagonal approximations, eq. (III-74) then appears as the following equation for kinetic energy density:

\[ \frac{\partial}{\partial t} (\rho E_k) + \nabla \cdot (\rho E_k \hat{u}) + \sum_i \nabla [\sum_j (p_{ij}^{(q)} + p_{ij}^{(t)}) u_j + F_i] \]

\[ = - \int dx' \left[ \rho(xt) \rho(x't') u_i(x't') + \frac{\hbar}{m} \Im g^{<(x't'; xt^+)} \nabla_i g^{<(xt; xt'^+)}} \nabla_i \psi(x, x') \right] \]

\[ + P_i(x, t) \]  

(III-76)

where the generalized kinetic energy density is given by
\[ E_k(x, t) = \frac{\hbar^2}{2m} \sum_{\lambda} n_{\lambda}(t) \nabla \psi_{\lambda}(x_t) \cdot \nabla \psi_{\lambda}^*(x_t) / \rho(x_t) \]
\[ = \sum_{\lambda} n_{\lambda} \left[ \frac{1}{2} m \phi_{\lambda}^2 \left( \nabla S_{\lambda} \right)^2 + \frac{\hbar^2}{2m} \left( \nabla \phi_{\lambda} \right)^2 \right] / \rho , \]

(III-76a)

and the generalized heat flux density is given by

\[ F_i(x, t) = \sum_{\lambda} n_{\lambda} \left\{ \left[ \frac{1}{2} m \phi_{\lambda}^2 \left( \nabla S_{\lambda} \right)^2 + \frac{\hbar^2}{2m} \left( \nabla \phi_{\lambda} \right)^2 \right] \left( \nabla_i S_{\lambda} - u_i \right) \right. \]
\[ + \left. \left[ \frac{\hbar^2}{4m} \nabla_i \phi_{\lambda} \phi_{\lambda}^2 + \frac{\hbar^2}{m} \nabla_i \phi_{\lambda} \nabla_j \phi_{\lambda} \right] \left( \nabla_j S_{\lambda} - u_j \right) \right. \]
\[ + \left. \frac{\hbar^2}{2m} \phi_{\lambda}^2 \nabla_i \phi_{\lambda} \nabla_j \phi_{\lambda} \left. \nabla^2 S_{\lambda} \right) \right\} . \]

(III-76b)

Similar to the case of Euler equation, eq. (III-76) is a direct generalization of the corresponding hydrodynamical equation for TDHF (which is described in Wo 77ii) except that the kinetic energy density, heat flux density and stress-tensor terms constitute components from all single-particle states which are weighted by the occupation numbers. Here, there is also an additional term \( P_I \) which is due to the particle collisions. It can be constructed from the collision matrix,

\[ P_I(x, t) \equiv -\frac{\hbar}{2m} \lim_{x' \to x} \left[ (\nabla - \nabla') I(xt; x't^+) \right] . \]

(III-77)

In the Markovian and diagonal approximations, this term can be expressed in terms of single-particle states,
\[ P_I(x, t) = \frac{\hbar^2}{2m_i} \sum \frac{1}{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \left[ \left( 1 - n_{\lambda_1} \right) \left( 1 - n_{\lambda_2} \right) n_{\lambda_3} n_{\lambda_4} - n_{\lambda_1} n_{\lambda_2} \left( 1 - n_{\lambda_3} \right) \left( 1 - n_{\lambda_4} \right) \right]^{-1} \]

\[ < \lambda_1 \lambda_2 \mid v' \mid \lambda_3 \lambda_4 > \]

\[ \int dx' \left[ \nabla \psi_{\lambda_1} (xt) \psi_{\lambda_2} (x't) \psi^{*}_{\lambda_3} (x't) \psi^{*}_{\lambda_4} (xt) - \psi_{\lambda_1} (xt) \psi_{\lambda_2} (x't) \psi^{*}_{\lambda_3} (x't) \psi^{*}_{\lambda_4} (xt) \right] \cdot \nabla v'(x, x') \]

(III-78)

As can be seen, the final form for \( P_I \) is somewhat complicated. Hence it will be of interest in the future numerical work to investigate whether there are dynamical conditions under which this may be approximated in a simple way in terms of local dynamical variables.

The conservation of total energy expressed in eq. (III-75) is an exact relation as far as the general ETDHF equation is concerned. In the Markovian and diagonal approximations, when eq. (III-76) is integrated over all space, the total energy can be shown to be conserved up to the second order in \( v' \), which is certainly consistent with the order of expansion in obtaining ETDHF. A consistent application of the basic ETDHF equations within the Markovian and diagonal approximations leads to higher-order terms which do not vanish identically. These residual terms, presumably small in magnitude (since they contain mainly off-diagonal matrix elements of the residual interaction of higher orders), originate from the Markovian approximation of the collision matrix. They do not affect the main conclusions reached in the present analytic
discussions of the theory. However, in the future implementations of ETDHF in practical calculations, these terms should be further investigated and hopefully removed all together by means of appropriate prescriptions such as the inclusion of the single-particle widths.
CHAPTER IV

FUNDAMENTALS OF NUCLEAR HYDRODYNAMICS
A From Microscopia to Macroscopia

The first part of this dissertation has been devoted to the formulation and study of a microscopic theory of nuclear dynamics. Briefly, starting from the exact hierarchy of Green's functions, an extended time-dependent Hartree-Fock approximation which includes particle-particle collisions due to the residual interaction has been obtained. This formalism not only generalizes the usual mean-field theory of time-dependent Hartree-Fock approximation, but also provides a convenient microscopic framework from which connections between the microscopic and macroscopic degrees of freedom in the dynamics can be made in a transparent manner. For example, the master equation of the ETDHF approximation presents an additional flexibility which is not found in the TDHF approximation. As particle scattering into or from any single-particle state has been incorporated in this scheme, dissipative phenomena can be described from and traced back to a microscopic basis. We have also seen how concepts like entropy and thermal equilibrium can be quantitatively introduced for a finite fermion system.

Up to the present time, most of the TDHF calculations on nuclear reactions have been performed at energies of a few MeV above Coulomb barrier (e.g. Ne 77, 78). Indeed, it has been argued by various workers that the mean-field picture is best applicable to low-energy phenomena, say up to about 10-20 MeV above Coulomb barrier (Bo-b 76, Da 78, St 78). In heavy-ion collisions, the momentum distribution of the nucleons in the collision region is approximately represented by a Fermi bisphere whose centers are separated by the relative momentum determined by the kinetic
energy per projectile nucleon. As the blocking due to the Pauli Principle becomes less prohibitive when the separation of the centers of the Fermi bisphere increases, the fraction of nucleons capable of making two-body collisions and the phase space for the final states increase with bombarding energy. One may expect that the relaxation time decreases with increasing bombarding energy, until the energy is so high that the bisphere separates into two disjoint Fermi spheres. Thereafter, the relaxation time becomes less sensitive to the change of the bombarding energy (Ra 79).

In Chapter III, we have seen how, following the Madelung procedure, macroscopic equations in a hydrodynamical form can be derived from the microscopic ETDHF equations. This strongly suggests the possibility of formulating a macroscopic dynamical theory in which one may forego the use of wavefunctions, but focus attention on a few macroscopic quantities such as the density field, current density (or velocity) field and energy density field. It is clear that in order that the hydrodynamical equations be self-contained, the auxiliary quantities $\rho_{ij}', \, \rho_{ij}^{(t)}$ and $F$ (see eqs. (III-55), (III-66) and (III-76)) must be expressed in terms of the basic local variables $\rho$, $J^\nu$ and $E_k$. There has been extensive theoretical investigations on the various approximations leading to hydrodynamics (e.g. Wo 75, 76, 77i-iii). But the most important physical condition necessary for the validity of nuclear hydrodynamics is that the time for setting up a local equilibrium, which occurs before global equilibrium, must be shorter than the characteristic reaction time scale. In the discussion of the microscopic ETDHF theory, a relaxation time scale has been estimated from the magni-
tude of the level density and interaction matrix elements:

\[ \tau_{\text{relax}} \sim \frac{4000}{A} \text{ fm/c} \]

Indeed, recent experimental studies (Ta-a 79) on the spectra of evaporated neutrons from deep-inelastic-collision fragments indicate that the relaxation time for thermal equilibrium of the compound system is approximately \(10^{-22}\) sec (\(\sim 30\text{ fm/c}\)). The above rough estimate drawn from ETDHF appears to agree well with these experimental results (for heavy systems \(A \sim 100\)).

Model calculations on the approach to equilibrium of an irregular momentum distribution leading to a temperature of 8.5 MeV also gives a relaxation time of about 40 fm/c (To 79, Wo 79iv).

As a summary of these general arguments, one should expect, from the phase space arguments, that nuclear hydrodynamics is best applied for energies between tens of MeV up to hundreds of MeV per nucleon. However, the discussions on level crossings and thermalization strongly suggest that even at very low energies of a few MeV per projectile nucleon, the high level density of heavy systems (say, uranium on uranium) is favorable for frequent particle-particle collisions and hence the validity of a hydrodynamical picture. In the last analysis, it is a direct confrontation of the hydrodynamical predictions with experiments that will give a definitive evaluation of this macroscopic approach.

Among the most important motivations for studying nuclear hydrodynamics, the following are the ones particularly emphasized in the present investigations:
(a) to develop a conceptually simple and computationally manageable model
with which nuclear collisions can be studied in a systematic way and over
an extensive energy range (a few MeV up to hundreds of MeV per
nucleon);

(b) to study the bulk properties and the equation of state of nuclear systems
and their manifestations in heavy-ion collisions;

(c) to provide a theoretical tool with which one can probe the various possi-
bilities of exotic phenomena under unusual conditions, particularly those
associated with the compression of nuclear matter.
B Basic Elements of Nuclear Hydrodynamics

In going from the microscopic to the macroscopic equations, attention is focused on the local quantities, namely, the (number) density field $\rho$, the current density field $\vec{J}$ (or velocity field $\vec{u}$) and the total energy density field $E_{\text{tot}}$ (or thermal energy density field $E_T$). Given an equation of state for the nuclear fluid, the underlying dynamics of the theory is specified by the time evolution of these macroscopic variables, from which all the observable quantities contained in the theory can be calculated.

One can follow the steps adopted by Frohlich (Fr 73, 74) and Wong et al (Wo 74, 75, 76, 77iiii), in which the basic equations transformed from the microscopic quantum equations by the Madelung procedure (see Chapter III and also Ma-a 26) are converted into a set of completely macroscopic, hydrodynamic equations by appropriately parametrizing the auxiliary quantities such as the pressure stress terms. However, from the standpoint of logical continuity, there is an alternative and perhaps more aesthetically satisfying approach. It will be briefly outlined in the following.

We have seen in Chapter II that starting from an exact many-body theory, an extended time-dependent Hartree-Fock approximation can be obtained. The basic equations of ETDHF have been written in a configuration space representation. But they can be expressed in the Wigner space (Wi 36). Next, one can go to a classical transcription (Wo 79ii) by interpreting the Wigner function in the ETDHF approximation as the classical distribution function. The resultant approximate Vlasov-Boltzmann equation forms the basis from which much statistical dynamics can be extracted. In a way very
analogous to that discussed in Chapter III, one can introduce the quantity of local entropy field and prove with the Vlasov-Boltzmann equation that the total time derivative of the local entropy never decreases. It is natural to consider the situation when the local entropy is stationary as the state of local thermal equilibrium. One can further prove that when local equilibrium is attained, the local momentum distribution is characterized by a Fermi-Dirac spherical distribution. The parameters in the distribution are: (1) the local temperature, (2) the local Fermi energy which is related to the local density, and (3) the displacement of the Fermi-Dirac spherical distribution from the origin which is related to the local velocity field. Thus, when there is local equilibrium, the local stress tensor, as obtained from the momentum distribution, depends only on the local temperature, the local density and the local velocity field. Then, by using the well known Enskog-Chapman procedure (Hu 63, Ha-d 71), one can approximate the Vlasov-Boltzmann equation by those of hydrodynamics (viscous). The geneology of nuclear hydrodynamics is summarized in figure 4.1.

1. Basic Equations

The basic equations of nuclear hydrodynamics are the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

(IV-1)

the Navier-Stokes equations,
\[
\frac{\partial (\rho u_i)}{\partial t} + \sum_{ij} \nabla_j \left[ \rho u_i u_j + \frac{1}{m} (p \delta_{ij} + p'_{ij}) \right]
\]

\[
= -\frac{n}{m} \nabla_i \int d^3 \rho \cdot V_L(r, t) \quad (IV-2)
\]

and the thermal energy density equation,

\[
\frac{\partial (\rho E_T)}{\partial t} + \nabla \cdot (\rho E_T \mathbf{u})
\]

\[
= -p_T \nabla \cdot \mathbf{u} + \nabla \cdot (\kappa \nabla T) - \sum_{ij} p'_{ij} \frac{\partial u_i}{\partial x_j} \quad (IV-3)
\]

The continuity equation describes the motion of the density field, while the Navier-Stokes equations are those for the current density. Here, \( m \) is the nucleonic mass and \( p \) is the total pressure term of the nuclear fluid, which can be obtained once the equation of state is given. The dissipative behavior of the system is accounted for by the viscosity stress-tensor \( p'_{ij} \), which is defined as:

\[
p'_{ij} = -\left\{ \eta \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{u} \right\} + \delta_{ij} \zeta \nabla \cdot \mathbf{u} \quad (IV-4)
\]

where \( \eta \) and \( \zeta \) are the transport coefficients of shear and compressional viscosity, respectively. The force-density term on the right side of eq. (IV-2) is due to the long-range part of the nucleon-nucleon interaction which is taken to be the sum of an effective Yukawa nuclear potential and the Coulomb interaction.

Because of technical convenience, the thermal energy density,
instead of the total energy density, has been chosen as a basic variable. In
the thermal energy density equation, \( p_T \) is the thermal pressure which can
be related to the equation of state, \( \kappa \) is the transport coefficient for thermal
conductivity and \( T \), the local temperature field of the nuclear fluid (dimension in MeV).

In the present investigation, the transport coefficients \( \eta \), \( \xi \) and \( \kappa \)
and the equation of state are inputs to the calculations. Ideally, they should
be derived from fundamental, microscopic theories and realistic nucleon-
nucleon interactions. But even at a more phenomenological level, estimates
based on known bulk properties of nuclei can be made.

2. Equation of State and Interaction Parameters

(a) Internal Energy Function for the Nuclear Fluid:

The use of an equation of state is a distinct characteristic of nuclear
hydrodynamics. Here, the internal energy (per nucleon) function of the
nuclear fluid assumes a mathematically simple, but physically plausible form:

\[
E_{\text{int}}(\rho, \sigma) = E_0(\rho) + E_T(\rho, \sigma),
\tag{IV-5}
\]

where \( E_0 \) is the zero-temperature part of the internal energy, assumed to
be dependent only on the local density field and \( E_T \), the thermal part which
depends on the local density field as well as the specific entropy \( \sigma \) (or tem-
perature). The zero-temperature part, which has been investigated by many
authors (e.g. Br-b 69, Be-b 71), is taken to be of the following form:
\[ E_0(\rho) = b_0 \rho^{2/3} + b_1 \rho^{3/3} + b_2 \rho^{4/3} + b_3 \rho^{5/3} \]  \hspace{1cm} (IV-6)

and the thermal part is parametrized by the Fermi gas model (Hu 63),

\[ E_T(\rho, \sigma) = \frac{1}{3} \frac{\hbar^2}{m} \left( \frac{6}{4\pi} \right)^{2} \sigma^2 \rho^{2/3} \]  \hspace{1cm} (IV-7)

where the spin-isospin degeneracy has been accounted for and \( E_T \) can be re-expressed in terms of the temperature field:

\[ E_T(\rho, T) = \frac{1}{3} \frac{m^2}{\hbar^2} \left( \frac{4\pi}{6} \right)^{2} \rho^{-2/3} T^2 \]  \hspace{1cm} (IV-8)

With this form of internal energy function, the coefficients \( b_i \) in eq. (IV-6) can be uniquely determined as follows. The leading coefficient \( b_0 \) is taken to be the same as that in the Fermi gas model:

\[ b_0 = 75.01 \text{ MeV-fm}^2 \]  \hspace{1cm} (IV-9)

The other coefficients can then be determined from three known bulk properties of nuclei:

the normal density at equilibrium \( \rho_o = 0.17 \text{ fm}^{-3} \),  \hspace{1cm} (IV-10)

the binding energy per particle \( E_B = -16.5 \text{ MeV} \)  \hspace{1cm} (IV-11)

and the coefficient of nuclear incompressibility at equilibrium density which is estimated from the giant monopole resonance of \(^{208}\text{Pb} \) (Wo 78i),

\[ K_c(\rho_o) = 186 \text{ MeV} \]  \hspace{1cm} (IV-12)

Almost the same value was obtained by Maher et al (Ma-b 69) in studies of
$^{16}O + ^{16}O$ collisions, suggesting the constancy of this value across the periodic table.

From the condition on the normal density, one has

$$
\left( \frac{\partial E_o}{\partial \rho} \right) \rho = \rho_o = 0
$$

(IV-13)

From the condition on binding energy, one has

$$
E_o(\rho_o) = b_0 \rho_o^{2/3} + b_1 \rho_o^{3/3} + b_2 \rho_o^{4/3} + b_3 \rho_o^{5/3}
$$

$$
= E_B
$$

(IV-14)

From the condition on nuclear incompressibility, one has

$$
K_c(\rho_o) = 9 \rho_o^2 \left( \frac{\partial^2 E_o}{\partial \rho^2} \right) \rho = \rho_o
$$

$$
= -2b_0 \rho_o^{2/3} + 4b_2 \rho_o^{4/3} + 10b_3 \rho_o^{5/3}
$$

(IV-15)

Eqs. (IV-13)-(IV-15) can be compactly written as:

$$
\begin{bmatrix}
1 & \rho_o^{1/3} & \rho_o^{2/3} \\
1 & \frac{4}{3} \rho_o^{1/3} & \frac{5}{3} \rho_o^{2/3} \\
0 & 4 \rho_o^{1/3} & 10 \rho_o^{2/3}
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
= \begin{bmatrix}
\rho_o^{-1} E_B - b_0 \rho_o^{-1/3} \\
-\frac{2}{3} b_0 \rho_o^{-1/3} \\
\rho_o^{-1} K_c(\rho_o) + 2b_0 \rho_o^{-1/3}
\end{bmatrix}
$$

(IV-16)

from which $b_1$, $b_2$ and $b_3$ can be uniquely determined.

With the equation of state given, the pressure terms can be obtained from standard thermodynamic relations. The total pressure of the nuclear fluid is given as:
\[ p(\rho, \sigma) = \rho^2 \frac{\partial E(\rho, \sigma)}{\partial \rho} \]
\[ = p_o(\rho) + p_T(\rho, \sigma), \quad (IV-17) \]

where the zero-temperature part of the pressure is given by
\[ p_o(\rho) = \rho^2 \frac{\partial E_o(\rho)}{\partial \rho} \]
\[ = \rho \left[ \frac{5}{3} \rho^2/3 + \frac{4}{3} \rho^{2/3} + \frac{5}{3} \rho^{3/3} \right] - 2S_e \quad (IV-18) \]

and the thermal part of the total pressure is given by:
\[ p_T(\rho, \sigma) = \rho^2 \frac{\partial E_T(\rho, \sigma)}{\partial \rho} \]
\[ = \frac{\rho}{3} \rho E_T. \quad (IV-19) \]

(b) Interaction Parameter:

For the effective nucleon-nucleon interaction, one can adopt a parameterization similar to that used in microscopic theories such as the TDHF approximation. The interaction is taken to constitute a short-range part \( V_S \) and a long-range part \( V_L \). The short-range component can be characterized by a delta-function with density dependence. This, however, can be incorporated into the internal energy part (Wo 77i, 77iii), while the long-range component is taken to be a sum of an effective Yukawa nuclear interaction and the Coulomb interaction. The Yukawa interaction is taken to be:
\[ V_Y(r, r') = \beta e^{-\alpha |r - r'|} \quad (IV-20) \]

where
\[ \beta = -85 \text{ MeV-fm} \quad (\text{IV-21}) \]

and

\[ \alpha = 1.5 \text{ fm}^{-1} \quad (\text{IV-22}) \]

These parameters \( \beta \) and \( \alpha \) are chosen to fit the binding energies and the root-mean-squared radii of nuclei along the \( \beta \)-stability line. It is this use of such Yukawa interaction which gives rise to a diffuse surface and an effective surface tension in a finite nuclear system within the hydrodynamical description. The Coulomb interaction is taken to be

\[ V_C(r, r') = \left( \frac{Z_p + Z_T}{A_p + A_T} \right)^2 \frac{e^2}{|r - r'|} \quad (\text{IV-23}) \]

where \((Z_p, A_p)\) and \((Z_T, A_T)\) are the atomic and mass numbers of the projectile and target nuclei, respectively, and \(e\) is the electronic charge.

The zero-temperature part of the internal energy function used in the present studies is plotted against the nuclear density in figure 4.2.
C Transport Coefficients of Nuclear Fluid

Not much is known concerning the transport coefficients for the nuclear fluid; at the present, they can only be estimated. Direct comparisons of the predictions obtained with different assumed sets of transport coefficients and confrontation with experimental results should eventually allow the determination of the transport coefficients.

The shear viscosity and thermal conductivity will be estimated from the Vlasov-Boltzmann equation which is embodied in the Landau Fermi liquid theory. The compressional viscosity will be estimated from the width of the giant monopole resonance in $^{208}\text{Pb}$.

1. Shear Viscosity and Thermal Conductivity

The transport coefficients can be evaluated when the matrix element for the cross-section in the Vlasov-Boltzmann equation is known. Using the Born approximation relating the matrix element $W$ to the cross-section and a procedure developed by Baym and Pethick (Ba-c 78) which was suggested earlier by Sykes and Brooker (Sy 70), one obtains the coefficient of thermal conductivity as:

$$\frac{k}{k_B} = \frac{1}{3\pi} \frac{p_F^3}{m_B^2 k_B T} \frac{1}{<d\sigma/d\Omega>} \sum_{\nu=2,4,\ldots} \frac{(2\nu+1)}{\nu(\nu+1) \left[\nu(\nu+1)-2\lambda_k\right]}$$

(IV-24)

where

$$<d\sigma/d\Omega> = \int \frac{d\Omega(\theta,\phi)}{4\pi} \left[ g \frac{d\sigma}{d\Omega} (\Theta_{13}(\theta,\phi)) \right] / 2\cos\frac{\theta}{2}$$

(IV-25)
\[
\lambda_k \langle \frac{d\sigma}{d\Omega} \rangle = \int \frac{d\Omega(\theta, \phi)}{4\pi} \left[ g \frac{d\sigma}{d\Omega} (\Theta_{13}(\theta, \phi)) \right] \frac{(1 + 2 \cos \theta)}{2 \cos \frac{\theta}{2}}
\]  

(IV-26)

In the above expression, \( \theta \) is the angle between \( \vec{p}_1 \) and \( \vec{p}_2 \) and \( \phi \) is the dihedral angle between the planes \( \vec{p}_1 \times \vec{p}_2 \) and \( \vec{p}_3 \times \vec{p}_4 \). \( \Theta \) is the angle between \( \vec{p}_1 \) and \( \vec{p}_3 \) and \( g \) is the degeneracy of the nuclear fluid which is 4 in this case. Upon taking a constant value of 1 fm\(^2\)/steradian for \( \frac{d\sigma}{d\Omega} \) from \((p, p)\) scattering (Pr 62) at a relative kinetic energy of \( \sim 40 \) MeV, one obtains

\[
\frac{\kappa}{k_B} = \frac{0.4224 \text{ MeV-c/fm}^2}{k_B T}
\]  

(IV-27)

One can follow the same procedure of Baym and Pethick (Ba-c 78) to obtain the shear viscosity coefficient

\[
\eta = \frac{1}{15 \pi^3} \frac{p_F^5}{m^2 (k_B T)^2} \sum_{\nu=1,3, \ldots} \frac{2\nu + 1}{\nu(\nu+1)} \frac{2\nu + 1}{[\nu(\nu+1) - 2\lambda] \eta}
\]  

(IV-28)

where \( \langle \frac{d\sigma}{d\Omega} \rangle \) is given as before and

\[
\lambda_\eta \langle \frac{d\sigma}{d\Omega} \rangle = \int \frac{d\Omega(\theta, \phi)}{4\pi} \left[ g \frac{d\sigma}{d\Omega} (\Theta_{13}(\theta, \phi)) \right] \frac{(-1 + 3 \sin \frac{\theta}{2} \sin^2 \phi)}{2 \cos \frac{\theta}{2}}
\]  

(IV-29)

Again, upon taking a constant value of 1.0 fm\(^2\)/steradian for \( \frac{d\sigma}{d\Omega} \), one obtains
\[ \eta = \frac{675.39 \text{ MeV}^3}{(k_B T)^2 \text{ fm}^2 \cdot c} \]  

The transport coefficients are therefore sensitive functions of the local temperature. It is known, however, that the temperature reached in the collisions of heavy-ions with a few hundred MeV per nucleon is about 30 MeV (Na 78). So, as a first estimate of an average transport coefficient, one can use 30 MeV as the temperature to obtain

\[ \kappa = 0.014 \text{ c/fm}^2 \]  

(IV-31)

and

\[ \eta = 0.75 \text{ MeV}/(\text{fm}^2 \cdot c) \]  

(IV-32)

2. Compressional Viscosity

To estimate the coefficient of compressional viscosity from the width of the giant monopole resonance, we can consider the equation of motion for small amplitude oscillations:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \]  

(IV-33a)

and

\[ \frac{\partial (\rho \vec{u})}{\partial t} + (\vec{u}^2) = -a^2 \rho + (\zeta - \frac{\eta}{3}) \nabla \cdot (\nabla \vec{u}) \]  

(IV-33b)

where \( a \) is the speed of sound related to the incompressibility \( K_c \) by

\[ a = \sqrt{\frac{K_c}{9m}} \]  

(IV-34)
From this set of equations, the monopole oscillation frequency becomes complex and is given by

\[
\omega = \omega_0 \left[ 1 - i \frac{(\zeta - \frac{2}{3} \eta) \pi^2}{2m \rho_0 \omega_0 R^2} \right]
\]  

where

\[
\hbar \omega_0 = \frac{\hbar \pi}{R} \sqrt{\frac{K_c}{9m}}
\]

is the eigen-energy of the monopole oscillation in the absence of compressional viscosity, \( R \), the radius of the nucleus and \( \rho_o \), the normal density of nuclear matter. Identifying the imaginary part of the frequency with the width \( \Gamma \) of the giant monopole resonance, one has

\[
\Gamma = \hbar \frac{(\zeta - \frac{2}{3} \eta) \pi^2 R^2}{2m \rho_0}.
\]

For \(^{208}\text{Pb}\), the width of the giant monopole resonance is 2.5 MeV (Ha-c 77).

Thus the coefficient of compressional viscosity is

\[
\zeta = 18.76 \text{ MeV/(fm}^2\text{-c).}
\]
Figure 4.1  The geneology of nuclear hydrodynamics. How nuclear hydrodynamics can be obtained from the many-body theory through various approximations is summarized.

Figure 4.2  The equation of state of nuclear fluid at zero-temperature.
Fig. 4.1

Geneology of Nuclear Hydrodynamics

MANY-BODY THEORY

TRUNCATION OF MARTIN-SCHWINGER HIERARCHY

EXTENDED TDHF APPROXIMATION

INTERPRETING WIGNER FUNCTION AS DISTRIBUTION FUNCTION

VLASOV–BOLTZMANN EQUATION

LOCAL ENTROPY
LOCAL H-THEOREM
LOCAL THERMAL EQUILIBRIUM

ENSKOG-CHAPMAN EXPANSION WHEN THERE IS LOCAL EQUILIBRIUM

NUCLEAR HYDRODYNAMICS
AN EQUATION OF STATE OF NORMAL NUCLEAR FLUID

Fig. 4.2

INTERNAL ENERGY AT ZERO TEMP. (MeV/A)

DENSITY (fm⁻³)

0.10 0.20 0.30 0.40 0.50 0.60

-10 0 10 20
CHAPTER V

A TWO-DIMENSIONAL MODEL OF NUCLEAR HYDRODYNAMICS
A Introductory Remarks

In the previous chapter, the motivations for and the basic concepts of nuclear hydrodynamics have been discussed. To facilitate detailed studies of nuclear collisions within the framework of this macroscopic approach, the complete set of equations (IV-1), (IV-2) and (IV-3) are solved to simulate the dynamics.

In the past hydrodynamical calculations applied to heavy-ion reactions by Amsden, Nix and co-workers (Am-b 77i, 77ii, 78), the numerical technique employed was the so-called particle-in-cell (PIC) method. In their work, although the kinematics has been treated properly, many realistic features of nuclear bulk properties cannot be incorporated; for example, the binding energy and Coulomb interaction are neglected. Furthermore, no effective long-range nucleon-nucleon interaction is used and the nuclear surface is thus taken to be sharp, which inevitably introduces the problem of negative pressure and makes necessary artificial prescriptions to circumvent numerical difficulties. In addition, viscosity and thermal conductivity of the nuclear system cannot be taken into account.

Since most of these past calculations were mainly concerned with heavy-ion collisions in the high-energy range between hundreds of MeV and several GeV (per nucleon), the neglect of some features such as the binding energy, Coulomb energy and surface diffuseness might be justifiable. However, the PIC method cannot be applied for the description of collision processes at low energies, say, from a few MeV above the Coulomb barrier up to tens of MeV per nucleon. Moreover, viscosity and thermal conductivity are responsi-
ble for dissipative mechanisms in the reactions. Depending on the actual magnitudes of these transport coefficients, dissipation can play an important role in determining the final dynamical state of a reaction. Accordingly, it is not at all obvious that the degrees of freedom associated with dissipation can be neglected, even in the medium-energy region of, say, 50–100 MeV per nucleon. If nuclear hydrodynamics has any validity in the description of heavy-ion collisions, it is of great interest to solve the problem by means of alternative methods by which these former shortcomings can be removed.

In the present investigation, the hydrodynamical equations are solved by means of the so-called flux-corrected-transport (FCT) method and the time-step-splitting (TSS) method of Boris and Book (Bo-c 73, Bo-d 75, Bo-c 76). The important aspect which distinguishes the present approach from that of Amsden and Nix is that realistic nuclear bulk properties, which otherwise cannot be easily incorporated in the PIC method, can now be readily treated by the FCT method in a simple manner. Here, with the possible difference in the values of input parameters, the low-, medium- and high-energy reactions can be studied on the same footing. In this chapter, we shall study the results of hydrodynamical calculations based on the FCT and TSS methods; details of the numerical problem will be given in Appendix V.

In order to develop a coherent picture and draw fruitful conclusions wherefrom, the results of the hydrodynamical calculations will be presented as follows. The emphasis will be placed on the characteristics of the hydrodynamical behavior of nuclear systems as manifested in the collision calculations and their possible experimental implications. The low dimensionality of
the two-dimensional (2D) problem renders possible fast and economic computations. Here, in a systematic study of a 2D model of nuclear hydrodynamics, the dependence of nuclear reactions on the following factors is analyzed: (i) bombarding energy, (ii) impact parameter and (iii) effects of viscosity and thermal conductivity. Special hydrodynamical features will be investigated for head-on cases of the three-dimensional problem, which will be the subject matter of the next chapter.

For convenience in the subsequent discussions, the following notations will be adopted hereafter. A bombarding energy, unless stated otherwise, is always given in terms of MeV per projectile nucleon with the target at rest.

The features observed from these calculations appear to suggest the following classification of energy regions. The interval between a few MeV per nucleon up to about 40 MeV per nucleon is taken to be the low-energy region. This includes the subsonic region (\(E/\text{AMU} < 10.3 \text{ MeV}\)) and a region in which supersonic features begin to emerge and dominate with increasing bombarding energies. The interval between 40 MeV and about 100 MeV is regarded as the medium-energy region in which features characteristic of shock phenomena are prominent. Energies above 100 MeV per nucleon constitute the high-energy region in which the nuclear matter undergoes considerable compression during the collision and the interaction often leads to a complete dissociation of the composite system.

A reaction can be classified kinematically according to the impact parameter. It is called head-on (central), near-head-on or peripheral, according to the following criteria:
The validity of a hydrodynamical picture depends critically upon whether local thermal equilibrium can be established within the reaction time of the system. It is clear from the outset that not all the nuclear collisions can be properly described by nuclear hydrodynamics. For example, peripheral and transfer reactions are unlikely to be treated well by nuclear hydrodynamics as the interaction time involved is too short for local equilibrium. Hence it is important to be selective in choosing the proper case for treatment. As the interaction time increases with decreasing impact parameter, head-on and near-head-on collisions of large nuclear systems clearly provide the most favorable cases in which nuclear hydrodynamics may be a valid description. Hence, it is on these cases that our attention will be focused.
B  Fundamentals of Two-Dimensional Nuclear Hydrodynamics

1. Geometry

In two-dimensional nuclear hydrodynamics, the motion of the nuclear fluid is restricted to the reaction plane which contains the centers of mass of the approaching nuclear systems. The motion of the fluid elements along the direction orthogonal to this reaction plane is frozen, for simplicity. Geometrically, this corresponds to the collision problem of two (infinite) parallel columns of nuclear matter and can be formalized by the following constraints:

\[ \rho(\vec{r}, t) = \rho(x, y, t), \quad (V-2a) \]
\[ u_x(\vec{r}, t) = u_x(x, y, t), \quad (V-2b) \]
\[ u_y(\vec{r}, t) = u_y(x, y, t), \quad (V-2c) \]
\[ u_z(\vec{r}, t) = 0 \quad (V-2d) \]

and

\[ E_T(\vec{r}, t) = E_T(x, y, t), \quad (V-2e) \]

where the xy-plane is parallel to the reaction plane and the z-axis is orthogonal to this plane.

The 2D model serves the following purposes:

(a) The 2D calculations treat the exact two-dimensional problem and solve the equations of motion for any scattering of arbitrary systems with good speed and accuracy. They are fast and economic because of the low dimensionality, but they are much less restricted than the previous one-dimensional studies (Wo 77). Consequently, much insight into the under-
lying dynamics can be gained from extensive calculations based on this 2D model. They result in reproducing and predicting many qualitative and semi-quantitative features of nuclear hydrodynamics.

(b) From a technical point of view, these 2D calculations have paved the way for realistic three-dimensional (3D) studies. In particular, the methods used here require only slight modifications in order to solve (exactly) the head-on cases and (approximately) the near-head-on cases of the 3D problem, which are the cases of greatest physical interests. This extension can be achieved without invoking the full complexity of the general 3D geometry.

2. Basic Equations of Two-Dimensional Nuclear Hydrodynamics

The basic 2D hydrodynamical equations are solved in the cartesian component form. They are:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \]  \hspace{1cm} (V-3)

\[ \frac{\partial (\rho \mathbf{u}_i)}{\partial t} + \sum_{j=1}^{2} \nabla_j \left[ \rho \mathbf{u}_i \mathbf{u}_j + \frac{1}{m} (p_{ij} + p'_{ij}) \right] = -\frac{n}{m} \nabla \cdot \mathbf{U} \]  \hspace{1cm} (V-4)

and

\[ \frac{\partial (\rho E_{T_i})}{\partial t} + \nabla \cdot (\rho E_{T_i} \mathbf{u}) = -p_T \nabla \cdot \mathbf{u} + \nabla \cdot (k \nabla T) \]

\[ - \sum_{i,j=1}^{2} p'_{ij} \frac{\partial u_i}{\partial x_j} \]  \hspace{1cm} (V-5)

In eqs. (V-3)-(V-5), the indices \( i \) and \( j \) run from 1 to 2 and all the
vectors and vector operators are understood to be two-dimensional. The 2D viscosity stress-tensor is now given as:

\[
P_{ij}^\prime = -\eta \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \delta_{ij} \nabla \cdot \vec{u} + \xi \delta_{ij} \nabla \cdot \vec{u} \right) \quad (V-5a)
\]

which, apart from the range of the tensor indices, differs from the general 3D viscosity stress-tensor (eq. IV-4) in that the factor in front of the \( \nabla \cdot \vec{u} \) term within the square bracket is unity instead of 2/3. This is due to the general condition that this tensor be traceless (see, for example, Landau and Lifshitz).

The logarithmic singularity of the Coulomb field at the origin in a one- or two-dimensional geometry introduces numerical problems. Hence, in the 2D model, the long-range potential \( V \) includes only a folding of the Yukawa interaction with the density. Stated differently, all bombarding energies used in the 2D model are, in principle, above Coulomb barrier.

3. Some Special Concepts of the 2D Calculations

Before each calculation, the static equilibrium density of each nuclear system is obtained separately by means of a self-consistent calculation (which is described in greater detail in Appendix V). The use of the effective Yukawa interaction between fluid elements leads to a diffuse edge of the density proﬁle which is a realistic feature for finite nuclei. Each system is then placed so that the surfaces of the interacting systems barely touch. They are then given velocity fields corresponding to the bombarding energy in question. This completes the establishment of the initial conditions.
Figure 5.1 presents the profiles of the static densities (at zero temperature) of a projectile and a target as used in a 2D calculation. This projectile system has a half-density radius of about 3.5 fm and corresponds, in a 3D configuration, to a system of $A \approx 20$ (from the usual $A^{1/3}$ rule). The target system has a half-density radius of about 6 fm which corresponds to a 3D system of $A \approx 200$. However, the 2D static solutions give a total particle number of 5.99 per fm along the z-direction for the projectile system and 17.41 for the target system. One should remember this 2D geometrical factor in order to put the 2D results in their appropriate perspectives. Hereafter, we shall designate 1 particle per fm along the z-direction as "one 2D mass unit", and similarly for other dynamical quantities.

In the remainder of the present chapter, attention will be focused on the calculations of this particular asymmetric system. It is chosen because it is a combination of a large target which has an essentially uniform interior and a smaller projectile for which there is as much interior as the surface. Comparisons of calculations on symmetric and asymmetric systems of the three-dimensional problem will be made in the next chapter.

The essential parametrizations appropriate for the hydrodynamical model have been discussed in Chapter IV. To investigate the importance of viscosity and thermal conductivity on the dynamics, 2D calculations based on a number of sets of transport coefficients of different orders of magnitude have been performed and the results studied. Here, we shall concentrate on the detailed comparisons of two sets of transport coefficients. They are:
\[ \eta, \zeta = 10^{-4} \text{MeV/(fm}^2\text{c)} \quad , \]
\[ \kappa = 10^{-4} \text{c/fm}^2 \] 

(V-6)

and

\[ \eta = 0.57 \text{MeV/(fm}^2\text{c)} \quad , \]
\[ \zeta = 2.21 \text{MeV/(fm}^2\text{c)} \quad , \]
\[ \kappa = 0.01 \text{c/fm}^2 \] 

(V-7)

Those 2D calculations based on the transport coefficients of (V-6) will be referred to as the "non-viscous" or "small viscosity" cases. Negligible thermal energy is generated in these cases. On the other hand, calculations done with the transport coefficients (V-7) will be referred to as the "viscous" or "large viscosity" cases, which are usually characterized by the production of considerable thermal energy.

Though many calculations have also been done for other values of the transport coefficients, those given by eqs. (V-6) and (V-7) are selected for detailed study because the solutions described by these systems appear to reproduce most of the features of heavy-ion collisions, for example, fusion, fusion-fission and deep-inelastic scatterings. With transport coefficients which are even one or two orders of magnitude larger than the values given in eq. (V-7), the final results of the collisions appear to be unrealistically dominated by fusion up to large relative angular momentum (corresponding to impact parameters larger than half of the nuclear radii). These cases, however, will not be separately discussed here.
Figure 5.1  Density profiles of the projectile and target systems in the two-dimensional model of nuclear hydrodynamics. From the half density radii and the $A^{\frac{1}{3}}$ rule, the projectile and target systems correspond to $A \approx 20$ and $A \approx 200$, respectively, in a three-dimensional configuration.
DENSITY PROFILES OF PROJECTILE AND TARGET IN 2D MODEL OF NUCLEAR HYDRODYNAMICS

Density (fm^3)

Radius (fm)

Fig. 5.1
C Low-Energy Region

1. Head-On Collisions

(a) Reactions at $E \leq 10\text{ MeV/AMU}$, $b=0\text{ fm}$ (Non-Viscous):

Figure 5.2 depicts the calculated time evolution of the density field and the current density field in the head-on collision of a two-dimensional system as described in the previous section. The subplots at the top are the cuts of nuclear density contours in the reaction plane graded in levels of $0.025\text{ fm}^{-3}$. To standardize the notations in subsequent discussions, the value $0.15\text{ fm}^{-3}$ is taken to be the "normal nuclear density", which is designated by the contour label "6" in the figure. Below each density subplot is a subplot of the current density field which indicates the flow pattern of the nuclear fluid. The current density ($\rho v$) at each point is represented by an arrow which originates from the point in question and has a length proportional to its magnitude.

The case shown in figure 5.2 corresponds to a bombarding energy of 10 MeV per projectile nucleon and the calculation is done with the set of small transport coefficients given in (V-6). The initial separation of the projectile and target centers has been chosen to be 14.4 fm.

A composite system begins to be formed at about $40\text{ fm/c}$. At $76\text{ fm/c}$ the newly formed composite system begins to demonstrate transfer of the relative kinetic energy to internal excitation energy, as indicated by the rather pronounced surface vibrations in the density field. The maximum density has increased to about $0.24\text{ fm}^{-3}$. The composite system maintains this central density between about $50\text{ fm/c}$ and $30\text{ fm/c}$. 
The flow pattern of the nuclear fluid illustrated by the current density plot at 76fm/c also shows how the system responds during the collision with its characteristic compression. The appearance of relatively large momentum fluxes in the directions between $90^0$ and $120^0$ (angles measured with respect to the beam direction) in the center-of-mass system can be interpreted as the result of the scattering of the nucleons from the projectile and target system. From the standpoint of the fluid dynamics, this is the consequence of the incompressibility of the nuclear fluid. This is followed by a relaxation and expansion of the compressed system along the transverse directions.

At about 101fm/c, the velocity field of the nuclear fluid has attained a high degree of collectivity in the following sense. The current density field vectors, largely parallel to one another at this time, are aligned along the transverse direction and away from the center-of-mass of the composite system.

It should be noted that the occurrence of such pronounced currents and subsequent transverse expansion in head-on (and also in near-head-on) collisions is a distinct feature of nuclear hydrodynamics. For example, this is in sharp contrast to the intermediate dynamical behaviors described by the mean-field theory of time-dependent Hartree-Fock approximation. Many TDHF calculations have been done for the head-on collisions of equal as well as unequal projectiles and targets, at bombarding energies of several MeV above the Coulomb barrier. They indicate that the evolution of the system during the interaction is mainly governed by the mean field in the entire
composite system, generated by all nucleons. This dynamical feature results in very little transfer of the longitudinal momentum to the transverse directions and leads to a prominent interpenetrability of the colliding nuclei (for example, Ko-a 76).

At 160 fm/c, the density field begins to form three clusters. In the subsequent time steps, there is a continual mass exchange among these increasingly pronounced clusters. The originally thick neck regions connecting these clusters gradually become thinner and the composite system eventually fragments into three approximately equal, self-bound masses at about 350 fm/c. Each of the "outer" fragments has 7.64 mass units and goes away in an angle of ±79° in the center-of-mass system, whereas the fragment in the "middle" has 8.12 mass units and recoils along the backward direction (180° in the center-of-mass system).

To investigate the energy dependence of low-energy head-on collisions, the above calculation was repeated for other bombarding energies. Figures 5.3, 5.4 and 5.5 depict the collisions at 5MeV, 2.5MeV and 1.25MeV per projectile nucleon, respectively. A comparison with figure 5.2 shows that the general dynamical features displayed in the 10 MeV reaction are typical of all these subsonic head-on collisions. However, in each of the stages, namely, the compression, the relaxation, the expansion, the clustering and the final fragmentation, the time interval involved increases slightly with decreasing bombarding energy.

In the 5 MeV case (fig. 5.3), the density field begins to form three clusters at 216 fm/c. However, in the subsequent time steps, these clusters
disappear because of the outward flow of the nuclear matter which tends to give rise to a roughly uniform density in the interior of the elongated system. At about 330 fm/c, this leads to the formation of two clusters, joined by a thick neck region with a density of about 60% to 70% of the normal nuclear density. This configuration finally breaks up into two equal, large clusters, each of 10.82 mass units, and a small fragment with 1.74 mass units in the center. The latter, however, has a maximum density of only about 30% of the normal density.

The dynamical evolution of the 2.5 MeV collision (fig. 5.4) is very similar to the 5 MeV case. The final result is a fragmentation into two equal clusters (each with 11.40 mass units), with a residual of nuclear matter of 0.59 mass units in the center. The 1.25 MeV collision (fig. 5.5), however, leads to a final breakup into two equal fragments.

The above calculations using small viscosity and thermal conductivity indicate that there appears to be a limiting energy below which the composite system eventually breaks up into two equal masses and above which, the final result is a three-body breakup. For the present system in question, this limit is about 1.25 MeV. In addition, fusion seems to be absent in these non-viscous head-on collisions.

(b) Reactions at E=10, 15 MeV/AMU (Viscous Case):

To study the effects of dissipation on the dynamics, the calculations for the low-energy head-on collisions were repeated, using the large values of transport coefficients given in (V-7). Figure 5.6 depicts the evolution of
the reaction at 10 MeV per projectile nucleon with large viscosity and thermal conductivity. It is of interest to compare this with the calculation of the same reaction with small viscosity and thermal conductivity (fig. 5.2 and 5.6).

Attention is particularly focused on the thermal energy degree of freedom, the maximum density reached, the flow pattern of the nuclear fluid and the final results of the collision.

With large viscosity and thermal conductivity, the projectile and target systems, which are originally at zero temperature before reaction, acquire quickly a temperature field after collision has taken place. The total thermal energy in the composite system reaches a maximum value of about 18 MeV (i.e. 18 MeV/fm in the 2D units) during the compression stage. In contrast, in the case with small viscosity and thermal conductivity, the total thermal energy generated remains less than 1 MeV during the entire reaction and hence the system remains essentially at zero temperature throughout the collision, thus justifying the term "non-viscous".

The maximum density reached in the viscous case of the 10 MeV is about 0.2 fm$^{-3}$, which is smaller than the corresponding reaction with small viscosity and thermal conductivity. This may be understood as a result of a diffusion effect reflecting the finite viscosity. This aspect of density compression will be again discussed later and in greater details.

At 10 MeV, the time evolution of the current density field in the viscous case is quite similar to that in the non-viscous case. The transverse currents resulting from the collision leads to the formation of two clusters and the elongation of the composite system along the transverse directions.
The final stage of this calculation is a fragmentation into two equal clusters, each with 10.56 mass units. There is also a small residual of 2.27 mass units in the center.

In a calculation of the 15 MeV head-on collision with large viscosity and thermal conductivity, the maximum density attained is about 0.24 fm\(^{-3}\). The maximum thermal energy generated is about 22 MeV and the composite system eventually breaks up into three roughly equal mass clusters.

At bombarding energies below 10 MeV per projectile nucleon, the viscous composite system breaks up into two equal fragments as the final state of the reaction. The interaction time involved is longer and in the presence of dissipation, the thermal energy is of the order of 10 MeV. The outgoing fragments have decreasing momenta as we go to lower bombarding energies (see fig. 5.6a for the 5 MeV case).

One can now summarize briefly our findings for these low-energy head-on collisions (E ≤ 15 MeV/AMU), for the cases with large and small transport coefficients. Fusion-fission appears to be a dominant feature in such reactions. For an asymmetric projectile-target system, there appears to be a threshold for the bombarding, below which the head-on collision leads to either a fragmentation into two equal masses or simply fusion. In the non-viscous case, this threshold for the system in the present discussions is about 2.5 MeV per projectile nucleon, but down to 1.25 MeV, there is no fusion observed. The presence of dissipation, however, shifts the threshold to a value near 10 MeV. Judging from the final kinetic energies in the viscous cases, it can be expected that with large viscosity and thermal conductivity,
fusion may likely occur at energies below 5 MeV per projectile nucleon in the head-on collisions.

Immediately above the respective thresholds of 2.5 MeV and 10 MeV, the non-viscous and viscous head-on collisions lead to a breakup into three subunits.

The production of thermal energy (~15–20 MeV in the viscous case as compared with an energy of less than 1 MeV in the non-viscous case) is clearly a consequence of dissipation. The maximum compression of the density, in general, is reduced in the viscous case, though in these low-energy reactions the qualitative features of the velocity field do not seem to be drastically changed in the presence of dissipation.

These two-dimensional results may have an interesting topological implication in the corresponding realistic, three-dimensional problem. In the 2D model, since the motion along the z-direction (perpendicular to the reaction plane) is suppressed, the clustering along the x- (symmetry) axis and the y-axis lead to fragmentations of a similar kind. However, in a three-dimensional configuration, while a clustering along the x-direction can imply a fission mode, that along the y-direction is, in general, associated with the formation of a ring of nuclear matter. Recently, a proposal (Wo 78iv) was made to examine this aspect of nuclear hydrodynamics, with the hope of ultimately devising critical experimental tests to distinguish the macroscopic theory of nuclear hydrodynamics and the microscopic theory of TDHF approximation in low-energy heavy-ion reactions. This aspect of forming nuclear tori in reactions will be discussed again in the next chapter on three-
Figures 5.2, 5.3, 5.4, 5.5

Head-on collisions in 2D nuclear hydrodynamics for the non-viscous case \( (\eta, \tau = 10^{-4} \text{ MeV/(fm}^2\text{-c}), \kappa = 10^{-4} \text{ c/fm}^2) \).

\[ E = 10, 5, 2.5, 1.25 \text{ MeV/AMU}. \]

Figure 5.6 Head-on collision in 2D nuclear hydrodynamics for the viscous case \( (\eta = 0.57 \text{ MeV/(fm}^2\text{-c)}, \tau = 2.21 \text{ MeV/(fm}^2\text{-c)}, \kappa = 0.01 \text{ c/fm}^2) \).

\[ E = 10 \text{ MeV/AMU}. \]
Fig. 5.2  2D NUCLEAR HYDRODYNAMICS
E=10 MeV  b=0 fm (non-viscous)

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2D NUCLEAR HYDRODYNAMICS
E=10 MeV  b=0 fm (non-viscous)

- $t=290.69 \text{ fm/c}$
- $t=315.35 \text{ fm/c}$
- $t=340.17 \text{ fm/c}$

- $t=352.16 \text{ fm/c}$
- $t=365.79 \text{ fm/c}$
- $t=377.79 \text{ fm/c}$
Fig. 5.3  2D NUCLEAR HYDRODYNAMICS
E = 5 MeV  b = 0 fm (non-viscous)

\[ t = 0.00 \text{ fm/c} \hspace{1cm} t = 53.23 \text{ fm/c} \hspace{1cm} t = 101.97 \text{ fm/c} \]

**DENSITY**

**CURRENT**

\[ t = 151.70 \text{ fm/c} \hspace{1cm} t = 202.90 \text{ fm/c} \hspace{1cm} t = 257.19 \text{ fm/c} \]
2D NUCLEAR HYDRODYNAMICS

\[ E = 5 \, \text{MeV} \quad b = 0 \, \text{fm} \, \text{(non-viscous)} \]

- \( t = 294.47 \, \text{fm/c} \)
- \( t = 330.59 \, \text{fm/c} \)
- \( t = 367.87 \, \text{fm/c} \)

- \( t = 392.09 \, \text{fm/c} \)
- \( t = 416.09 \, \text{fm/c} \)
- \( t = 440.15 \, \text{fm/c} \)
### Fig. 5.4  2D NUCLEAR HYDRODYNAMICS

\[ E = 2.5 \text{ MeV} \quad b = 0 \text{ fm (non-viscous)} \]

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2D NUCLEAR HYDRODYNAMICS

$E=2.5 \text{ MeV} \quad b=0 \text{ fm (non-viscous)}$

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Fig. 5.5  2D NUCLEAR HYDRODYNAMICS
E=1.25 MeV  b=0 fm (non-viscous)

\begin{tabular}{|c|c|c|}
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\textbf{DENSITY} & \textbf{CURRENT} & \\
\hline
\textbf{t=0.00 fm/c} & \textbf{t=64.93 fm/c} & \textbf{t=126.49 fm/c} \\
\hline
\textbf{t=190.23 fm/c} & \textbf{t=255.42 fm/c} & \textbf{t=319.12 fm/c} \\
\hline
\end{tabular}
2D NUCLEAR HYDRODYNAMICS

E = 1.25 MeV  b = 0 fm (non-viscous)

\[ t = 359.96 \text{ fm/c} \quad t = 421.15 \text{ fm/c} \quad t = 481.92 \text{ fm/c} \]
Fig. 5.6 2D NUCLEAR HYDRODYNAMICS
E=10 MeV  \( b=0 \text{ fm (viscous)} \)

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2D NUCLEAR HYDRODYNAMICS

\( E = 10 \text{ MeV} \) \( b = 0 \text{ fm (viscous)} \)

\[
\begin{array}{ccc}
\text{t = 312.78 fm/c} & \text{t = 348.81 fm/c} & \text{t = 384.80 fm/c} \\
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\text{CURRENT} & \text{CURRENT} & \text{CURRENT} \\
\end{array}
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\[
\begin{array}{ccc}
\text{t = 420.80 fm/c} & \text{t = 456.82 fm/c} & \text{t = 493.07 fm/c} \\
\text{DENSITY} & \text{DENSITY} & \text{DENSITY} \\
\text{CURRENT} & \text{CURRENT} & \text{CURRENT} \\
\end{array}
\]
dimensional nuclear hydrodynamics.

(c) Reactions at $E = 20, 40 \text{ MeV/AMU}$, $b = 0 \text{ fm}$ (Non-Viscous Case):

It is of interest to study the characteristics of collisions in a supersonic region, that is, at bombarding energies greater than $10.3 \text{ MeV per projectile nucleon}$. Figure 5.7 depicts the time evolution of the head-on collision at $20 \text{ MeV}$ and with small transport coefficients. Several features stand out when this reaction is compared with the subsonic reactions discussed in the previous sections (see fig. 5.2-5.6). These supersonic features become increasingly pronounced as one goes to higher energies, as will be seen later.

Firstly, the more violent collision leads to a larger value of the maximum density attained ($0.27 \text{ fm}^{-3}$ at $20 \text{ MeV}$). The density field of the central compressed region is now marked by a profile with rather sharp "edges". For example, at about $50 \text{ fm/c}$, the projectile and target systems have completely fused to form a compressed composite system. There is a large density gradient as one goes from the vacuum on the left side to the interior of the compressed region. This is clearly indicated by the rather closely spaced density contours in the density plots.

Secondly, the unequal masses of the projectile and target and the response of the system to compression results in some peculiar intermediate shapes of the system. For example, at $50 \text{ fm/c}$ (fig. 5.7), there are relatively large currents going outwards in an approximately $110^0$ direction, due to the collision of nuclear matter from the projectile and target. In the next $10 \text{ fm/c}$ or so, the system develops into a trapezoidal shape (which is a frustrum in a three-dimensional case). At about $64 \text{ fm/c}$, the system begins
to expand outwards in all directions, but in such a way that the density contours maintain this trapezoidal shape for about 50 to 60 fm/c. As will be seen later, this intermediate geometrical configuration is typical of the collisions of unequal projectile and target at medium- and high-energies.

Thirdly, when the composite system relaxes and expands, there is a strong tendency for the density field to form clusters, leading to many rather complicated transient configurations of the density contours. In contrast, the evolution of the current density field during this expansion stage is quite simple. Roughly speaking, there are two main groups of currents, namely, strong transverse currents and some relatively weaker longitudinal currents. The transverse currents of nuclear matter diverge outwards in directions between about $90^\circ$ and $120^\circ$ in the center-of-mass system, whereas the longitudinal currents oscillate forwards and backwards. This behavior of the density field and the current density field result in a pronounced stretching of the system before its final fragmentation.

A further illustration of these features is shown in figure 5.8 in which the calculated time evolution of the 40 MeV head-on collision is depicted. The maximum density in this reaction reaches $0.31 \text{ fm}^{-3}$ and the density profile of the compressed region (trapezoidal) has even sharper "edges".

These supersonic features have special physical significance. We have seen that even at subsonic energies, there are pronounced transverse fluxes of nuclear matter produced in the reaction. In the supersonic region, depending on the actual bombarding energy, the transverse flow of nuclear matter can lead to clusters of all sizes and also dissociated single nucleons. Hence,
experimental identifications of these fragments and particles may provide a definitive test for nuclear hydrodynamics. It is well known in classical gas and fluid dynamics that large density compressions and sharp density profiles are among the distinguishing characteristics of shock phenomena (Landau and Lifshitz). Recently, there has been considerable speculations and studies on the possibilities of generating nuclear shock waves in heavy-ion collisions (Wo 74, Sc-b 72, Am-77ii). Such phenomena, if they can occur in nature, are not only of great interests of their own right, but will also provide much important dynamical information about the properties and the stability of nuclear matter under large compression. For such investigations, it is clear that a hydrodynamical description is a natural framework.

Nuclear hydrodynamics now stands between a theory and a purely phenomenological model. On one hand, we have argued from the geneology of nuclear hydrodynamics that it can be obtained as an approximation from the "exact" many-body theory (provided that the assumption about local thermal equilibrium is satisfied). On the other hand, the rigorous derivations of the equation of state from nucleon-nucleon interactions and also the transport coefficients from microscopic calculations are still open problems. However, without any further assumptions, our ability to solve the hydrodynamical equations explicitly leads to the conclusion that the supersonic head-on collisions are characterized by the occurrence of shock waves. A relevant question to ask appears to be: what are their experimental signatures and what are the most favorable conditions to observe them? This problem will be revisited in later discussions.
(d) Reactions at \( E = 20 - 40 \text{ MeV/AMU}, \ b = 0 \text{ fm} \) (Viscous Cases):

The calculations discussed in the previous section were all done with small values of transport coefficients \((eq. (V-6))\), resulting in only negligible amounts of thermal energy being generated during the collision. However, at bombarding energies of tens of MeV's, energy dissipation can constitute an important degree of freedom in the dynamics and hence should be considered in proper treatment.

To study the effects of dissipation on the features which have been discussed in the previous section, it is of interest to compare the calculations of the head-on collisions based on the large and small transport coefficients, in the energy region between 20 MeV and 40 MeV. Apart from some superficial and qualitative similarities of the viscous and non-viscous cases, the presence of dissipation, in general, reduces the maximum density compression, leads to thermal energy production and decreases the total kinetic energy in the final state.

Figure 5.9 depicts the time evolution of the head-on collision at 40 MeV, with large viscosity and thermal conductivity. In contrast to the corresponding non-viscous case (fig. 5.8), the density profile of the compressed region has smoother edges. The maximum density reached is about \(0.29 \text{ fm}^{-3}\) and the maximum thermal energy generated is about 50 MeV.

The system initially undergoes a compression process very similar to that in the non-viscous case. The compressed nuclear matter begins to expand outwards at about 60 fm/c. In the next 100 fm/c or so, the composite system becomes a very elongated, sausage-shaped configuration. Unlike the
non-viscous case in which the viscous system breaks up into three sub-units at about 300 fm/c, the system expands and the density field clusters oscillate along the transverse directions. At about 160 fm/c, the central density decreases from the normal value. It drops to below 60% of the normal value in 60 fm/c, but rises again to a smaller central density value than before. Such a damped density oscillation can repeat many times and the calculation was terminated before many cycle had taken place. As the system gradually expands and the kinetic energy decreases, it is likely that such a system would be unstable against particle emission and dissociation.

In order to make more quantitative comparisons of the viscous and non-viscous cases, one can consider the total kinetic energy (in the center-of-mass system) as a function of time. The kinetic energy curves in the viscous and non-viscous cases are given in figures 5.10 and 5.11, respectively. In both cases, the curves rise slightly in the first 10 fm/c or so. This is an effect of the attractive Yukawa field which leads to an initial acceleration of the projectile and target towards each other (with the Coulomb field neglected in the 2D model). Then the curves drop rapidly, by about 60 MeV, to minima near 45 fm/c.

It is not accidental that these minima should coincide with the time at which the systems both undergo maximum density compression. A considerable fraction of the initial relative kinetic energy is lost at the expense of density compression of the nuclear matter. In fact, at 45 fm/c, the compressed region of the composite system is essentially at rest in the center-of-mass system (see fig. 5.8 and 5.9). The main contribution to the total
The kinetic energy of the system comes from the currents entering the compressed region from the target and also to some extent, from the transverse currents.

The rise of the curves beyond the maximum compression stage is due to the ejection of the nuclear matter from the compressed region along the transverse directions. The curve of the non-viscous case regains the value of the initial kinetic energy before it drops again and levels off to about 110 MeV. Because of the energy dissipation, the curve of the viscous case rises by only about 30 MeV from the minimum. It finally drops and levels off at about 70 MeV at 200 fm/c.

During the entire reaction, the curve of the non-viscous case remains consistently higher than that for the viscous case. The difference between the curves equals approximately the thermal energy produced. For comparison, the kinetic energy in the non-viscous case for the impact parameter of 3 fm is given as the dashed curve in fig. 5.11. This curve is very similar to the one for zero impact parameter (solid line), except that on the whole it is slightly higher. This reflects that there is less kinetic energy loss due to a smaller contact of the projectile and target systems in the near-head-on collision than the head-on case.

The thermal energy in the 40 MeV collision (viscous case) for three impact parameters (b=0, 3, 6 fm) are plotted as functions of time in figure 5.12. The curve of the zero impact parameter rises from zero to a maximum value of about 50 MeV in 50 fm/c. It fluctuates before levelling off when the whole system has attained a global equilibrium. The curve for b=3 fm has the same pattern, except it rises to a maximum value of about 43 MeV. The
curve for $b=6\text{ fm}$ rises to about $30\text{ MeV}$ and then levels off after $50\text{ fm/c}$.

Figure 5.12a presents the thermal energy curves for the head-on collisions at 80, 100, 150, 200, and 250 MeV/AMU. The rate at which the system acquires thermal energy at the beginning of the reaction increasing rapidly with higher bombarding energies. An interesting feature of these curves is that they all level off at approximately the same energy ($\sim 60\text{ MeV}$).
Figures 5.7, 5.8

Head-on collisions in 2D nuclear hydrodynamics for the non-viscous case ($\eta, \gamma = 10^{-4}$ MeV/(fm$^2$), $\kappa = 10^{-4}$ c/fm$^2$).

$E = 20, 40$ MeV/AMU.

Figure 5.9

Head-on collision in 2D nuclear hydrodynamics for the viscous case ($\eta = 5.7$ MeV/(fm$^2$), $\gamma = 2.21$ MeV/(fm$^2$), $\kappa = 0.01$ c/fm$^2$).

$E = 40$ MeV/AMU.

Figure 5.10

Collective kinetic energy from 2D nuclear hydrodynamics for the viscous case ($\eta = 0.57$ MeV/(fm$^2$), $\gamma = 2.21$ MeV/(fm$^2$), $\kappa = 0.01$ c/fm$^2$).

$E = 40$ MeV/AMU, $b = 0$ fm.

Figure 5.11

Collective kinetic energy from 2D nuclear hydrodynamics for the non-viscous case ($\eta, \gamma = 10^{-4}$ MeV/(fm$^2$), $\kappa = 0.01$ c/fm$^2$).

$E = 40$ MeV/AMU, $b = 0, 3$ fm.

Figure 5.12

Thermal energy from 2D nuclear hydrodynamics ($\eta = 0.57$ MeV/(fm$^2$), $\gamma = 2.21$ MeV/(fm$^2$), $\kappa = 0.01$ c/fm$^2$).

$E = 40$ MeV/AMU, $b = 0, 3, 6$ fm.

Figure 5.12a

Thermal energy from 2D nuclear hydrodynamics ($\eta = 0.57$ MeV/(fm$^2$), $\gamma = 2.21$ MeV/(fm$^2$), $\kappa = 0.01$ c/fm$^2$).

$E = 80, 100, 150, 200, 250$ MeV/AMU, $b = 0$ fm.
Fig. 5.7  2D NUCLEAR HYDRODYNAMICS  
$E = 20 \text{ MeV} \ b = 0 \text{ fm} \ (\text{non-viscous})$

<table>
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<td>----------------</td>
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<th>t = 52.40 fm/c</th>
<th>t = 64.54 fm/c</th>
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<tbody>
<tr>
<td>Current</td>
<td>---------------</td>
<td>----------------</td>
<td>---------------</td>
</tr>
</tbody>
</table>
2D NUCLEAR HYDRODYNAMICS
E=20 MeV  b=0 fm (non-viscous)

Density plots at:
- t = 115.57 fm/c
- t = 153.04 fm/c
- t = 197.87 fm/c

Current plots at:
- t = 238.12 fm/c
- t = 277.91 fm/c
- t = 318.42 fm/c
Fig. 5.8  2D NUCLEAR HYDRODYNAMICS
E=40 MeV  b=0 fm  (non-viscous)

\begin{align*}
\text{DENSITY} & \quad \text{CURRENT} \\
\text{t}=0.00 \text{ fm/c} & \quad \text{t}=14.51 \text{ fm/c} & \quad \text{t}=26.55 \text{ fm/c} \\
\text{t}=38.72 \text{ fm/c} & \quad \text{t}=51.15 \text{ fm/c} & \quad \text{t}=64.35 \text{ fm/c}
\end{align*}
2D NUCLEAR HYDRODYNAMICS
E=40 MeV b=0 fm (non-viscous)

\begin{tabular}{ccc}
\textbf{DENSITY} & \textbf{DENSITY} & \textbf{DENSITY} \\
$t=78.54$ fm/c & $t=91.81$ fm/c & $t=103.98$ fm/c \\
\textbf{CURRENT} & \textbf{CURRENT} & \textbf{CURRENT} \\
\end{tabular}
Fig. 5.9  2D NUCLEAR HYDRODYNAMICS
E=40 MeV  b=0 fm (viscous)

<table>
<thead>
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</thead>
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<td>t=14.35 fm/c</td>
</tr>
<tr>
<td>- - 5 fm</td>
<td>- - 0.02 c/fm³</td>
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<td>t=50.47 fm/c</td>
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<tr>
<td>---------</td>
<td>---------</td>
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<td>t=0.00 fm/c</td>
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<tr>
<td>- - 5 fm</td>
<td>- - 0.02 c/fm³</td>
</tr>
<tr>
<td>t=38.47 fm/c</td>
<td>t=50.47 fm/c</td>
</tr>
</tbody>
</table>
2D NUCLEAR HYDRODYNAMICS
E = 40 MeV b = 0 fm (viscous)

$E = 40$ MeV $b = 0$ fm (viscous)

$t = 107.83$ fm/c $t = 135.64$ fm/c $t = 160.66$ fm/c

DENSITY

CURRENT

$t = 188.07$ fm/c $t = 212.12$ fm/c $t = 236.24$ fm/c

DENSITY

CURRENT
KINETIC ENERGY FROM 2D NUCLEAR HYDRODYNAMICS

$E = 40 \text{ MeV}$

$b = 0 \text{ fm}$

(viscous)
KINETIC ENERGY FROM 2D NUCLEAR HYDRODYNAMICS

$E = 40 \text{ MeV}$

(non-viscous)

$b = 0 \text{ fm}$ --

$b = 3 \text{ fm}$ ---

Fig. 5.11
THERMAL ENERGY FROM 2D NUCLEAR HYDRODYNAMICS

$E = 40$ MeV

TOTAL THERMAL ENERGY (MeV)

TIME (fm/c)
THERMAL ENERGY FROM 2D NUCLEAR HYDRODYNAMICS

TOTAL THERMAL ENERGY (MeV)

TIME (fm/c)

- 250 MeV
- 200 MeV
- 150 MeV
- 100 MeV
- 80 MeV
2. Non-Head-On Collisions

In order to substantiate the analysis on nuclear hydrodynamics, it is necessary to extend the studies to non-head-on collisions. There are at least two reasons for this. Head-on collisions are in fact rare events in heavy-ion reactions (zero probability, from a strictly mathematical point of view!). It is therefore of great interest to see whether the features observed in head-on collisions are present in the more general non-head-on cases. Moreover, the axial symmetry imposes a severe restriction which preclude most two-body scatterings.

However, it has been previously argued that not every non-head-on collision can be properly described by nuclear hydrodynamics. From the standpoint of interaction time, only head-on and near-head-on collisions (defined in (V-1)) provide favorable conditions for which nuclear hydrodynamics can be expected to have good validity. Hence, for the two-dimensional asymmetric system in the present discussions, it is necessary to focus our attention on those cases with impact parameters less than 9 fm (see also fig. 5.1).

(a) Reaction at $E=10\text{MeV/AMU}$, $b=3\text{fm}$ (Non-Viscous Case):

Figure 5.13 depicts the time evolution of a collision at 10 MeV per projectile nucleon and with an impact parameter of 3 fm, which corresponds to a total angular momentum of $9.3^+$ in the center-of-mass system. This calculation is done with the set of small transport coefficients given by eq. (V-6).
The projectile and target systems are fused at about 40 fm/c to form a compound system. At about 50 fm/c, the interior of the composite system reaches a maximum density of $1.4 \rho_0$. As a result of the collision, there are surface vibrations with large amplitudes in the composite system. In addition, the current density field indicates that the system rotates as a whole in a clockwise direction, a direct consequence of the conservation of the initial angular momentum.

After about 70 fm/c, the compressed system begins to relax and the interior density drops from the maximum value to the normal value in 20 to 30 fm/c. The formation of two unequal clusters in the density field signifies the tendency to fission. The velocity field is in such a way that these clusters move apart. The connecting neck region thins out as time proceeds, until finally, the clusters are completely separated at about 300 fm/c.

Knowledge of the density and current fields permits computations of observable quantities. The smaller fragment has 5.55 mass units and leaves the interaction region at an angle of $+63^\circ$ in the center-of-mass system ($\theta_{\text{lab}} = +34^\circ$). The larger fragment has 17.84 mass units and moves away at an angle of $-117^\circ$ in the center-of-mass system ($\theta_{\text{lab}} = -32^\circ$).

The projectile and target systems have initial kinetic energies of 33.1 MeV and 11.4 MeV, respectively, in the center-of-mass system. Here, the total kinetic energy for a system or a fragment is simply the sum of the collective kinetic energy contributed from all fluid elements:

$$E_k(t) = \frac{1}{2} m \int_{\text{fragment}} d^3 r \ n(\vec{r}, t) \ |\vec{u}(\vec{r}, t)|^2 . \quad (V-8)$$
where $\mathbf{u}$ is the velocity field, relative to the center-of-mass system or the laboratory system as the case may be. The outgoing projectile-like fragment has a total kinetic energy of 11.1 MeV in the center-of-mass system. The larger fragment has a kinetic energy of 11.8 MeV.

It is clear that since the velocity field of the composite system is "randomized" because of the scatterings of the fluid elements during the collision, it is meaningful to define a kinetic energy associated with the translational motion of a fragment:

$$ E_{tr}(t) = \frac{1}{2} \frac{P^2(t)}{M} \quad (V-9) $$

where $M$ is the total mass of the fragment and $\mathbf{P}$, the linear momentum of the fragment. In the center-of-mass system, the small fragment has a translational kinetic energy of 5.7 MeV and the large one, 3.4 MeV.

At any time, the kinetic energy of a fragment associated with its translational motion cannot exceed its total kinetic energy. The two quantities are equal only when all the velocity field vectors in the system are all parallel, that is, loosely speaking, when the velocity field is at a "maximum order". Hence a measure of the "randomization" of the fragment is its intrinsic kinetic energy (the total kinetic energy relative to its center-of-mass):

$$ E_{intr}(t) = \frac{1}{2} m \int_{\text{fragment}} d^3 r \left| n(r, t) \left( \mathbf{u}(r, t) - \mathbf{u}_{cm}(t) \right) \right|^2 \quad (V-10) $$

where $\mathbf{u}_{cm} (= \mathbf{P}/M)$ is the velocity of the center-of-mass of the fragment. This is easily seen to be related to $E_k$ and $E_{tr}$ as:
$E_{\text{intr}} = E_k - E_{\text{tr}}$. \hspace{1cm} (V-11)

As a brief summary, the main result to be drawn from this calculation is that the final fragments have essentially the same masses of the initial projectile and target. Approximately 50% of the initial kinetic energy is lost. However, the remaining final kinetic energy is so distributed that a considerable part of it goes to random motion of the fluid elements. The lost kinetic energy goes into the internal excitation of the fragments which are very much deformed at the time of separation. This calculation thus provides a typical example of a two-body deep-inelastic collision.

(b) Reaction at $E=10\text{MeV}/\text{AMU}, \ b=3\text{fm}$ (Viscous Case):

Figure 5.14 depicts the calculation of the 10 MeV collision, with an impact parameter of 3 fm and with large transport coefficients. Compared with the corresponding non-viscous case (fig. 5.13), the density field undergoes a similar sequence of intermediate shapes. But the final fragmentation takes place at about 500 fm/c. The two fragments are each self-bound and have 8.75 and 14.63 mass units.

The presence of dissipation, however, is responsible for several features which distinguish the viscous and non-viscous cases. Firstly, the total interaction time in the viscous case is about 170 fm/c longer than that of the non-viscous case. Secondly, the light and heavy fragments in the viscous case carry thermal energies of 1.2 Mev and 1.9 Mev, respectively. But the non-viscous system has less than 0.01 MeV during the entire collision. Thirdly, the dissipation directly affects the velocity field of the nuclear fluid,
which in turn affects the momenta and scattering angles of the fragments.

In the viscous case, the small and large fragments have total (collective) kinetic energies of 7.2 MeV and 5.9 MeV, respectively, which are smaller than the kinetic energies of the fragments from the non-viscous case. They have intrinsic kinetic energies of 4.3 MeV and 5.3 MeV. Hence, over 70% of the final kinetic energy goes into the random motion of the fluid elements. (In the non-viscous case, about 50% of the final kinetic energy goes into random motion.)

At the time of separation of the two fragments at about about 507 fm/c, the large fragment (target-like) has a center-of-mass velocity of less than 0.001c and that of the small fragment (projectile-like) is about 0.02c. It is worthy to note that even a 1 MeV nucleon already has a velocity of 0.05c. These small velocities are at the limit of the accuracy of the present calculations. For practical purposes, the fragments in this case should be taken as being at rest in the center-of-mass system at the time of separation.

The dynamical situation is now quite clear. The presence of viscosity decreases the final kinetic energy but increases the production of thermal energy. It also increases the "randomness" of the velocity field and thus decreases the translational kinetic energy of the fragments. In this viscous case, the fragments are at rest in the center-of-mass system, but due to the sole effect of the mutual Yukawa attraction, they begin to move towards each other to form a fused compound system again. In a more realistic three-dimensional case, it is easy to envisage that fusion may be prevented by the Coulomb repulsion of the fragments.
Figure 5.13 Non-head-on collision in 2D nuclear hydrodynamics for the non-viscous case ($\eta, \zeta = 10^{-4} \text{MeV}/(\text{fm}^2 \cdot c), \kappa = 10^{-4} \text{c}/\text{fm}^2$).
E = 10 MeV/AMU, $b = 3 \text{ fm}$.

Figure 5.14 Non-head-on collision in 2D nuclear hydrodynamics for the viscous case ($\eta = 0.57 \text{MeV}/(\text{fm}^2 \cdot c), \zeta = 2.21 \text{MeV}/(\text{fm}^2 \cdot c), \kappa = 0.01 \text{c}/\text{fm}^2$).
E = 10 MeV/AMU, $b = 3 \text{ fm}$. 
Fig. 5.13  2D NUCLEAR HYDRODYNAMICS
E=10 MeV  b=3 fm (non-viscous)

\[
\begin{array}{ccc}
\text{t=0.00 fm/c} & \text{t=52.08 fm/c} & \text{t=100.48 fm/c} \\
\hline
\text{DENSITY} & \text{DENSITY} & \text{DENSITY} \\
\text{0.02 c/fm}^3 & \text{0.02 c/fm}^3 & \text{0.02 c/fm}^3 \\
\text{CURRENT} & \text{CURRENT} & \text{CURRENT} \\
\end{array}
\]
2D NUCLEAR HYDRODYNAMICS
E=10 MeV  b=3 fm (non-viscous)

DENSITY

CURRENT

\begin{align*}
 t &= 284.19 \text{ fm/c} \\
 t &= 308.18 \text{ fm/c} \\
 t &= 332.35 \text{ fm/c}
\end{align*}
Fig. 5.14 2D NUCLEAR HYDRODYNAMICS
E = 10 MeV b = 3 fm (viscous)

<table>
<thead>
<tr>
<th>Density</th>
<th>Current</th>
</tr>
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<tbody>
<tr>
<td>t = 0.00 fm/c</td>
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<tr>
<td>t = 185.27 fm/c</td>
<td>t = 246.33 fm/c</td>
</tr>
</tbody>
</table>

Note: The diagram shows the evolution of density and current over time.
2D NUCLEAR HYDRODYNAMICS
E=10 MeV  b=3 fm (viscous)

<table>
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<th>Density</th>
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<td>t=506.75 fm/c</td>
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(c) Summary of Viscous and Non-Viscous Cases at Low Energies:

In the present two-dimensional asymmetric system, collisions of below 30 MeV per projectile nucleon generally lead to either fusion or fragmentation into two or more smaller, self-bound masses. In the latter cases, the final states of the reactions can be characterized by the observable quantities such as the fragment masses, energies and scattering angles.

The results of these calculations for the viscous and non-viscous cases are summarized in tables 5.1 and 5.2, respectively. Here, fusion is defined as the case when the composite system rotates by at least 180° about the center of mass (orbiting phenomena) or when the composite system remains in a bound state without breakup for a time interval longer than some typical time scale, say, 400-500 fm/c. A fusion-fission event, however, is characterized by the masses, scattering angles and momenta of the outgoing fragments. There are several intermediate cases marked by asterisks. In these cases, the fragments barely break apart; but due to the absence of the Coulomb field in the 2D model, they are stationary in the center-of-mass system or are simply attracted back by the mutual Yukawa attraction. The scattering angles of in the center-of-mass system of such cases are calculated from the axis joining the mass centers of the separated clusters. But the scattering angles in the laboratory system are not well defined.

Fusion occurs in the viscous case for collisions at 5 MeV and below and for non-zero impact parameters. For example, figure 5.15 depicts a fusion event. Here, the bombarding energy is 5 MeV per projectile nucleon with an impact parameter of 6 fm. The compound system rotates
while the velocity field becomes randomized due to viscosity so that fragmentation is prevented. Note that for the non-viscous case, the compound system breaks into two masses. Hence, comparisons of systematic hydrodynamical calculations and the energy dependence of fusion cross-section from experimental data will provide very valuable information concerning the role of dissipation mechanisms in these low-energy collisions.

Tables 5.1 and 5.2 also indicate that the scattering angles are sensitive to the presence of dissipation. The following features can be observed from a comparison of the viscous and non-viscous cases. In the center-of-mass system, the scattering angles are large ( \( >30^\circ \) ), especially for cases of small impact parameters. This directly reflect the consequence of large transverse currents of nuclear matter during the collision, as has been discussed in the previous sections. This is a characteristic feature of nuclear hydrodynamics, subject to experimental tests.

Dissipation also affects the scattering angles in the laboratory system. The angles corresponding to the viscous cases are more forward peaked than those of the non-viscous cases. This can be understood. Dissipation not only reduces the total collective kinetic energy of a fragment, it also converts a larger fraction of this energy into random motion of the fluid elements in the system, thus resulting in a smaller translational momentum of the fragment. Accordingly, given the same motion of the center of mass, the presence of a large dissipation leads to smaller laboratory scattering angles.
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<td>$\theta_{\text{scatt}}$</td>
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<td>0 fm</td>
<td>11.12</td>
</tr>
<tr>
<td></td>
<td>1.15</td>
<td>$180^\circ$</td>
</tr>
<tr>
<td></td>
<td>3 fm</td>
<td>6.52</td>
</tr>
<tr>
<td></td>
<td>6 fm</td>
<td>5.16</td>
</tr>
<tr>
<td>10</td>
<td>0 fm</td>
<td>7.64</td>
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<td></td>
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<td>4.47</td>
</tr>
<tr>
<td>20</td>
<td>0 fm</td>
<td>6.14</td>
</tr>
<tr>
<td></td>
<td>11.11</td>
<td>$180^\circ$</td>
</tr>
<tr>
<td></td>
<td>3 fm</td>
<td>4.25</td>
</tr>
<tr>
<td></td>
<td>6 fm</td>
<td>3.72</td>
</tr>
<tr>
<td>Energy</td>
<td>Fragment</td>
<td>( M , \text{cm} )</td>
</tr>
<tr>
<td>---------</td>
<td>----------</td>
<td>-----------------</td>
</tr>
<tr>
<td>5 MeV</td>
<td>0 fm</td>
<td>11.69 (+ 90^\circ)</td>
</tr>
<tr>
<td></td>
<td>3 fm</td>
<td>fusion</td>
</tr>
<tr>
<td></td>
<td>6 fm</td>
<td>fusion</td>
</tr>
<tr>
<td></td>
<td>9 fm</td>
<td>fusion</td>
</tr>
<tr>
<td>10 MeV</td>
<td>0 fm</td>
<td>10.89 (+ 110^\circ) (+ 23^\circ)</td>
</tr>
<tr>
<td></td>
<td>1 fm</td>
<td>180^\circ 0^\circ</td>
</tr>
<tr>
<td></td>
<td>3 fm</td>
<td>8.75 (+ 73^\circ)</td>
</tr>
<tr>
<td></td>
<td>6 fm</td>
<td>4.52 (+ 34^\circ) (+ 4^\circ)</td>
</tr>
<tr>
<td>15 MeV</td>
<td>0 fm</td>
<td>8.89 ( \sim 90^\circ) (+ 20^\circ)</td>
</tr>
<tr>
<td></td>
<td>1 fm</td>
<td>180^\circ 0^\circ</td>
</tr>
<tr>
<td></td>
<td>3 fm</td>
<td>6.03 (+ 66^\circ) (+ 26^\circ)</td>
</tr>
<tr>
<td></td>
<td>6 fm</td>
<td>4.89 (+ 33^\circ) (+ 13^\circ)</td>
</tr>
<tr>
<td>20 MeV</td>
<td>0 fm</td>
<td>7.50 (+ 88^\circ) (+ 26^\circ)</td>
</tr>
<tr>
<td></td>
<td>1 fm</td>
<td>180^\circ 0^\circ</td>
</tr>
<tr>
<td></td>
<td>3 fm</td>
<td>5.07 (+ 63^\circ) (+ 33^\circ)</td>
</tr>
<tr>
<td></td>
<td>6 fm</td>
<td>4.20 (+ 33^\circ) (+ 5^\circ)</td>
</tr>
<tr>
<td>30 MeV</td>
<td>0 fm</td>
<td>4.44 (+ 72^\circ) (+ 27^\circ)</td>
</tr>
<tr>
<td></td>
<td>1 fm</td>
<td>180^\circ 0^\circ</td>
</tr>
<tr>
<td></td>
<td>6 fm</td>
<td>4.57 (+ 30^\circ) (+ 16^\circ)</td>
</tr>
</tbody>
</table>
Figure 5.15 A fusion event described by 2D nuclear hydrodynamics.

\[ E = 5 \text{ MeV/AMU}, \quad b = 6 \text{ fm}, \]
\[ n = 0.57 \text{ MeV/(fm}^2\text{-c)}, \quad \tau = 2.21 \text{ MeV/(fm}^2\text{-c)}, \quad \kappa = 0.01 \text{ c/fm}^2. \]
Fig. 5.15 2D NUCLEAR HYDRODYNAMICS
E = 5 MeV  b = 6 fm (viscous)

<table>
<thead>
<tr>
<th>Density</th>
<th>Current</th>
</tr>
</thead>
<tbody>
<tr>
<td>t = 0.00 fm/c</td>
<td>t = 63.71 fm/c</td>
</tr>
<tr>
<td>t = 186.43 fm/c</td>
<td>t = 247.91 fm/c</td>
</tr>
</tbody>
</table>

- 5 fm
- 0.02 c/fm$^3$
2D NUCLEAR HYDRODYNAMICS
E = 5 MeV  b = 6 fm  (viscous)

\[ t = 368.83 \text{ fm/c} \quad t = 445.74 \text{ fm/c} \quad t = 523.72 \text{ fm/c} \]
D Medium- and High-Energy Collisions

In the two extreme domains of low energies of a few MeV per nucleon (above the Coulomb barrier) and high energies of several hundred MeV per nucleon, the characteristic features of the collisions as described by nuclear hydrodynamics are very different. In the previous section, we have found that the low-energy collisions \( (E \leq 40 \text{ MeV/AMU}) \) are characterized by fusion and fragmentation. The features of deep-inelastic events are reproduced in these calculations. These aspects of hydrodynamics can be directly checked from the observable final reaction products.

In the medium-energy region between 40 MeV/AMU and 100 MeV/AMU, some of the supersonic behaviors observed in the 30 MeV and 40 MeV reactions become more pronounced, in particular, the formation of compressed nuclear matter confined in a shock region with "sharp edges". The intermediate stages of these reactions are usually characterized by the competing processes of density clusters and particle dissociation from the compound system. The final states in the head-on and near-head-on collisions involve small (and often unstable) clusters along the transverse directions in the center-of-mass system and particles ejected along the forward and backward directions.

It is clear that the medium-energy region is a transitional region beyond which the available kinetic energy in the compound system is too large to lead to self-bound sub-systems in the final state. It is useful to determine how this boundary depends on the other kinematical factors of the reaction. It is physically meaningful to consider the total kinetic energy in the center-of-mass system which is given by:
where \( E_p \) is the initial total kinetic energy of the projectile system (in the laboratory frame); \( A_P \) and \( A_T \) are the mass numbers of the projectile and target, respectively. Here, the nucleonic mass is simply taken to be the average of the proton mass and neutron mass and the Coulomb energy is neglected. Since the binding energy per nucleon in nuclear matter is about 16 MeV, the critical bombarding energy per projectile nucleon is now estimated to be:

\[
E_{\text{crit}} / A_P = \frac{16 (A_P + A_T)^2}{A_P A_T} \text{ MeV}
\]  

At energies much greater than this limit, the compound system is unstable against complete dissociation. In the present two-dimensional asymmetric system, this energy is about 84 MeV/AMU. Indeed, calculations show that in both the viscous and non-viscous cases, the composite undergoes a complete fragmentation at the end of the reaction at a bombarding energy of 100 MeV/AMU or above.

(a) Density Compression

Figure 5.16 depicts the time evolution of a head-on collision at 250 MeV per projectile nucleon. The sequence of events shown here is very typical of the high-energy central collisions.

At about 10 fm/c, a region of compressed nuclear matter is rapidly built up from the influxes from both the projectile and target systems. The
sharp density profile of this compressed region, indicated by the closely spaced density contours, displays the early stage of a shock zone. In the center-of-mass system, this shock zone moves in the forward direction, but eventually comes to a stop at about $20 \text{fm/c}$. This is the time at which the maximum density of $\sim 0.49 \text{fm}^{-3}$ is reached. After about $30 \text{fm/c}$, the compressed system begins to relax and expand as a whole in all outward directions. There are small currents directed at the forward and backward angles in the center-of-mass system, but the largest currents lie mostly in the $\sim 120^\circ$ direction.

The central density of the expanding compound system continues to decrease in an irretrievable manner. For such case, we adopt the operational criterion that the calculation is to be terminated once the maximum density drops to below 50% of the normal nuclear matter density. This is a reasonable prescription. When the maximum density drops below 50% of the normal density, the shortest separation between nucleons increases by 26% of the equilibrium separation. Hence the average separation between nucleons increases even more than this fraction. Because of the short-range nature of the nuclear interaction, the typical distance between nucleons in this expanding system is too large for the nucleon-nucleon interaction to be operative. Accordingly, the the final system can be regarded as a free fermion gas and the result of this reaction is interpreted to be a complete dissociation into single particles. The knowledge of the density and current density fields at the end of the calculation allows one to compute the angular and energy distributions of the outgoing nucleons. The characteristic features of these distributions will be discussed in the next section.
A quantity of great interest is the compression ratio (maximum density/normal nuclear density) attained in the head-on collisions at medium and high energies. Under simplifying assumptions about the dynamical conditions and geometrical configurations of these compressed zones, analytic expressions relating various geometrical factors (such as the shock cone angle) with the compressed density by applying the so-called shock relations (Tab 78, Landau and Lifshitz). But the values of the compression ratios obtained from full 2D calculations are given in figure 5.17, where they are plotted against the bombarding energy per projectile nucleon. It is easy to see that the compression ratio increases monotonically with bombarding energy. It is also clear that the presence of dissipation decreases the density compression. Compression ratio curves for realistic three-dimensional systems will be shown in the next chapter.
Figure 5.16  Head-on collision in 2D nuclear hydrodynamics.

\[ E = 250 \text{ MeV/AMU} \quad (\eta, \zeta = 10^{-4} \text{ MeV/(fm}^2\text{-c}), \kappa = 10^{-4} \text{ c/fm}^2). \]

Figure 5.17  Compression ratios from 2D nuclear hydrodynamics for viscous and non-viscous cases.
Fig. 5.16 2D NUCLEAR HYDRODYNAMICS
E = 250 MeV b = 0 fm (non-viscous)

t = 0.00 fm/c  t = 11.95 fm/c  t = 23.95 fm/c

DENSITY

= 0.02 c/fm³

CURRENT

| t = 35.95 fm/c | t = 47.95 fm/c | t = 60.31 fm/c |

DENSITY

CURRENT
DENSITY COMPRESSION RATIO
FROM 2D NUCLEAR HYDRODYNAMICS

Fig. 5.17

DENSITY COMPRESSION RATIO

PROJECTION LABORATORY ENERGY (MeV/A)

(non-viscous)

(viscous)
(b) Angular and Energy Distributions

In order that the hydrodynamical behavior of heavy-ion collisions be susceptible to definitive tests, it is essential that measurable quantities can be constructed from the theory. In the case of medium- and high-energy reactions, the distributions of the final dissociated nucleons are directly observable and the hydrodynamical predictions of these can be confronted with experiments.

Figures 5.18a and 5.18b show the angular distributions $dN/d\theta$ and the energy distributions $d^2N/dE_d\theta$, respectively, of the head-on collision at 100 MeV per projectile nucleon. Here, $N$ is the number distribution function of the outgoing nucleons (normalized to the total particle number in the system), $\theta$ is the angle measured with respect to the beam direction and $E$ is the kinetic energy of the nucleons. In the two-dimensional geometry, $\theta$ assumes both positive and negative values ($-180^\circ \leq \theta \leq 180^\circ$). For example, in fig. 5.16, the collision axis divides each subplot into two halves. Those angles measured in the upper half above the axis are assigned positive values, while those in the lower half plane are defined to be negative.

Figure 5.18a presents angular distributions of the non-viscous case of the 100 MeV collision. In the center-of-mass distribution, there are prominent peaks at $\pm 106^\circ$, which apparently come from the transverse currents of nuclear matter during the collisions. In the laboratory distribution, due to the center-of-mass motion, the peaks occur at smaller angles and there is no contribution in the backward angle ($\pm 180^\circ$). The underlying components of the laboratory distribution are further analyzed by the energy distributions
given in fig. 5.18b. There is a small contribution in the forward angle direction between $-4^{\circ}$ and $+4^{\circ}$. In the angular range between $28^{\circ}$ and $56^{\circ}$, the spectrum has peaks at 9 MeV and 19 MeV. But in the angular range between $56^{\circ}$ and $96^{\circ}$, the energy spectrum comes from the more energetic particles of 20-50 MeV. This result, as in many other cases, indicates that the transverse currents constitute most of the energetic nucleons.

For comparison, figure 5.19a and 5.19b present the angular and energy distributions for the 200 MeV collision (non-viscous). The distributions of the 250 MeV case are given in figures 5.20a and 5.20b. With larger bombarding energies, the peaks in the angular distributions tend to shift towards larger angles and the features are increasingly clear cut.

It is of interest to study the effects of dissipation on these distributions. The viscous cases of 100 MeV, 200 MeV and 250 MeV are given in figures 5.21a-b, 5.22a-b and 5.23a-b. It can be seen that the angular distributions are sensitive to the presence of dissipation. The angular peaks tend to move towards smaller angles with dissipation and the difference in the two cases is particularly clear in the higher energies. For example, in the 250 MeV reaction with large viscosity and thermal conductivity, the distribution in the center-of-mass (fig. 5.23a) has narrow peaks at $\pm 90^{\circ}$ and a broad distribution between $90^{\circ}$ and $180^{\circ}$ which peaks at about $122^{\circ}$. In the corresponding distribution of the non-viscous case, the shape is different and it peaks at $\pm 118^{\circ}$. The positions of the angular peaks in the two cases also differ significantly. Such viscosity dependence of the angular distributions will be seen again in the next chapter on three-dimensional nuclear hydrodynamics.
Angular Distributions $dN/d\theta$ and Energy Distributions $d^2N/dE_d\theta$ from 2D Nuclear Hydrodynamics:

These distributions are given in arbitrary units. To convert any distribution into absolute units, i.e. particle number/str for the angular distribution and particle number/(MeV-str) for the energy distribution, multiply it by a factor of $180/\pi$.

**Non-Viscous Cases** ($\eta, \zeta = 10^{-4}\text{MeV}/(\text{fm}^2\cdot\text{c}), \kappa = 10^{-4}\text{c}/\text{fm}^2$)

- Figures 5.18a, 5.18b $E = 100\text{ MeV/AMU}$
- Figures 5.19a, 5.19b $E = 200\text{ MeV/AMU}$
- Figures 5.20a, 5.20b $E = 250\text{ MeV/AMU}$

**Viscous Cases** ($\eta = 0.57\text{ MeV}/(\text{fm}^2\cdot\text{c}), \zeta = 2.21\text{ MeV}/(\text{fm}^2\cdot\text{c}), \kappa = 0.01\text{c}/\text{fm}^2$)

- Figures 5.21a, 5.21b $E = 100\text{ MeV/AMU}$
- Figures 5.22a, 5.22b $E = 200\text{ MeV/AMU}$
- Figures 5.23a, 5.23b $E = 250\text{ MeV/AMU}$
Fig. 5.18a  
2D NUCLEAR HYDRODYNAMICS  
E=100 MeV  b=0 fm (non-viscous)
Fig. 5.18b

2D NUCLEAR HYDRODYNAMICS
E=100 MeV b=0 fm (non-viscous)

\[ -4^\circ < \theta < 4^\circ \]

\[ 28^\circ < \theta < 56^\circ \]

\[ 56^\circ < \theta < 96^\circ \]

\[ d^2 N / dE d\theta \]

\[ E_{LAB} (\text{MeV}) \]
Fig. 5.19a  2D NUCLEAR HYDRODYNAMICS
E = 200 MeV  b = 0 fm (non-viscous)
Fig. 5.19b

2D NUCLEAR HYDRODYNAMICS
E=200 MeV b=0 fm (non-viscous)

-4° < θ < 4°

8° < θ < 88°

88° < θ < 104°
Fig. 5.20a  2D NUCLEAR HYDRODYNAMICS:
E=250 MeV b=0 fm (non-viscous)

\[
\frac{dN}{d\theta}
\]

\[
\theta_{CM} (\text{deg})
\]

\[
\frac{dN}{d\theta}
\]

\[
\theta_{LAB} (\text{deg})
\]

\[
t=60.30 \text{ fm/c}
\]
Fig. 5.20b

2D NUCLEAR HYDRODYNAMICS
E=250 MeV b=0fm (non-viscous)

-4°<θ<4°

12°<θ<48°

48°<θ<88°

88°<θ<116°

\( \frac{d^2N}{dEd\theta} \)

\( E_{LAB} (MeV) \)
Fig. 5.21a  2D NUCLEAR HYDRODYNAMICS
E=100 MeV b=0 fm (viscous)

\[ \frac{dN}{dt} \]

\[ E = 100 \text{ MeV}, \quad b = 0 \text{ fm (viscous)} \]

\[ \theta_{CM} (\text{deg}) \]

\[ \theta_{LAB} (\text{deg}) \]

\[ t = 72.90 \text{ fm/c} \]
Fig. 5.21b

2D NUCLEAR HYDRODYNAMICS
E=100 MeV b=0 fm (viscous)

-4° < θ < 4°

-4° < θ < 36°

36° < θ < 64°

64° < θ < 88°

\[ \frac{d^2 N}{dE d\theta} \]
Fig. 5.22a 2D NUCLEAR HYDRODYNAMICS
$E=200\ \text{MeV} \ \ b=0\ \text{fm/c (viscous)}$

$\frac{dN}{d\theta}$

$\theta_{\text{CM}}\ (\text{deg})$

$\theta_{\text{LAB}}\ (\text{deg}) \ t=54.0\ \text{fm/c}$
Fig. 5.22b

2D NUCLEAR HYDRODYNAMICS

\[ E = 200 \text{ MeV} \quad b = 0 \text{ fm} \quad \text{(viscous)} \]

\[ -4^\circ < \theta < 4^\circ \]

\[ 8^\circ < \theta < 48^\circ \]

\[ 48^\circ < \theta < 68^\circ \]

\[ 68^\circ < \theta < 104^\circ \]

\( E_{\text{LAB}} \) (MeV)
Fig. 5.23a  2D NUCLEAR HYDRODYNAMICS
E=250 MeV  b=0 fm (viscous)
Fig. 5.23b

2D NUCLEAR HYDRODYNAMICS
E=250 MeV b=0 fm (viscous)

\[ \frac{d^2N}{dE d\theta} \]

\[ E_{LAB} (\text{MeV}) \]

-4° < \theta < 4°
52° < \theta < 80°
56° < \theta < 124°
80° < \theta < 124°
CHAPTER VI

CENTRAL COLLISIONS

BY

THREE-DIMENSIONAL NUCLEAR HYDRODYNAMICS
A Introductory Remarks

In this chapter, calculations which solve the problem of head-on collisions within a three-dimensional nuclear hydrodynamical model will be discussed.

It has been argued from the standpoint of the interaction time that not every nuclear collision may be properly treated by nuclear hydrodynamics. In the applications of this macroscopic theory, attention should be focused on the head-on and near-head-on cases.

In the general two-dimensional problem, solved and discussed in the previous chapter, the techniques developed there, in principle, can be extended in a straightforward manner to the completely three-dimensional calculation. The main consideration of such extension is an unavoidable and formidable computer storage problem. In practice, a compromise can be made by a decrease of the number of grid-points in each direction with a corresponding increase of the mesh-size, this being done at the expense of the accuracy. Preliminary results from such investigations were reported by Tang and Wong (Ta-b 77) and Maruhn (Ma-d 77).

In the exactly central collisions, however, this complex computational problem can be circumvented by means of an exploitation of the axial symmetry which reduces substantially the original equations to an effective set of two-dimensional equations. This has been achieved without any sacrifice of speed and accuracy (Appendix V). However, it can be expected that the conclusions drawn from these central collisions can be representative of the other cases of non-zero, but small impact parameters.
It should be noted in passing that an approximate method based on an
approximate axial symmetry has been developed to solve near-head-on collisions.
A brief discussion on this method is given in Appendix V. It will be tested, with
the hope of extending the present calculations to other $\theta$-values near zero.

The discussions in the present chapter will concentrate on the head-on
collisions of $^{20}\text{Ne} + ^{197}\text{Au}$ and also $^{208}\text{Pb} + ^{208}\text{Pb}$ at various bombarding
energies from 50 MeV up to 400 MeV per projectile nucleon. In the sub­
sequent discussions, the former case will be referred to as the "asymmetric"
collision (or system) and the latter, the "symmetric" collision (or system).

The notations in this chapter are essentially the same as those given in
the previous two chapters. To study the effects of dissipation on the dynamics,
calculations based on two different sets of transport coefficients are compared.

They are:

\begin{align*}
\eta, \zeta &= 10^{-4} \text{MeV/(fm}^2\text{-c)}, \\
\kappa &= 10^{-4} \text{c/fm}^2
\end{align*}

(VI-1)

which, as in the two-dimensional studies, will be referred to as the "non-
viscous" or "small viscosity" case, and the finite values of transport coefficients
estimated in Chapter IV,

\begin{align*}
\eta &= 0.75 \text{MeV/(fm}^2\text{-c)}, \\
\zeta &= 18.76 \text{MeV/(fm}^2\text{-c)}, \\
\kappa &= 0.014 \text{c/fm}^2
\end{align*}

(VI-2)

which will be referred to as the "viscous" or "large viscosity" case.
Figure 6.1 depicts the time evolution of the density field in the center-of-mass system for the collision for 250 MeV per projectile \( \text{Ne} \) nucleon on \( \text{Au} \). The calculation is done with the small values of transport coefficients given in eq. (VI-1).

At about 10 fm/c, a region of compressed nuclear matter is formed and has attained a density of about 1.5 times the normal value (0.15 fm\(^{-3}\)). The density continues rising in this region until a maximum value of 0.35 fm\(^{-3}\) is reached at about 25 fm/c. During the stage of compression, the density profile of this region has a very sharp jump as one goes from this region to the rest of the system of normal density. The thickness of the shock front is only slightly greater than the mesh size of 0.6 fm. The compressed region undergoes a forward displacement to the right as a whole as time proceeds. At about 25 fm/c, the Ne nucleus is entirely embedded in the target system and the maximum density of 0.35 fm\(^{-3}\) is also reached. The compressed region maintains this maximum density for a few fm/c and then begins to relax. At about 32 fm/c, due to the size of the target and the consequent time delay in transmitting the impulse of the collision, roughly half of the target system is still unperturbed. There are currents going into the compressed region directed from the target. However, this current encounters the forward propagation of the shock region and gives rise to an ejection of nuclear matter along a "side-wing" direction which is approximately 120° in the center-of-mass system. This "side-wing" expands outwards as the density in the compressed region drops from the maximum value. After 60 fm/c,
the central density of the composite system drops back to the normal value and continues to decrease monotonically in time, while the volume of the system expands outwards irretrievably. This feature in the collision is similar to that observed in the two-dimensional studies (Chapter V) and also to what was obtained previously (Am-b 75). As in the two-dimensional studies, we adopt the operational criteria that for such a case, the calculation is to be terminated once the maximum density drops to below 50% of the normal nuclear matter density. At this stage, the system can be considered as a dilute gas, subject only to the Coulomb interaction. The final result of this reaction is interpreted as a complete dissociation into single particles. The knowledge of the density and velocity fields at the end of the calculation allows one to compute the angular and energy distributions of the outgoing nucleons. For the present calculations of angular and energy distributions, we even neglect the final state Coulomb interaction and the thermal velocities of the particles. They will probably shift and broaden the peaks in the angular and energy distributions slightly, but will not change the main features of the distributions.

Figure 6.2 shows the angular distributions $dN/d\Omega$ in the center-of-mass and laboratory systems. The main structure of the center-of-mass distribution has prominent forward and backward peaks which are, respectively, one and two orders of magnitude higher than the rest of the distribution. There is also a peak at about $90^\circ$ with a width of about $10^\circ$ and a rather broad peak which extends from $100^\circ$ to $120^\circ$ and centers at about $110^\circ$. In the angular distribution in the laboratory system, apart from the prominent
peaks in the forward and backward directions (which are, respectively, two and one orders of magnitude higher than the rest), there is a rather broad maxima which extends from $66^\circ$ to $100^\circ$ and peaks at $85^\circ$. The underlying components of this distribution can be analyzed in greater detail from the energy distributions $2N/dEdC$, given in figures 6.3a-6.3c. The forward peak comes from very slow particles with less than $5\,\text{MeV}$. They arise as a consequence of the stopping of the small projectile nucleus by a much larger target. The backward peak in the laboratory system (fig. 6.3a) consists mainly of low-energy particles with less than $10\,\text{MeV}$ and also some slightly more energetic particles with more than $15\,\text{MeV}$. This arises from the expansion of the shock region to the backward direction. As the expansion of the shock region during the early collision stage ($\sim 25\,\text{fm/c}$) is unhindered in this direction (in contrast to the forward direction), slightly more energetic particles - of about $15\,\text{MeV}$ up to $30\,\text{MeV}$ - are also found in the backward direction, whereas the forward peak is restricted to particles with below $5\,\text{MeV}$. Figures 6.3a-6.3b are the calculated energy distributions covering the angular range from $74^\circ$ to $100^\circ$ with an angular bin of $10^\circ$ and less. Figure 6.3c is the calculated energy distribution for the range between $66^\circ$ and $100^\circ$. The spectrum of particles of the last distribution falls roughly into two groups. There is a large group of particles below $40\,\text{MeV}$ which shows peaks at about $19\,\text{MeV}$ and $31\,\text{MeV}$, and a much smaller group of particles above $50\,\text{MeV}$ which peaks at about $65\,\text{MeV}$.

To study the effects of viscosity and thermal conductivity on the dynamics, the calculation for $^{20}\text{Ne} + ^{197}\text{Au}$ at $250\,\text{MeV/AMU}$ is repeated with the larger values of the transport coefficients given in eq. (VI-2). The
time evolution of the density field for this calculation is depicted in figure 6.4. In this case, though some features appear to be similar to the small viscosity case, there are, in fact, several distinct differences. Firstly, in the small viscosity case, the thermal energy remains negligible (about 1 MeV at the most) throughout the entire reaction. In contrast, the larger viscosity and thermal conductivity lead to the generation of considerable thermal energy during the compression stage, with a maximum value of about 1400 MeV. The second difference lies in the density field. With large viscosity, the density profile is smoother and the maximum density reached (0.23 fm\(^{-3}\)) is smaller. That this is a viscosity effect can be understood as a consequence of the Navier-Stokes equations. There, the compressional viscosity gives a diffusion term associated with the current density field. Therefore, the greater the compressional viscosity, the greater is the rate of diffusion of the current density field. It is clear that when the compressional viscosity is large, any effect of density accumulation in a region due to the influx of nuclear matter from the outside will be reduced due to a competing diffusion process.

With small viscosity, the central density begins to drop below 0.075 fm\(^{-3}\) at about 82 fm/c, while with large viscosity, it begins to drop below this value at about 97 fm/c. These are the times at which the angular and energy distributions are extrapolated from the corresponding calculations. Some interesting insight can be gained from detailed comparisons of these distributions for the non-viscous and viscous cases. For the large viscosity, the angular distributions (in both the center-of-mass and laboratory
systems) are given in figure 6.5. The center-of-mass distribution has pronounced and narrow peaks in the forward and backward directions with a width of about $2^\circ$. There is also a peak at $90^\circ$ with a width of about $15^\circ$. This is a simpler structure than the corresponding distribution in the small viscosity case (fig. 6.2) for which there is a broad spectrum between $90^\circ$ and $135^\circ$. The laboratory angular distribution has peaks in the forward and backward directions, and also a broad distribution between $50^\circ$ and $90^\circ$ with a peak at about $67^\circ$. In fact, except for the positions of the various peaks which are shifted by about $18^\circ$, this is very similar in shape to the corresponding distribution between $60^\circ$ and $100^\circ$ of the "small" viscosity calculation. (Compare the laboratory distributions in fig. 6.2 and 6.5).

Accordingly, the effect of viscosity in this case appears to give an overall shift of the laboratory angular distributions towards smaller angles. This effect of the dissipation will be systematically examined at other energies and will be discussed in the following paragraphs.

In the viscous case, the energy distributions in the laboratory system of the forward and backward peaks come from slow particles of below $5$ MeV and $10$ MeV, respectively. Figure 6.6 presents the energy distribution for the angular range between $58^\circ$ and $90^\circ$. This is a broad spectrum which first peaks at about $15$ MeV, but drops slowly after that in an approximately exponential manner and extends to about $70$ MeV. One can compare this energy distribution for the viscous case with that for the non-viscous case (fig. 6.3c). Although the latter has also a peak at roughly between $15$ MeV and $30$ MeV, the shape is entirely different. In the non-viscous case, the
distribution is narrower and peaks at about 40 MeV and 65 MeV.
Figure 6.1  Time evolution of the density field for $^{20}\text{Ne} + ^{197}\text{Au}$ at 250 MeV/AMU with small viscosity.

Figure 6.2  Angular distributions for $^{20}\text{Ne} + ^{197}\text{Au}$ at 250 MeV/AMU with small viscosity. This and all other angular distributions hereafter are given in the same, but arbitrary units. To convert a given angular distribution into the absolute unit (nucleon number/sr), multiply it by the factor $\frac{180}{\pi}$.

Figures 6.3a, 6.3b, 6.3c  
Energy distributions in the laboratory system for $^{20}\text{Ne} + ^{197}\text{Au}$ at 250 MeV/AMU with small viscosity. This and all other energy distributions are given in the same, but arbitrary units. To convert a given energy distribution into the absolute unit (nucleon number/(MeV-sr)), multiply it by a factor of $\frac{180}{\pi}$.

Figure 6.4  Time evolution of the density field for $^{20}\text{Ne} + ^{197}\text{Au}$ at 250 MeV/AMU with large viscosity.

Figure 6.5  Angular distributions for $^{20}\text{Ne} + ^{197}\text{Au}$ at 250 MeV/AMU with large viscosity.

Figure 6.6  Energy distribution for $^{20}\text{Ne} + ^{197}\text{Au}$ at 250 MeV/AMU with large viscosity.
Fig. 6.1

$^{20}\text{Ne}+^{197}\text{Au}$

$E=250\text{ MeV} \ b=0\text{ fm (non-viscous)}$

t = 0.00 fm/c  

$E > \frac{5}{\mu}\text{MeV}$

t = 10.82 fm/c  

t = 21.41 fm/c  

$E > \frac{5}{\mu}\text{MeV}$

t = 32.00 fm/c  

t = 42.73 fm/c  

t = 53.39 fm/c  

$E > \frac{5}{\mu}\text{MeV}$

t = 55.37 fm/c  

t = 58.90 fm/c  

t = 62.43 fm/c  

$E > \frac{5}{\mu}\text{MeV}$

t = 66.41 fm/c  

t = 71.71 fm/c  

t = 76.84 fm/c
$^{20}\text{Ne} + ^{197}\text{Au}$

$E = 250 \text{ MeV} \ b = 0 \text{ fm (non-viscous)}$

\[\theta_{\text{LAB}} (\text{deg}) \quad t = 81.79 \text{ fm/c}\]
Fig. 6.3a

$^{20}$Ne + $^{197}$Au

$E = 250$ MeV $b = 0$ fm (non-viscous)

$174.0^\circ < \theta < 180.0^\circ$

$74.0^\circ < \theta < 84.0^\circ$

$d^2N/dE d\Omega$

$E_{LAB}$ $t = 81.79$ fm/c
Fig. 6.3b

$^{20}\text{Ne} + ^{197}\text{Au}$

$E=250$ MeV  $b=0$ fm (non-viscous)

$84.0^\circ < \phi < 90.0^\circ$

$90.0^\circ < \phi < 100.0^\circ$

$E_{\text{LAB}}$

$t=81.79$ fm/c
Fig. 6.3c

$^{20}\text{Ne} + ^{197}\text{Au}$

$E = 250 \text{ MeV}$ $b = 0 \text{ fm (non-viscous)}$

$E_{\text{LAB}} (\text{MeV})$

t = 81.79 \text{ fm/c}$
$^{20}\text{Ne} + ^{197}\text{Au}$

$E = 250 \text{ MeV} \quad b = 0 \text{ fm (viscous)}$

Fig. 6.4
Fig. 6.5

\[ ^{20}\text{Ne} + ^{197}\text{Au} \]
\[ E = 250 \text{ MeV} \quad b = 0 \text{ fm (viscous)} \]

\[
\begin{array}{c}
\begin{array}{c}
\frac{dN}{d\Omega} \\
\theta_{\text{CM (deg)}}
\end{array}
\end{array}
\]
Fig. 6.6

\[ \text{20}^{\text{Ne}} + \text{197}^{\text{Au}} \]

\[ E = 250 \text{ MeV} \; b = 0 \text{ fm (viscous)} \]

\[ 58.0^\circ < \phi < 90.0^\circ \]

\[ d^2N/dEd\Omega \]

\[ E_{\text{LAB}} \text{ (MeV)} \]

\[ t = 96.87 \text{ fm/c} \]
Figure 6.7 depicts the reaction of $^{208}\text{Pb} + ^{208}\text{Pb}$ at 100 MeV per projectile nucleon and for the small values of viscosity and thermal conductivity. At about 18 fm/c, a central region of compressed nuclear matter is formed ($\sim 0.25\text{fm}^{-3}$). This is the beginning of the compression stage. The rest of the projectile and target nuclei are essentially unperturbed. They continue to come toward each other and maintain a constant influx of nuclear matter into the central compressed region. The maximum density reached in this reaction is about $0.30\text{fm}^{-3}$, which is two times the normal nuclear density. Similar to the non-viscous case of $^{20}\text{Ne} + ^{197}\text{Au}$ discussed earlier, the density profile of the shock region also has very sharp edges. A detailed study of the intermediate steps between 18 fm/c and 82 fm/c shows that the diffuseness of the "edge" of this shock region actually undergoes oscillation. A hint of such oscillation can be found by comparing the level spacing of the contours from time steps $35.76\text{fm/c}$ to $107.06\text{fm/c}$ in fig. 6.7. This is due to the tendency of the central region to expand outwards even during the compression stage. Beginning at about 36 fm/c, there is ejection of nuclear matter in the $90^\circ$ direction in the center-of-mass system and eventually the composite system becomes disc-like before the final overall expansion. The angular distributions in the center-of-mass and laboratory systems are given in figure 6.8. The distribution in the center-of-mass system has peaks at the forward and backward directions and at $90^\circ$. The main structure of the laboratory distribution has peaks at the forward direction at $7^\circ$ and at $39^\circ$. Figure 6.9 shows the energy distributions in the
laboratory system for the forward angle direction and for the angular range between $35^\circ$ and $55^\circ$. The energy distribution for the forward angle has a group of particles of between 12 MeV and 38 MeV, with a peak at about 23 MeV. The energy distribution for the angular range between $35^\circ$ and $55^\circ$ has a broad spectrum from 30 MeV up to 60 MeV, with a peak at about 39 MeV.

Figure 6.10 depicts the reaction $^{208}\text{Pb} + ^{208}\text{Pb}$ at 100 MeV/AMU with the set of large coefficients of viscosity and thermal conductivity. The thermal energy generated during the compression stage ($\sim \text{20 fm/c - 70 fm/c}$) reaches a maximum of about 1300 MeV. The maximum density attained in this reaction is $0.23 \text{ fm}^{-3}$. A comparison of the density plots with those of the small viscosity case (fig. 6.7) reveals some differences. In the viscous case, the density profile remains very smooth during the entire compression stage. The maximum density is smaller than that in the viscous case. But once the maximum density is reached at about 36 fm/c, the system sustains this density for a longer period ($\sim \text{36 fm/c - 68 fm/c}$). The center-of-mass angular distribution (fig. 6.11) is very similar to the small viscosity case, that is, with sharp peaks at $90^\circ$, $0^\circ$ and $180^\circ$. However, the laboratory distribution (fig. 6.11) displays some differences. The distribution between $30^\circ$ and $70^\circ$ has a shape which is roughly the same as that between $20^\circ$ and $60^\circ$ for the small viscosity case, with the exception that the position of the peaks are all shifted to a slightly larger angle.

The energy distributions in the laboratory system are given in figures 6.12a and 6.12b. The forward peak comes from a relatively narrow spectrum
of particles which peaks at 23 MeV with a width of about 5 MeV. In the angular range between $24^\circ$ and $44^\circ$, the particles have kinetic energies between 25 MeV and 50 MeV, with the energy spectrum peaking at about 31 MeV. The energy distribution for the angular range between $44^\circ$ and $64^\circ$ (fig. 6.12b) is a broad spectrum extending from 30 MeV up to 100 MeV, peaking at about 5 MeV. There is also a small group of energetic particles of 90–100 MeV. This broad distribution is in contrast to the relatively narrower distribution in the non-viscous case for the angular range between $35^\circ$ and $55^\circ$. 
Figure 6.7  Time evolution of the density field for $^{208}\text{Pb} + ^{208}\text{Pb}$ at 100 MeV/AMU with small viscosity.

Figure 6.8  Angular distributions for $^{208}\text{Pb} + ^{208}\text{Pb}$ at 100 MeV/AMU with small viscosity.

Figure 6.9  Energy distributions in the laboratory system for $^{208}\text{Pb} + ^{208}\text{Pb}$ at 100 MeV/AMU with small viscosity.

Figure 6.10  Time evolution of the density field for $^{208}\text{Pb} + ^{208}\text{Pb}$ at 100 MeV/AMU with large viscosity.

Figure 6.11  Angular distributions for $^{208}\text{Pb} + ^{208}\text{Pb}$ at 100 MeV/AMU with large viscosity.

Figures 6.12a, 6.12b  Energy distributions in the laboratory system for $^{208}\text{Pb} + ^{208}\text{Pb}$ at 100 MeV/AMU with large viscosity.
Fig. 6.7

$^{208}\text{Pb} + ^{208}\text{Pb}$

$E = 100 \text{ MeV}, b = 0 \text{ fm} \ (\text{non-viscous})$

t = 0.00 \text{ fm/c} 

$\begin{array}{c}
t = 18.24 \text{ fm/c} \\
t = 35.76 \text{ fm/c} \\
t = 50.26 \text{ fm/c} \\
t = 66.32 \text{ fm/c} \\
t = 81.58 \text{ fm/c} \\
t = 100.99 \text{ fm/c} \\
t = 104.00 \text{ fm/c} \\
t = 107.06 \text{ fm/c} \\
t = 110.06 \text{ fm/c} \\
t = 113.06 \text{ fm/c} \\
t = 116.06 \text{ fm/c}
\end{array}$
Fig. 6.8

$^{208}\text{Pb} + ^{208}\text{Pb}$

$E=100$ MeV $b=0$ fm (non-viscous)

$\theta_{\text{CM}}$ (deg)

$\theta_{\text{LAB}}$ (deg) $t=122.06$ fm/c
Fig. 6.9

$^{208}\text{Pb} + ^{208}\text{Pb}$

$E = 100 \text{ MeV} \quad b = 0 \text{ fm (non-viscous)}$

$0.0^\circ < \theta < 2.0^\circ$

$35.0^\circ < \theta < 55.0^\circ$

$E_{\text{LAB}}(\text{MeV}) \quad t = 122.06 \text{ fm/c}$
Fig. 6.10

$^{208}\text{Pb} + ^{208}\text{Pb}$

$E = 100 \text{ MeV} \ b = 0 \text{ fm (viscous)}$

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<td>119.97</td>
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</tbody>
</table>
Fig. 6.11

$^{208}\text{Pb} + ^{208}\text{Pb}$

$E = 100\text{ MeV}$ $b = 0\text{ fm (viscous)}$

$\theta_{\text{CM}}$ (deg)

$\theta_{\text{LAB}}$ (deg) $t = 133.54\text{ fm/c}$
Fig. 6.12a

\[ ^{208}\text{Pb} + ^{208}\text{Pb} \]

\( E=100 \text{ MeV} \ b=0 \text{ fm (viscous)} \)

\[ 0.0^\circ \lessgtr \theta < 2.0^\circ \]

\[ 21.0^\circ \lessgtr \theta < 44.0^\circ \]
Fig. 6.12b

$^{208}\text{Pb} + ^{208}\text{Pb}$

$E = 100\text{ MeV} \ b = 0\text{ fm (viscous)}$

$44.0^\circ < \theta < 64.0^\circ$

$d^2N/dE d\Omega$

$E_{\text{LAB}} (\text{MeV})$

$t = 133.54\text{ fm/c}$
D Viscosity and Energy Dependence of Angular Distributions

In the detailed study of $^{20}\text{Ne} + ^{197}\text{Au}$ at 250 MeV/AMU and also of $^{208}\text{Pb} + ^{208}\text{Pb}$ at 100 MeV/AMU, the angular distributions have been found to be sensitive to the presence of dissipation. To firmly establish this dependence as a special feature of the hydrodynamical description and also to study further the energy dependence of the angular distributions, the calculations have been extended to other bombarding energies.

Figures 6.13a and 6.13b show the angular distributions for the reaction $^{20}\text{Ne} + ^{197}\text{Ne}$ at 100 MeV/AMU for the viscous and non-viscous cases, respectively. In the viscous case (fig. 6.13a), apart from the forward and backward peaks, there is a broad distribution extending from about $10^\circ$ up to $75^\circ$. In the non-viscous case (fig. 6.13b), there are peaks in the forward angles and the distribution between $\sim 50^\circ$ and $\sim 100^\circ$ is approximately symmetrical and peaks at about $73^\circ$. This distribution is distinctly different from that of the viscous case. Similarly, other calculations for the high-energy collisions of Ne on Au result in viscosity-dependent angular distributions. (For example, the 400 MeV calculation has been reported in Tab 79.) In general, the energy distributions are broadened by the presence of large dissipation.

Figures 6.14a and 6.14b are the angular distributions of $^{208}\text{Pb} + ^{208}\text{Pb}$ at 50 MeV/AMU for the viscous and non-viscous cases, respectively. Figures 6.15a and 6.15b are the angular distributions for the same system at 250 MeV/AMU, and figures 6.16a and 6.16b, those at 400 MeV/AMU. Apart from differences in the detailed shapes and heights of the peaks, the
positions of the angular peaks are also slightly different in the viscous and non-viscous cases.

From these systematic comparisons of the results of $^{20}\text{Ne} + ^{197}\text{Ne}$ and $^{208}\text{Pb} + ^{208}\text{Pb}$, one observes that the shift of the positions of the angular peaks due to viscosity is a clear cut feature, especially for the asymmetric system. The broadening of the underlying energy spectra is also a consequence of dissipation in the system.
Figures 6.13a, 6.13b
Laboratory angular distributions for \(^{20}\text{Ne} + ^{197}\text{Au}\) at 100 MeV/AMU with large and small viscosity.

Figures 6.14a, 6.14b
Laboratory angular distributions for \(^{208}\text{Pb} + ^{208}\text{Pb}\) at 50 MeV/AMU with large and small viscosity.

Figures 6.15a, 6.15b
Laboratory angular distributions for \(^{208}\text{Pb} + ^{208}\text{Pb}\) at 250 MeV/AMU with large and small viscosity.

Figures 6.16a, 6.16b
Laboratory angular distributions for \(^{208}\text{Pb} + ^{208}\text{Pb}\) at 400 MeV/AMU with large and small viscosity.
$^{20}\text{Ne} + ^{197}\text{Au}$

$E = 100 \text{ MeV} \ b = 0 \text{ fm (viscous)}$

$\theta_{\text{LAB}} (\text{deg})$ 

$t = 223.40 \text{ fm/c}$
Fig. 6.13b

$^{20}\text{Ne} + ^{197}\text{Au}$

$E = 100 \text{ MeV} \, b = 0 \text{ fm (non-viscous)}$

$\frac{dN}{d\Omega}$ vs $\theta_{LAB}(\text{deg})$

$t = 132.27 \text{ fm/c}$
Fig. 6.14a

$^{208}\text{Pb} + ^{208}\text{Pb}$

$E = 50 \text{ MeV} \ b = 0 \text{ fm (viscous)}$

$\frac{dN}{d\Omega}$

$\theta_{\text{LAB}} (\text{deg})$

t = 217.16 \text{ fm/c}
Fig. 6.14b

$^{208}\text{Pb} + ^{208}\text{Pb}$

$E = 50 \text{ MeV}$  $b = 0 \text{ fm}$ (non-viscous)

![Graph showing angular distribution of reaction products.]

$dN/d\Omega$

$\theta_{\text{LAB}} \text{ (deg)}$  
$t = 226.50 \text{ fm/c}$

$3.24 \times 10^3$
Fig. 6.15a

$^{208}\text{Pb} + ^{208}\text{Pb}$

$E=250 \text{ MeV} \ b=0 \text{ fm (viscous)}$

$1.39 \times 10^2$

$\frac{dN}{d\Omega}$

$7.5$

$\text{LAB (deg)}$

$t=90.58 \text{ fm/c}$
Fig. 6.15b

\[ 208_{\text{Pb}} + 208_{\text{Pb}} \]

\[ E = 250 \text{ MeV} \quad b = 0 \text{ fm (non-viscous)} \]

\[ \begin{align*}
    \frac{dN}{d\Omega} \\
    \theta_{\text{LAB (deg)}} \\
    t = 79.98 \text{ fm/c}
\end{align*} \]
Fig. 6.16a

$^{208}$Pb + $^{208}$Pb

$E = 400$ MeV $b = 0$ fm (viscous)

$\theta_{LAB}$ (deg)

$\frac{dN}{d\Omega}$
Fig. 6.16b

$^{208}\text{Pb} + ^{208}\text{Pb}$

$E = 400\text{ MeV}$ $b = 0\text{ fm (non-viscous)}$

$\theta_{\text{LAB}}(\text{deg})$

$t = 66.23\text{ fm/c}$
E Compression Ratio and Other Features

Since the formation of compressed nuclear matter and its consequences are among the most important motivations for medium- and high-energy heavy-ion physics, we shall now summarize some of the results with these questions in mind. The main results of various three-dimensional calculations are compactly presented in figure 6.17. The curves are the compression-ratios (maximum density/normal density) achieved in the $^{20}\text{Ne} + ^{197}\text{Au}$ and $^{208}\text{Pb} + ^{208}\text{Pb}$ reactions, plotted against the bombarding energy. Each curve corresponds to one set of transport coefficients. It is clear from the figure that the compression ratio increases monotonically with increasing energy. Also, a high compression appears to be favored more by large projectile-target systems. This can be understood as being due to, at least partly, greater energies available in the center-of-mass system.

The role played by dissipation is rather significant. Large viscosity and thermal conductivity results in the production of considerable thermal energy. Not only is the density profile smoothed out by a large viscosity, but the maximum density reached is also substantially reduced.

Though this chapter is mainly concerned with the high-energy reactions, it is noteworthy to remark that there are also many other special features of nuclear hydrodynamics which manifest themselves, particularly at lower energies. A notable example is that for an expanding compound system, there is a tendency in the density field to form clusters when the colliding energy is less than a certain limit. Above this energy, the nuclear matter is so compressed in the shock region that when it expands, it keeps
on expanding without collapsing back into the clusters with normal nuclear matter density. This kind of behavior can be understood in terms of the shape of the equation of state (internal energy function $E(n)$). For small deviations of the density from equilibrium value, $E(n)$ is approximately a parabola. However, for large values of deviation, it has a high wall for large densities (unless there are other minima) but an open and relatively flat region for low densities. Highly compressed nuclear matter, after relaxing to the equilibrium density and going to the low density region, will have too high a kinetic energy in the density degree of freedom to come to a turning point.

Systematic studies of a two-dimensional model indicate that an energy of $E \geq 60 \text{ MeV/AMU}$ lead to a dissociation of the composite system, while collisions at $E \leq 40 \text{ MeV/AMU}$ lead to a fragmentation into self-bound clusters. In the three-dimensional head-on collision, a clustering in the direction of the collision axis corresponds to a fission-like behavior, but a clustering perpendicular to the collision axis corresponds to the formation of a ring of nuclear matter. In order to investigate this aspect, calculations for $^{20}\text{Ne} + ^{197}\text{Au}$ at 50 MeV with zero impact parameter are performed. Figure 6.18 depicts the result for the non-viscous case. At about $70 \text{ fm/c}$, the composite system relaxes in response to the compression of nuclear matter. At about $114 \text{ fm/c}$, we begin to see some clustering effect in the radial direction. At $198 \text{ fm/c}$, this eventually leads to the formation of a vortex ring of nuclear matter, with a density of about $0.075 \text{ fm}^{-3}$. The central cluster has a normal density. At the end of the calculation, this
torus goes toward the right in the center-of-mass system, while the central cluster goes in the opposite direction. The calculation is terminated here since we do not know whether this configuration is a stable one. Non-axial sausage-type instability is a very important mode (Wo 73). Future calculations for very small but non-zero impact parameters will help to determine more definitively the detailed dynamics beyond this stage. The same calculation done with a large viscosity shows that at the comparable time scale, the "clustering effect" and the ring formation are much less pronounced.

When the energy of the projectile $^{20}$Ne is increased to 100 MeV/AMU, the density contour indicates that the density of the compound system keeps on decreasing after the shock region relaxes. There is no restriction of the nuclear matter density back to the equilibrium value, as was the case for the central cluster in the case of 50 MeV/AMU. One infers that 100 MeV per projectile nucleon is approximately the density above which a head-on collision of the projectile $^{20}$Ne with $^{197}$Au will lead to a complete dissociation of the composite system. For the collision of $^{208}$Pb on $^{208}$Pb, one finds that a head-on collision with an energy above 50 MeV/AMU will lead to a complete dissociation of the composite system.
Figure 6.17  Density compression ratios from 3D nuclear hydrodynamics.

Figure 6.18  Time evolution of the density field for $^{20}$Ne + $^{197}$Au at 50 MeV/AMU with small viscosity.
Density Compression Ratio from 3D Nuclear Hydrodynamics

\[ \frac{n_{\text{max}}}{0.15 \text{ fm}^3} \]

- \[ ^{208}\text{Pb} + ^{208}\text{Pb} \] (non-viscous)
- \[ ^{20}\text{Ne} + ^{197}\text{Au} \] (non-viscous)
- \[ ^{208}\text{Pb} + ^{208}\text{Pb} \] (viscous)
- \[ ^{20}\text{Ne} + ^{197}\text{Au} \] (viscous)

Fig. 6.17
**Fig. 6.18**

$^{20}\text{Ne} + ^{197}\text{Au}$

$E = 50 \text{ MeV} \quad b = 0 \text{ fm (non-viscous)}$

$t = 0.00 \text{ fm/c}$  
$t = 38.03 \text{ fm/c}$  
$t = 74.19 \text{ fm/c}$

$t = 114.13 \text{ fm/c}$  
$t = 155.31 \text{ fm/c}$  
$t = 198.90 \text{ fm/c}$
CHAPTER VII

CONCLUSIONS
A Summary

The main results obtained in this thesis involve studies of nuclear dynamics along two directions.

By means of the field-theoretic language of Green's functions, the microscopic many-body theory has been examined. This has led to the formulation of an extended time-dependent Hartree-Fock approximation which includes particle collisions due to the residual interaction and contains the usual mean-field theory of time-dependent Hartree-Fock approximation as a collisionless limit. This generalization not only gives a better understanding of TDHF, but also provides valuable insight into the questions concerning particle-particle correlations in the dynamical problems of nuclear systems.

The formal structure and the physical consequences of ETDHF have been investigated, with the following main results:

1) The basic equations of motion can be conveniently expressed as a set of modified time-dependent Hartree-Fock equations, coupled to a master equation for the occupation probability amplitudes of the single-particle states. In principle, the entire set of single-particle states is included in ETDHF; a particle can be scattered into or from any state provided energy conservation is satisfied.

2) The master equation, as an analogue of the quantum Boltzmann equation in configuration space, provides a convenient basis from which to study dissipative phenomena of finite systems. Concepts of entropy and local and global thermal equilibrium have been quantitatively introduced and an H-theorem has been proven. Analytic solutions to simple cases of the master
equation have been obtained. They illustrate the salient features of the approach of a finite nuclear system to thermal equilibrium by means of new level-crossing formulas. In addition, estimates have been made on the thermal relaxation time scale of excited heavy ions within the framework of ETDHF.

3) Macroscopic equations in a hydrodynamical form have been obtained from ETDHF. The effects of particle collisions are demonstrated by terms derived from a collision matrix. The associated conservation theorems have also been proven.

Nuclear hydrodynamics, a different approximation to the many-body problem, has been studied from extensive calculations on a two-dimensional model and special cases of the three-dimensional problem. In the earlier works of Nix et al along similar lines, only the kinematics could be properly treated. As a sharp contrast, the present investigations solve the complete set of hydrodynamical equations for a compressible, viscous and thermally conducting nuclear fluid, which is endowed with realistic nuclear features such as the binding energy and surface diffuseness. These calculations cover a wide range of energies - from a few MeV to hundreds of MeV per nucleon. The main results are as follows.

1) Systematic calculations on a two-dimensional model have reproduced qualitatively the features of low-energy collisions such as fusion, fission and deep-inelastic scatterings. From the scattering angles and the kinetic energies of the final fragments, it has been shown that dissipation plays an important role in the dynamics. Special features of hydrodynamics are
observed and their experimental tests discussed. Calculations on the medium- and high-energy collisions have shown features characteristic of shock phenomena.

2) Head-on collisions of the three-dimensional problem have been calculated for $^{20}\text{Ne} + ^{197}\text{Au}$ and $^{208}\text{Pb} + ^{208}\text{Pb}$, from 50 MeV/AMU up to 400 MeV/AMU. Prominent features have been found in the angular and energy distributions of the dissociated nucleons from these reactions, which can be confronted with experiments in the near future. The effects of dissipation have also been investigated and are found to play some role in the detailed shapes of the angular distributions. Direct comparisons of hydrodynamical calculations with experiments will provide a unique means of determining the nuclear transport coefficients.
B Discussions

The formulation and studies of the ETDHF have provided a first step towards including higher-order correlations in a nuclear dynamical theory. With more advanced computing facilities and more precise experimental measurements, nuclear many-body theory can now be subjected to increasingly stringent tests. Hence, implementations of such generalizations as ETDHF in practical calculations are clearly a worthy pursuit in theoretical studies. Numerical studies based on ETDHF have been started by Wong and coworkers. An alternative approach of the so-called couple-cluster approximation has been advocated by Negele and coworkers, the numerical implementation of which is also currently under investigation. It can be expected that results in these studies in the near future will lead to a step forward in low-energy nuclear physics.

There are many advantages in the use of Green's functions. In the study of higher-order correlations, powerful diagrammatic techniques can be developed. In addition, as a field-theoretic language, the Green's functions will be most appropriate for the construction of covariant theories in future relativistic generalizations. Such relativistic extensions are intimately related to high-energy phenomena of nuclear collisions and are, at the present, still an open question.

In the low-energy region of below tens of MeV per nucleon, nuclear hydrodynamics may be regarded as a special limit (near thermal equilibrium) contained in the logical framework of the microscopic many-body theory. From this standpoint, hydrodynamics is best applied to cases where the dynamical
evolution of the nuclear system is dominated by collective features. But this is not the only purpose of nuclear hydrodynamics. In the intermediate energy region where there is no reliable, basic theories for nuclear dynamics, hydrodynamics can be applied and expected to be a useful phenomenological model with which to study the behavior of nuclear matter under unusual conditions. It is a question of great theoretical interest to see how, starting from the relativistic Green's functions, one can arrive at the macroscopic equations of hydrodynamics under appropriate conditions. If successful, this will provide a rigorous justification of relativistic hydrodynamics applied to heavy-ion collisions.

The studies of nuclear hydrodynamics in this thesis have led to several questions of interest. From a technical point of view, it is of great interest to extend the present head-on calculations of the three-dimensional problem to non-zero impact parameters. We have argued that the validity of hydrodynamical treatments is best tested for near-head-on collisions. The formidable computer storage problem for full-fledged three-dimensional calculations has been circumvented by a technique which exploits axial symmetry. It is now being tested and hopefully, the more general results will be available in the near future. The peculiarities of different equations of state, such as ones which include second minima in the internal energy curve, or the pion degree of freedom, can now be conveniently studied, hydrodynamical calculations being feasible. Studies along this line will shed more light on the intrinsic properties of nuclear matter.

In conclusion, the present investigations have led to some useful re-
results concerning the microscopic and macroscopic aspects of nuclear dynamics and over different energy regions. From these studies, a class of problems have emerged. Investigations of these in the future will lead us to an understanding of nuclear phenomena at a level that is more satisfying.
APPENDIX I:
TIME-PATH ORDERING OF GREEN'S FUNCTIONS

The objective of this appendix is to examine the origin of the time-path ordering in expansions of generalized Green's functions. First, it is convenient to consider some fundamental relations of operators in the second-quantized form.

Lemma

Given a Hermitian operator $\hat{H}(t)$, which is a combination of an equal number of creation and annihilation field operators, the following operator identities hold:

1) $T[\hat{H}(t_1)\hat{H}(t_2)\ldots\hat{H}(t_n)]^\dagger = \tilde{T}[\hat{H}(t_1)\hat{H}(t_2)\ldots\hat{H}(t_n)]$

where $T$ is the usual time-ordered operator, a more general (path-dependent) form of which has been defined in eq. (II-12), and $\tilde{T}$ is the anti-time-ordered operator, defined for a product of creation and annihilation fermion operators as:

$T[\hat{O}_1^{(1)}\hat{O}_2^{(2)}\ldots\hat{O}_n^{(n)}]$

$$= \sum_P (-)^P \theta(t_{\alpha_1} - t_{\alpha_2}) \theta(t_{\alpha_2} - t_{\alpha_3}) \ldots \theta(t_{\alpha_{n-1}} - t_{\alpha_n})$$

$$\hat{O}_{\alpha_n}^{(\alpha_n)} \hat{O}_{\alpha_{n-1}}^{(\alpha_{n-1})} \ldots \hat{O}_{\alpha_1}^{(\alpha_1)}$$

where $\theta$ is the usual step-function.
Eqs. (A-1-1) and (A-1-2) can be readily established from the definitions of $T$ and the formal expansion of the time-ordered exponential functions.

Intermediate Representation

The Schroedinger representation of operators and states vectors is often used in ordinary quantum mechanics. But the Heisenberg and Dirac (interaction) representations are the more frequently used ones in quantum field theory and the abstract formulation of many-body problems. For the present purposes, it is convenient to introduce a general, intermediate representation of which the usual Dirac picture is a special case.

Consider a problem in which the system is specified by a full Hamiltonian $\hat{H}$. For simplicity, consider a closed system and $\hat{H}$ to be time-independent. One can start from the Schroedinger picture and consider an arbitrary decomposition into two Hermitian components:

$$\hat{H} = \hat{H}_0(t) + \hat{V}(t). \quad (A-1-3)$$

Here, $\hat{H}_0$ has the appearance of an unperturbed (and also one-body) Hamiltonian, but it is in fact more general. Since $\hat{H}_0$ is allowed to vary in time, it may be chosen to be some appropriate model dynamical Hamiltonian with some prescribed time evolution. Then, corrections due to particle-particle correlations will be contained in $\hat{V}$. The validity of the present results does not depend on $\hat{H}_0$ being a one-body operator.
The state vectors in the intermediate representation are defined as follows. To every state vector \( \tilde{\Psi}(t) \) in the Schroedinger picture corresponds one in the intermediate picture \( \tilde{\Psi} \):

\[
|\tilde{\Psi}(t)\rangle \equiv \hat{U}_I(t,t_o) |\Psi_S(t)\rangle,
\]

where the operator \( \hat{U}_I \) is defined as:

\[
\hat{U}_I(t,t_o) = \begin{cases} 
T \left[ \exp \left\{ \frac{i}{\hbar} \int_{t_o}^{t} d\tau \hat{H}_o(\tau) \right\} \right] & \text{if } t > t_o \\
T \exp \left[ -\frac{i}{\hbar} \int_{t_o}^{t} d\tau \hat{H}_o(\tau) \right] & \text{if } t < t_o
\end{cases}
\]

(\ A-I-4a \)

**Theorem**

Given a Hermitian operator \( \hat{H}(t) \), the following operators are unitary and they are Hermitian conjugates of each other:

\[
\hat{T} \exp \left[ \frac{i}{\hbar} \int_{t_o}^{t} d\tau \hat{H}(\tau) \right] \quad \text{and} \quad T \exp \left[ -\frac{i}{\hbar} \int_{t_o}^{t} d\tau \hat{H}(\tau) \right]
\]

for any limits \( t \) and \( t_o \). This establishes the unitarity and the usual properties of an evolution operator for \( \hat{U}_I \). The theorem can be proven by means of algebraic methods by considering the formal expansions of these time-ordered and anti-time-ordered exponential functions. The proof is straightforward and will not be reproduced here.

It is now clear that an operator \( \hat{O}_I \) in the intermediate picture is related to \( \hat{O}_S \) in the Schroedinger picture as:

\[
\hat{O}_I(t) = \hat{U}_I(t,t_o) \hat{O}_S \hat{U}_I(t,t_o)\dagger
\]

(\ A-I-5 \)
**Time-Path**

The rest of the arguments is very similar to those in the conventional perturbation theory (Fe 71). For example, we can consider the Green's function

\[ \langle \hat{\psi}(1) \hat{\psi}^\dagger(2) \rangle \]

There is the usual conversion from the Heisenberg picture to the intermediate picture (because the field operators in the exact Green's functions are given in Heisenberg picture). But we shall be concerned with matrix elements like (Cr 68):

\[ \langle \bar{m}_t | \hat{U}_I(t_1, t_2) \hat{\psi}_I(t_2, t_f) \hat{U}_I(t_f, t) \hat{\psi}_I^\dagger(t_1, t) | n_{t_1} \rangle \]

It is important to note that this expression can be compactly written as

\[ \langle \bar{m}_t | T_P \left\{ \exp \left[ \frac{-i}{\hbar} \int_{t_f}^{t_1} d\tau V(\tau) \right] \hat{\psi}_I(2) \hat{\psi}_I^\dagger(1) \right\} | n_{t_1} \rangle \]  

(A-I-6)

where \( T_P \) is introduced so that \( t_1 \) is taken to be on the forward branch and \( t_2 \), on the backward branch, time-paths which has been introduced in Chapter II. The expression in (A-I-6) can be obtained after some tedious, but rather straightforward algebraic steps.
APPENDIX II: TDHF IN DENSITY-MATRIX AND SINGLE-PARTICLE FORMS

It is well known that starting from the TDHF single-particle equations, the usual TDHF equation in one-body density matrix form can be obtained. Moreover, such a density matrix is idempotent. In the literature, this property and the two forms of the TDHF equation are used interchangeably. However, the proper converse theorem that the TDHF equation in the density matrix form leads uniquely to the single-particle equations, as far as the author is aware, has never been explicitly proven. For this reason and also for the reason that a thorough understanding of the converse theorem can be helpful for future generalizations in cases where the one-body density matrix equation is of a more general form and the density matrix need not be idempotent, it is necessary to analyze under what conditions the two forms of the TDHF equation are equivalent.

Following the spirit of the present thesis, one can consider the Green's functions of which the usual density matrices are special equal-time limits. It is straightforward to show, as has been done in the literature, that a single Slater determinantal wavefunction leads to an idempotent one-body density matrix. This appendix concerns the converse. First, one proves that under appropriate conditions, a one-body density matrix has the usual bilinear representation in terms of a finite set of orthonormal single-particle basis. Then, by construction, one proves that the TDHF single-particle equations can be uniquely determined from a density matrix form.
A. Representation Theorem

Given a one-body density matrix $N$ which satisfies the following properties:

1. Idempotent Property:

$$\int dx'' N(x;x'';t) N(x'';x';t) = N(x;x';t) \quad (A-II-1)$$

2. Hermiticity Condition:

$$N^\dagger(x;x';t) = N(x';x;t) \quad (A-II-2)$$

3. Finite Normalization Condition:

$$\int dx N(x;x;t) < \infty \quad (A-II-3)$$

then, $N$ defines a finite-dimensional subspace and has a representation

$$N(x;x';t) = \sum_{\lambda=1}^{A} \psi_\lambda(x,t) \psi_\lambda^*(x',t) \quad (A-II-4)$$

where $\psi_\lambda$ are orthonormal basis functions which span this subspace and $A$ is some integer.

Proof: From the idempotent condition, the one-body density matrix has the properties of a projection operator. It follows from standard theorems in the operator theory (Ha-b 57 ) that it decomposes the functional space into two orthogonal subspaces, $S(N)$ and $S'(N)$, where $S(N)$ contains eigen-vectors of $N$ corresponding to the eigen-value 1 and $S'(N)$ contains eigen-vectors corresponding to the eigen-value 0. There are then two possibilities to consider. The subspace $S(N)$ is either finite-dimensional or infinite-dimensional. For the latter case of infinite dimensionality, $S(N)$ is inconsistent with the condition (A-II-3) of finite normalization and hence must be excluded from
further consideration. Hence, for some integer $A$, one has
\[ \dim S(N) = A \quad (A-II-5) \]

Let $\psi_{\lambda}$ be any orthonormal basis which spans $S(N)$. One can construct a one-body density matrix as:
\[ \tilde{N}(x; x'; t) = \sum_{\lambda} \psi_{\lambda}(x, t) \psi_{\lambda}^*(x', t) \]

It is easy to prove that $\tilde{N}$ satisfies conditions (A-II-1), (A-II-2) and (A-II-3). Moreover, operations of $N$ and $\tilde{N}$ on any arbitrary single-particle wavefunction lead to the same result, thus implying that $N$ and $\tilde{N}$ are in fact identical. This completes the proof of the bilinear representation of idempotent one-body density matrices.

(Q. E. D.)

B Converse Theorem

Starting with the density matrix equation:
\[ i\hbar \frac{\partial}{\partial t} N(x; x'; t) = -\frac{\hbar^2}{2m} (\nabla^2 - \nabla'^2) N(x; x'; t) \]
\[ + \int dx'' [v(x, x'') - v(x', x'')] \left[ N(x; x'; t) N(x''; x''; t) - N(x; x''; t) N(x''; x'; t) \right] \quad (A-II-6) \]

the usual TDHF single-particle equations can be recovered.

Proof: An application of the bilinear expansion of (A-II-4) and eq. (A-II-6) leads to the following:
where $\psi_{\lambda}$ are the basis functions which span the subspace defined by $N$ at time $t$ and the single-particle operator $h$ is, in fact, the usual TDHF Hamiltonian which can be defined in a representation-independent manner by means of the density matrix one starts with:

$$ [h \psi_{\lambda}](x, t) $$

$$ = -\frac{\hbar^2}{2m} \nabla^2 \psi_{\lambda}(x, t) + \int dx'' v(x, x'') N(x'';x'';t) \psi_{\lambda}(x, t) $$

$$ - \int dx'' v(x, x'') N(x; x''; t) \psi_{\lambda}(x'', t) $$

(A-II-8)

In what follows, it is relevant to note that while the correspondence between $N$ and the subspace is unique, so far, the choice of the single-particle basis is arbitrary. For example, the basis functions in eq. (A-II-4) under a phase transformation

$$ \psi_{\lambda}(x, t) \rightarrow \psi_{\lambda}(x, t) = e^{i \theta(t)} $$. (A-II-9)

leaves the density unchanged.

Now, construct the following $N \times N$, time-dependent matrix:

$$ C_{\lambda \lambda'}(t) = \langle \psi_{\lambda'} | i\hbar \frac{\partial}{\partial t} - h | \psi_{\lambda} \rangle $$. (A-II-10)

Since both $h$ and the subspace $S(N)$ (and hence the basis functions) vary with time continuously, $C_{\lambda \lambda'}$ is also a continuous matrix function of time. A comparison of both sides of eq. (A-II-7) shows that the matrix $C_{\lambda \lambda'}$ is hermitian. In addition, the operator $(i\hbar \frac{\partial}{\partial t} - h)$
leaves the subspace $S(N)$ invariant. Accordingly, at each time $t$, there exists a diagonalization for the matrix $C_{\lambda\lambda'}$ with real eigenvalues which are also continuous functions of time.

Without loss of generality, one can now take $\psi_{\lambda}$ to be such a basis which is diagonal with respect to $(i\hbar \frac{\partial}{\partial t} - h)$, so that

$$[i\hbar \frac{\partial}{\partial t} - h] \psi_{\lambda}(xt) = \epsilon_{\lambda}(t) \psi_{\lambda}(xt) , \quad (A-II-11)$$

where $\epsilon_{\lambda}(t)$ are the real eigenvalues of the matrix $C_{\lambda\lambda'}$.

Now, one can define the following phase transformation of the single-particle wavefunctions:

$$\psi_{\lambda}(xt) \rightarrow \tilde{\psi}_{\lambda}(xt) = \exp \left[ -i \int_{t_0}^{t} \epsilon_{\lambda}(\tau) \, d\tau \right] \psi_{\lambda}(xt) \quad (A-II-12)$$

It is now easy to see that

$$[i\hbar \frac{\partial}{\partial t} - h] \tilde{\psi}_{\lambda}(xt) = 0 \quad (A-II-13)$$

which is the usual single-particle form of TDHF. This completes the proof of the equivalence of the density matrix and single-particle representations of the TDHF problem.

(Q. E. D.)
APPENDIX III: 
CONSERVATION OF HERMITICITY AND NORMALIZATION IN ETDHF 

In the ETDHF approximation, as in any other approximation in the many-body theory, the one- and two-body Green's functions constructed are not the exact functions. Hence, many properties of the Green's functions which can be established from the exact definitions should be checked for the approximate functions. In this appendix, the hermiticity and normalization conditions will be shown to be conserved in time in the ETDHF approximation.

Proof: Consider the general one-body equation (II-48) in the symmetric form. Both sides of eq. (II-48) can be regarded as functional operators acting on a c-number function of \((x_t; x'_t')\). It is trivial to verify that an operation of eq. (II-48) on \(g(x_t;x'_t')^*\) results in an equation which is the same form as that for \(- g(x'_t';x_t)\). Now, consider again the equal-time limits. If the initial conditions are such that

\[
g^< (x_t=0; x'_t'=0) = - g^< (x'_t'=0; x_t=0)^* \quad (A-III-1)
\]

and

\[
g^> (x_t=0; x'_t'=0) = - g^> (x'_t'=0; x_t=0)^* \quad (A-III-2)
\]

which are equivalent to eqs. (II-60a) and (II-60b), then conditions of (A-III-1) and (A-III-2) are satisfied for later times \(t=t'>0\). Equivalently, conditions given in eqs. (II-60a) and (II-60b) are preserved in time.

An integration over all space, after taking the equal-space and equal-time limits \(x'_t' \rightarrow x_t\), reduces eq. (II-48) to the following:
\[ \frac{\partial}{\partial t} \int dx \, g^<(x,t;x',t) = 0 \]

where the usual boundary conditions that $g^<$ and $g^>$ vanish at infinity has been applied. Hence, the normalization condition (II-61) is preserved in time.

(Q. E. D.)
APPENDIX IV: QUANTUM H-THEOREM

An entropy $S$ can be defined for ETDHF in terms of single-particle occupation numbers. One can prove an H-theorem in a way analogous to the classical Boltzmann H-theorem.

From eq. (III-5) and a direct calculation, the rate of change of entropy can be obtained:

$$\frac{dS(t)}{dt} = \frac{-\pi k_B}{\hbar} \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \left| \langle \lambda_1 \lambda_2 | v' | \lambda_4 \lambda_3 - \lambda_3 \lambda_4 \rangle \right|^2$$

$$\left[ (1-n_{\lambda_1})(1-n_{\lambda_2}) n_{\lambda_3} n_{\lambda_4} - n_{\lambda_1} n_{\lambda_2} (1-n_{\lambda_3})(1-n_{\lambda_4}) \right]$$

$$\ln \left[ \frac{n_{\lambda_1}}{1-n_{\lambda_1}} \right]$$

(A-IV-1)

where, the master equation (II-80) has been applied to eliminate $\hbar$. Appropriate exchanges among the indices $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$ in eq. (A-IV-1) result in different representations of the same expression on the right side. A combination of these different forms leads to the following symmetric expression:

$$\frac{dS(t)}{dt} = \frac{-\pi k_B}{4\hbar} \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} \xi(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4) \left| \langle \lambda_1 \lambda_2 | v' | \lambda_4 \lambda_3 - \lambda_3 \lambda_4 \rangle \right|^2$$

$$\left[ (1-n_{\lambda_1})(1-n_{\lambda_2}) n_{\lambda_3} n_{\lambda_4} - n_{\lambda_1} n_{\lambda_2} (1-n_{\lambda_3})(1-n_{\lambda_4}) \right]$$

$$\ln \left[ \frac{n_{\lambda_1} n_{\lambda_2} (1-n_{\lambda_3})(1-n_{\lambda_4})}{(1-n_{\lambda_1})(1-n_{\lambda_2}) n_{\lambda_3} n_{\lambda_4}} \right]$$

(A-IV-2)
For any combination of $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$ in the summation in eq. (A-IV-2), the product of the n-factor and the logarithmic term is of the form $(y-x)\ln(x/y)$, which is negative definite for any $x, y > 0$. Hence, one has, for any time,

$$\frac{dS(t)}{dt} \geq 0.$$ (A-IV-3)

Moreover, from this argument, it is clear that $dS/dt = 0$ if and only if for any $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_4$,

$$\ln\left[\frac{(1-n)\lambda_1 n\lambda_3 (1-n)\lambda_4}{(1-n)\lambda_2 n\lambda_4 (1-n)\lambda_3 (1-n)\lambda_2 n\lambda_4}\right] = 0,$$ (A-IV-4)

which, by the master equation, is equivalent to the condition

$$\frac{dn(\lambda)}{dt} = 0,$$

for all $\lambda$. This completes the proof for the H-theorem. (Q.E.D.)

As a further refinement, it is obvious that the above proof of the H-theorem remains valid even in the more general case in which the delta-function in the master equation is replaced by any other distribution function $D(\lambda_1\lambda_2;\lambda_3\lambda_4)$ of the single-particle energies, which is symmetrical under the exchange of indices $\lambda_1 \rightarrow \lambda_2$, $\lambda_3 \rightarrow \lambda_4$ and $(\lambda_1 \rightarrow \lambda_3$, $\lambda_2 \rightarrow \lambda_4)$. 


APPENDIX V:
NUMERICAL ASPECTS OF NUCLEAR HYDRODYNAMICS

The numerical problem of nuclear hydrodynamics has been solved by the development of computer codes for two- and three-dimensional calculations, the results of which have been discussed in Chapters V and VI. The fundamental algorithms used are based on the recently developed flux-corrected-transport (FCT) and time-step-splitting (TSS) methods of Boris and Book (Bo-c 73, 76, Bo-d 75). These numerical techniques have now found broad and successful applications in other fields, notably plasma physics.

The complete set of hydrodynamical equations is solved for a compressible, viscous and thermally conducting fluid. As a sharp contrast to the particle-in-cell (PIC) method employed in some earlier hydrodynamical calculations of heavy-ion collisions (Am-b 75, 77i, 77ii), the FCT method can easily include the realistic features like the binding energy and Coulomb interaction. Moreover, the diffuseness of nuclear surfaces can be incorporated by the introduction of an effective Yukawa interaction.

The basic concepts and the schematic structure of the FCT and TSS methods will be discussed in this appendix. Generalizations of these methods and comparisons with other techniques can be found in the literature and will not discussed here. But the numerical aspects which are specific of the nuclear problems will be given.
A Flux-Corrected-Transport Method (FCT)

The FCT method is a numerical approach (but not a particular numerical algorithm) to the non-linear problem of hydrodynamic-like equations. It leads to a class of Eulerian finite-difference algorithms with the following features:

(a) The propagation of a dynamical variable is achieved in two stages: a transport or convective stage, followed by an anti-diffusive or convective stage.

(b) The equations are solved in the conservative form. That is, symbolically, they are cast in the following form:

\[ \frac{\partial X}{\partial t} + \nabla \cdot (Xu) = S(X) \]  \hspace{1cm} (A-V-1)

where \( X \) is the dynamical variable to be propagated in time. It can be a scaler quantity such as the density and energy density, or it can be a vector component of the current density. The quantity \( u \) is the local velocity field and \( S(X) \) is the source term associated with \( X \). In both the transport and anti-diffusive stages, the conservation of the total particle number and the non-negativity of the number and energy densities are enforced.

(c) These algorithms are particularly designed for the study of shock phenomena which are generally characterized by discontinuities of the dynamical variables across certain surfaces (the so-called shock fronts). The FCT algorithms are known to be numerically stable and can give accurate description of many dynamical features, the bulk
properties at least.

We shall now illustrate the essential concepts of the FCT method by means of the one-dimensional continuity equation. At the beginning of each time step, the number density and the velocity field are known at the uniformly spaced Eulerian grid-points, \( \{ n_j^{(0)}, u_j^{(0)} : j = 1, 2, \ldots, N \} \). They are used as the initial values for the present time step. The basic problem is to construct a numerical operator by which \( \{ n_j^{(0)} \} \) can be propagated in a time interval \( \delta t \).

In passing, it should be noted that this example of the continuity equation only serves to demonstrate the basic features of designing FCT algorithms. In actual numerical implementations, many generalizations and modifications are possible in order to cope with the problem in question. For example, the equations need not be solved in cartesian coordinates. Also the choice of uniformly spaced grid-points is basically one of convenience; this restriction may be relaxed if necessary. The boundary conditions and the geometric configuration can be rather general.

First, at each grid-point, we introduce a dimensionless geometrical factor:

\[
\epsilon_j = \left| \frac{u_j \delta t}{\delta x} \right| \quad \text{, (A-V-2)}
\]

where \( \delta x \) is the distance between two adjacent grid-points. It is convenient to choose a time interval \( \delta t \) such that it satisfies the criteria:

\[
\max \{ \epsilon_j : j = 1, 2, \ldots, N \} \leq 0.5 \quad \text{. (A-V-3)}
\]
This constraint prevents any accidental density accumulation at any grid-point, which may lead to large errors.

At the beginning of the transport stage, the motion of a typical fluid element is followed for one time step in a Lagrangian manner. The resultant transient density field is then constructed in such a way that the total particle number is rigorously conserved. Geometrically, this procedure is represented by figures A5.1a and A5.1b. The initial density profile is schematically depicted in figure A5.1a and the subsequent profile after the Lagrangian transport by figure A5.1b. The conservation condition (conservation of the areas of the shaded trapezoids in the diagrams) uniquely determines the transient density. The original densities at the j-th and (j+1)-th grid-points become, respectively:

\[ n_j^{(+)} = n_j \delta x / [ \delta x + (u_{j+1} - u_j) \delta t ] \]  

(A-V-4a)

and

\[ n_j^{(-)} = n_{j+1} \delta x / [ \delta x + (u_{j+1} - u_j) \delta t ] \]  

(A-V-4b)

To complete the transport stage, this transient density profile is extrapolated back to the original Eulerian grid-points by the following prescription. For the cell boundary between the j-th and (j+1)-th grid-points (shown by the dotted lines between them in fig. A5.1), the fluid mass represented by the area of the trapezoid to the right of the boundary is calculated and assigned to the (j+1)-th grid-point, while that to the left of the boundary, to the j-th grid-point. This procedure is repeated for every pair of adjacent points. After some
straightforward computations, the following transported solution is obtained:

\[
n_j^{(1)} = \frac{1}{2} Q_-(n_{j-1}^{(0)} - n_j^{(0)}) + \frac{1}{2} Q_+(n_{j+1}^{(0)} - n_j^{(0)}) + (Q_+ + Q_-) n_j^{(0)}
\]

(A-V-5)

where

\[
Q_\pm = \frac{0.5 \pm u_{j} \delta t / \delta x}{1 \pm (u_{j+1} - u_{j}) \delta t / \delta x}.
\]

(A-V-5a)

This completes the transport stage of the algorithm.

To understand the necessity for a second stage, we can consider the simple case of a uniform velocity field. The transported solution (A-V-5) is reduced to

\[
n_j^{(1)} = \frac{1}{2} \epsilon (n_{j+1}^{(0)} - n_j^{(0)}) + (\frac{1}{2} + \frac{1}{2} \epsilon^2) (n_{j+1}^{(0)} - 2n_j^{(0)} + n_{j-1}^{(0)})
\]

(A-V-6)

It is important to recognize that on the right side of eq. (A-V-6), what is within the square bracket is a convective term. The second expression is a velocity-dependent diffusion term which does not vanish identically even in the limit of zero velocity field. Accordingly, it is necessary to construct a correction or anti-diffusion term to correct for this strong diffusion. This is the objective of the second stage in an FCT algorithm.

In the zero velocity limit, eq. (A-V-6) is simplified to the following:

\[
n_j^{(1)} = n_j^{(0)} + \frac{1}{8} (n_{j+1}^{(0)} - 2n_j^{(0)} + n_{j-1}^{(0)})
\]

(A-V-7)

An intuitive anti-diffusion equation would be:
This simple correction, however, can introduce additional numerical errors due to new unphysical maximas and minimas (Bo-c 73, 76, Bo-d 75). Hence, it is necessary to generalize eq. (A-V-8). This can be done by the introduction of anti-diffusion mass fluxes:

\[
\hat{n}_j^{(1)} = n_j^{(1)} - \frac{1}{2} (n_{j+1}^{(1)} - n_{j-1}^{(1)})
\]  \hspace{1cm} (A-V-8)

where the anti-diffusion mass fluxes are defined as:

\[
f_{j \pm \frac{1}{2}} = \pm \frac{1}{2} (n_{j \pm 1}^{(1)} - n_j^{(1)})
\]  \hspace{1cm} (A-V-9a)

This choice of anti-diffusion mass fluxes guarantees positivity and the absence of new maximas and minimas. To ensure strict conservation in a simple way, the anti-diffusion mass fluxes can be corrected by imposing the restriction that no anti-diffusive flux transfer of mass can push the density at any grid-point to beyond the adjacent points. The corrected fluxes \( \{ f^c_{j \pm \frac{1}{2}} \} \) are then given by:

\[
f^c_{j \pm \frac{1}{2}} = \text{sgn} \Delta_{j \pm \frac{1}{2}} \max \{ 0, \min(\Delta_{j-\frac{1}{2}} \text{sgn} \Delta_{j+\frac{1}{2}}, \gamma |\Delta_{j+\frac{1}{2}}|, \Delta_{j+\frac{3}{2}} \text{sgn} \Delta_{j+\frac{1}{2}}) \}
\]  \hspace{1cm} (A-V-10)

where

\[
\Delta_{j \pm \frac{1}{2}} = n_{j \pm 1}^{(1)} - n_j^{(1)}
\]  \hspace{1cm} (A-V-10a)

and the factor \( \gamma \) is \( \frac{1}{6} \) in the present case, which, in principle, can be further improved by the inclusion of other smaller corrections, at the expense
of additional computational effects.

The numerical procedures discussed above can be summarized and generalized to include a source term in the continuity equation. The steps to propagate \( \{ X_i^{(0)} \} \) with given velocities \( \{ u_i^{(0)} \} \) and values of the source term \( \{ S_i \} \) are as follows:

(a) The untransported fluxes and diffused but untransported solution are:

\[
\varphi^{(0)}_{i+\frac{1}{2}} = X^{(0)}_{i+1} - X^{(0)}_i \tag{A-V-11a}
\]

and

\[
X^D_i = X^{(0)}_i + (\varphi^{(0)}_{i+\frac{1}{2}} - \varphi^{(0)}_{i-\frac{1}{2}}) \tag{A-V-11b}
\]

(b) The transport convection factors are computed:

\[
\epsilon_i^\pm = \frac{1}{2} \pm u_i \delta t / \delta x \tag{A-V-12a}
\]

\[
Q_i^+ = \frac{\epsilon_i^+}{\epsilon_i^{+1} + \epsilon_i^-} \tag{A-V-12b}
\]

and

\[
Q_i^- = 1 - Q_i^+ \tag{A-V-12c}
\]

(c) A transported diffused solution is then constructed:

\[
X^TD_i = 4 (Q_i^+ \varphi^{(0)}_{i+\frac{1}{2}} - Q_i^- \varphi^{(0)}_{i-\frac{1}{2}}) + Q_i^+ (X^{(0)}_i - S_{i+\frac{1}{2}}) + Q_i^- (X^{(0)}_i - S_{i-\frac{1}{2}}) \tag{A-V-13}
\]

(d) Modifications in the fluxes are made and auxiliary quantities are computed:

\[
\varphi^T_{i+\frac{1}{2}} = \varphi^{(0)}_{i+\frac{1}{2}} + \frac{1}{8} (X^T_{i+1} - 2X^T_i + X^T_{i-1}) \tag{A-V-14a}
\]
\[ \Delta X_{i+\frac{1}{2}}^{TD} = X_{i+\frac{1}{2}}^{TD} - X_{i}^{TD} \quad (A-V-14b) \]

(e) The flux correction is then made:

\[ \phi_{i+\frac{1}{2}}^* = \text{sgn} \phi_{i+\frac{1}{2}} \max \{ 0, \min [\text{sgn} \phi_{i+\frac{1}{2}} \cdot \Delta X_{i-\frac{1}{2}}^{TD}, |\phi_{i+\frac{3}{2}}|, \text{sgn} \phi_{i+3/2}] \} \]

\[ (A-V-15) \]

(f) The final solution is then given as:

\[ X_{i}^{(1)} = X_{i}^{TD} - (\phi_{i+\frac{1}{2}}^* - \phi_{i-\frac{1}{2}}^*) \quad (A-V-16) \]
B Time-Step-Splitting Method (TSS)

The FCT operator solves the equations in one dimension. The multidimensional problem is solved by the implementation of a time-step-splitting method. The basic principles are as follows. Every time step is divided into two halves. A dynamical variable is propagated in one direction through the first half cycle, at which the velocity field is evaluated. Then, using the initial values at the beginning of the time step and the velocity estimated at the half cycle, the variable is propagated through the entire time step by means of the one-dimensional operator along this direction. This procedure is then repeated for the other directions.

The mathematical formulation of this scheme is as follows. Let \( \hat{\theta} \) be, symbolically, the operator which propagates a one-dimensional generalized continuity equation such that

\[
X^{(\text{new})} = \hat{\theta} [X^{(\text{old})}, u^{(\text{old})}, S^{(\text{old})}, \delta x, \delta t] \quad (A-V-17)
\]

That is, given the "old" values of \( X \), \( u \) and the source term \( S \), \( \hat{\theta} \) propagates the values of \( X \) over the time interval \( \delta t \) and with a grid-size \( \delta x \). For simplicity, we consider a two-dimensional problem (in cartesian coordinates). An extension to the most general three-dimensional problem will be straightforward.

In a general two-dimensional problem, a complete propagation of the dynamical variables over a time step can be subdivided into four stages, that is, two half cycles for each direction. The sequence of numerical constructions in each stage is given below and all symbols are self-explanatory.
1. First Half Cycle For the x-Direction:

The initial conditions $n^{(0)}$, $u_x^{(0)}$, $u_y^{(0)}$ and $E_T^{(0)}$ are either given as in the case of the first time step, or have been calculated from a previous time step. The required source terms are first computed:

$$S^{(0)}(n_x) = S[n^{(0)}, u_x^{(0)}, u_y^{(0)}]$$  \hspace{1cm} (A-V-18a)

and

$$S^{(0)}(nE_T) = S[n^{(0)}, u_x^{(0)}, u_y^{(0)}, E_T^{(0)}]$$  \hspace{1cm} (A-V-18b)

Then the one-dimensional propagator is applied for the half cycle in the following sequence:

$$n^{(\frac{1}{2}x)} = \theta[n^{(0)}, u_x^{(0)}, 0, \delta x, \frac{1}{2} \delta t]$$  \hspace{1cm} (A-V-19a)

$$(n_x^{(\frac{1}{2}x)})^{(1/2)x} = \theta[(n_x^{(0)}), u_x^{(0)}, S^{(0)}(n_x^{(0)}), \delta x, \frac{1}{2} \delta t]$$  \hspace{1cm} (A-V-19b)

$$(n_y^{(\frac{1}{2}x)}) = \hat{\theta}[(n_y^{(0)}), u_x^{(0)}, 0, \delta x, \frac{1}{2} \delta t]$$  \hspace{1cm} (A-V-19c)

and

$$(nE_T^{(\frac{1}{2}x)}) = \hat{\theta}[(nE_T^{(0)}), u_x^{(0)}, S^{(0)}(nE_T^{(0)}), \delta x, \frac{1}{2} \delta t]$$  \hspace{1cm} (A-V-19d)

Then, the velocities and the thermal energy (per particle) at the half cycle are computed:

$$u_x^{(\frac{1}{2}x)} = (n_x^{(\frac{1}{2}x)}) / n^{(\frac{1}{2}x)}$$  \hspace{1cm} (A-V-20a)

$$u_y^{(\frac{1}{2}x)} = (n_y^{(\frac{1}{2}x)}) / n^{(\frac{1}{2}x)}$$  \hspace{1cm} (A-V-20b)

and
This completes the first half cycle in the $x$-propagation. The second half cycle consists of a very similar sequence of constructions.

2. Second Half Cycle in the $x$-Direction:

$$E_T^{(\frac{3}{2}x)} = \frac{(nE_T)(x)}{n(x)} \quad . \quad \text{(A-V-20c)}$$

This completes the first half cycle in the $x$-propagation. It is followed by a similar propagation along the $y$-direction.

3. First Half Cycle in the $y$-Direction:
\[ S^{(x)}(n_{y}) = S[n^{(x)}, u^{(x)}, u^{(x)}] , \quad (A-V-24a) \]
\[ S^{(x)}(nE_{T}) = S[n^{(x)}, n^{(x)}, u^{(x)}, E^{(x)}] , \quad (A-V-24b) \]
\[ n^{(y)} = \hat{\theta}[n^{(x)}, u^{(x)}, 0, \delta y, \frac{1}{2}\delta t] , \quad (A-V-25a) \]
\[ (n_{y})^{(\frac{1}{2}y)} = \hat{\theta}[(n_{y})^{(x)}, u^{(x)}, S^{(x)}(n_{y}), \delta y, \frac{1}{2}\delta t] , \quad (A-V-25b) \]
\[ (n_{x})^{(\frac{1}{2}y)} = \hat{\theta}[(n_{x})^{(x)}, u^{(x)}, 0, \delta y, \frac{1}{2}\delta t] , \quad (A-V-25c) \]
\[ (nE_{T})^{(\frac{1}{2}y)} = \hat{\theta}[(nE_{T})^{(x)}, u^{(x)}, S^{(x)}(nE_{T}), \delta y, \frac{1}{2}\delta t] , \quad (A-V-25d) \]
\[ u^{(\frac{1}{2}y)} = (n_{y})^{(\frac{1}{2}y)} / n^{(\frac{1}{2}y)} , \quad (A-V-26a) \]
\[ u^{(\frac{1}{2}y)} = (n_{x})^{(\frac{1}{2}y)} / n^{(\frac{1}{2}y)} \quad (A-V-26b) \]

\[ E^{(\frac{1}{2}y)} = (nE_{T})^{(\frac{1}{2}y)} / n^{(\frac{1}{2}y)} . \quad (A-V-26c) \]

This completes the propagation in the first half cycle for the y-direction.

4. Second Half Cycle in the y-Direction:

\[ S^{(\frac{1}{2}y)}(n_{y}) = S[n^{(\frac{1}{2}y)}, u^{(\frac{1}{2}y)}, u^{(\frac{1}{2}y)}] , \quad (A-V-27a) \]
\[ S^{(\frac{1}{2}y)}(nE_{T}) = S[n^{(\frac{1}{2}y)}, u^{(\frac{1}{2}y)}, E^{(\frac{1}{2}y)}] , \quad (A-V-27b) \]
\[ n^{(new)} = n^{(y)} = \hat{\theta}[n^{(x)}, u^{(\frac{1}{2}y)}, 0, \delta y, \delta t] , \quad (A-V-28a) \]
\[ (n_{y})^{(new)} = (n_{y})^{(y)} = \hat{\theta}[(n_{y})^{(x)}, u^{(\frac{1}{2}y)}, S^{(\frac{1}{2}y)}(n_{y}), \delta y, \delta t] , \quad (A-V-28b) \]
\[(n_{u_x})^{\text{new}} = (n_{u_x})^y = \hat{\theta}[(n_{u_x})(x), u_{y}^{\frac{1}{2}y}, 0, \delta y, \delta t] \quad \text{(A-V-28c)}\]

\[(n_{E_T})^{\text{new}} = (n_{E_T})^y = \hat{\theta}[(n_{E_T})(x), u_{y}^{\frac{1}{2}y}, S^{\frac{1}{2}y}(n_{E_T}), \delta y, \delta t] \quad \text{(A-V-28d)}\]

\[u^{\text{new}}_{y} = u^y_{y} = (n_{u_y})^{\text{new}} / n^{\text{new}} \quad \text{(A-V-29a)}\]

\[u^{\text{new}}_{x} = u^x_{x} = (n_{u_x})^{\text{new}} / n^{\text{new}} \quad \text{(A-V-29b)}\]

and

\[E^{\text{new}}_{T} = E^y_{T} = (n_{E_T})^{\text{new}} / n^{\text{new}} \quad \text{(A-V-29c)}\]

This completes the propagation of the dynamical variables in both directions and for the entire time step. The new values \{n^{\text{new}}, u^{\text{new}}_{x}, u^{\text{new}}_{y}, E^{\text{new}}_{T}\} are then used as initial values for the next time step and the steps described above will be repeated.
The present section is concerned with certain numerical aspects of two-dimensional (2D) and three-dimensional (3D) nuclear hydrodynamics, namely, the establishment of appropriate initial conditions and the construction of the Yukawa and Coulomb fields. Finally, remarks will be made with regard to the efficiency and accuracy of the computations.

1. Initial Conditions:

The initial density field of a nuclear system (a cylinder of nuclear matter in the 2D case and a spherical nucleus in the 3D problem) is obtained from a self-consistent calculation which leads to a static solution.

An alternative procedure can be adopted and has been found to give sufficiently accurate static solutions for the present purposes. An initial trial density of a Woods-Saxon form, endowed with a zero velocity field, is propagated a number of times (typically ten to twenty times):

\[ n_{\text{init}}(r) = n_0 \left( 1 + \exp\left[ \frac{(r - R_o)}{a} \right] \right)^{-1} \quad (A-V-30) \]

The parameters in eq. (A-V-30) are chosen to be the same as those for real nuclei:

\[ n_0 = 0.17 \text{ fm}^{-3} \quad (A-V-30a) \]

\[ R_o = 1.2 A^{3/5} \quad (A-V-30b) \]

and

\[ a = 0.6 \text{ fm} \quad (A-V-30c) \]
In the two-dimensional problem, the primary objective is to achieve a density profile which resembles that of a real nucleus and this parametrization is adequate. In the more realistic three-dimensional case, the density given by eq. (A-V-30) is integrated over all space and then normalized to give the correct total particle number of the system in question.

The complete set of hydrodynamical equations are propagated in time for ten to twenty steps. The final density profile differs only slightly from the original one (usually less than 1% at each radial point). In addition, the very small velocity field at each grid-point guarantees that this density is indeed a good approximate static solution.

With the static solutions thus obtained for the projectile and target systems, the two systems are so placed that their (diffuse) surfaces barely touch. The mutual distortions due to the Yukawa and Coulomb interactions are neglected.

Each system is then given a velocity field corresponding to the bombarding energy in question, that is, for a bombarding energy of $E/A_P$ (in MeV/projectile nucleon), the velocities of the projectile and target in the center-of-mass system are, respectively:

$$v_P = \frac{A_T}{A_P + A_T} v$$

(A-V-31)

and

$$v_T = \frac{A_P}{A_P + A_T} v$$

(A-V-32)

where
with
\[ \mu = \frac{A_P A_T}{A_P + A_T} m. \]  

(\text{A--V--33a})

Here, \( r_0 \) is the initial separation between the nuclear centers. The difference in the expressions of \( v \) for the 2D and 3D cases is due to the fact that the Coulomb interaction is not included in the two-dimensional model. This completes the discussion on the initial conditions.

2. Yukawa and Coulomb Fields:

The Yukawa and Coulomb fields, generated by an arbitrary density field \( \rho(\mathbf{r}, t) \), are respectively given by the following expressions:

\[ V_Y(\mathbf{r}, t) = \beta \int d^3r' \rho(\mathbf{r}', t) \frac{e^{-\alpha |\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \]  

(\text{A--V--34})

and

\[ V_C(\mathbf{r}, t) = \left[ \frac{Ze}{A} \right]^2 \int d^3r' \rho(\mathbf{r}', t) \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \]  

(\text{A--V--35})

Two different numerical methods are used to compute these fields, namely, the fast-Fourier-transform (FFT) method and the conjugate-gradient method (CGM). Mathematically, the Coulomb field can be formally regarded as a limiting case of the Yukawa field \( (\alpha \to 0) \). Hence, these methods treat the two cases on the same footing.
By the Fourier convolution theorem, the Fourier transform \( \tilde{V}_Y(k) \) of the Yukawa field is related to the Fourier transform \( \tilde{\rho}(k) \) of the density field according to the following

\[
\tilde{V}_Y(k) = \frac{4\pi \beta \tilde{\rho}(k)}{k^2 + \alpha^2} .
\]  
(A-V-36)

The FFT method consists of the computation of the Fourier transform \( \tilde{\cdot} \) and the inverse transform of \( \tilde{V}_Y \) given in eq. (A-V-36). This standard technique is applied in the two-dimensional calculations in which, however, the Coulomb interaction is not included.

The CGM computes the Yukawa and Coulomb fields by means of solving their corresponding Helmholtz and Poisson equations by iteration:

\[
\{\nabla^2 - \alpha^2\} V_Y(r) = -4\pi \beta \tilde{\rho}(r) \quad (A-V-37)
\]

and

\[
\nabla^2 V_C(r) = -4\pi \left[\frac{Z\epsilon}{A}\right]^2 \rho(r) \quad (A-V-38)
\]

Full details of the procedures of the CGM have been given in the paper by Flocard, Koonin and Weiss (Fl 78). The method has been slightly modified for the calculations described in Chapter VI where there is axial symmetry in order that the actual storage and computation time can be saved.
3. Additional Remarks

The two-dimensional code solves the complete set of hydrodynamical equations in the cartesian coordinates. It has been optimized and for a configuration of 64 x 64 grid-points, with a grid-size of 0.6 to 1.5 fm, it achieves a speed of, depending on the actual input and output, 3.0 to 3.5 cpu seconds per time step on the 360/91 computer at Oak Ridge National Laboratory. This allows the study of very large systems with little difficulty. For low energies (tens of MeV per nucleon), a typical reaction takes about 400 to 600 time steps for one energy and one impact parameter. But for high energy collisions (hundreds of MeV), a reaction takes about only 100 to 200 steps.

Originally, a general three-dimensional code, which was an extension of the two-dimensional code, was developed. It treats the three-dimensional problem for arbitrary systems and arbitrary impact parameters. Preliminary results have been reported (Ta-b 77). However, there is still an unavoidable storage problem (and hence a problem of computation times), which renders this general code non-trivial to apply.

To circumvent this storage problem, a new 3D code has been developed. It is modified from the 2D-code and solves the hydrodynamical equations in the cylindrical coordinates (with the z-axis coincident with the axis joining the mass centers). Accordingly, in the special case of head-on collisions (no $\phi$-dependence in the equations), it effectively solves a 2D problem, but in fact treats a three-dimensional geometry. The calculations discussed in Chapter VI were done with this method. At the completion of the present thesis work, this
special 3D-code has been extended to cases with small impact parameters (with an approximation). This will be tested in the near future so that results for non-head-on, three-dimensional collisions can be obtained to substantiate the results so far obtained in this thesis.

This special 3D code is based on the following consideration. In near-head-on collisions, the configuration of the density field is one with approximate axial symmetry. This is observed in calculations on 2D near-head-on collisions (where the problem is treated exactly). The result of a typical case is presented in figure A5.2. In the 3D case, at each time step, we search for an approximate symmetry axis about which the density field is then axially symmetrized. All dynamical variables are solved on the reaction plane. In this way, the original three-dimensional problem is treated by a set of two-dimensional equations which include additional terms to take into account other degrees of freedom in an approximate way. In the limit of zero impact parameter, however, this method is exact for both collisions of symmetric and asymmetric systems.

As a consequence of these special considerations, this special 3D code achieves a speed comparable to the 2D code, but requires no additional storage. For a typical configuration of $64 \times 64$ grid-points, with a grid-size of 0.6 to 1.2 fm and a time step interval of 0.6 to 1.5 fm/c, depending on the actual input, output and other special implementations, this 3D code takes only 3 to 3.5 cpu seconds per time step on the 360/91 computer at Oak Ridge National Laboratory. Hence, this can easily accommodate studies of heavy systems like $^{208}\text{Pb} + ^{208}\text{Pb}$. 
Figure A5.1 Schematic representation of the transport stage of an FCT-algorithm.

Figure A5.2 Non-head-on collision of a symmetric system in 2D nuclear hydrodynamics.
E = 10 MeV/AMU, b = 3 fm.
TRANSPORT STAGE IN FCT-ALGORITHM

(a) \[ t = 0 \]

(b) \[ t = \delta t \text{ after Lagrangian transport} \]
Fig. A5.2  2D NUCLEAR HYDRODYNAMICS
E=10 MeV  b=3 fm

\[ t = 0.00 \text{ fm/c} \quad t = 66.03 \text{ fm/c} \quad t = 126.50 \text{ fm/c} \]

\[ t = 126.50 \text{ fm/c} \quad t = 187.79 \text{ fm/c} \quad t = 248.17 \text{ fm/c} \]
2D NUCLEAR HYDRODYNAMICS

$E = 10 \text{ MeV}$, $b = 3 \text{ fm}$

$t = 163.09 \text{ fm/c}$
$t = 199.79 \text{ fm/c}$
$t = 236.03 \text{ fm/c}$
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