In Memory of time at Yale

Sukjoon Lee
BEGINNING WITH A RENORMALIZABLE LAGRANGIAN, WHICH IS CONSTRUCTED FROM NUCLEON AND MESON FIELDS, A RELATIVISTIC HARTREE-FOCK APPROXIMATION IS DERIVED UTILIZING MEAN FIELD TECHNIQUES. THE GROUND STATE OF THE NUCLEUS IS TAKEN TO BE A SLATER DETERMINANT OF SINGLE-NUCLEON STATES, AND THE MESON FIELDS ARE TREATED AS QUANTIZED FIELDS. NEGLECTING THE EXCHANGE (FOCK) TERM, THE RESULTS REDUCE TO A RELATIVISTIC HARTREE APPROXIMATION IN WHICH THE MESON FIELDS ARE TREATED CLASSICALLY.

DEFORMED NUCLEI ARE INVESTIGATED UTILIZING THIS APPROXIMATION. TO DO THIS, A METHOD FOR CARRYING OUT RELATIVISTIC HARTREE CALCULATIONS FOR AXIALLY SYMMETRIC NUCLEI IS DEVELOPED. THE VALIDITY OF THE METHOD IS CHECKED BY COMPARING CALCULATIONS WITH THIS METHOD AND EXISTING CALCULATIONS FOR SPHERICAL NUCLEI. THE AXIAL HARTREE CALCULATIONS SHOW THAT BINDING ENERGIES AND BULK PROPERTIES OF DEFORMED NUCLEI CANNOT BE EXPLAINED SIMULTANEOUSLY. THE MORE DEFORMED FROM THE SPHERICAL SHAPE, THE LARGER THE DISCREPANCY. POSSIBLE RESOLUTIONS OF THIS DIFFICULTY ARE DISCUSSED.
RELATIVISTIC MEAN FIELD METHODS AND APPLICATION

TO AXIALLY DEFORMED NUCLEI

A Dissertation
Presented to the Faculty of the Graduate School
of
Yale University
in Candidacy for the Degree of
Doctor of Philosophy

by
Suk-Joon Lee
December 1986
ACKNOWLEDGEMENTS

It is a pleasure to acknowledge the support, help, contributions, and instruction of many people. First of all, I am grateful to my thesis advisor, Dr. Michael R. Strayer, for excellent guidance, assistance, and support throughout these years.

I would especially like to thank Professors F. Iachello and D. A. Bromley for their continuous help and financial assistance. I am also grateful to Dr. A. B. Balantekin for many helpful discussions and collaboration in diverse topics.

Dr. J. B. McGrory and the members of the Physics Division at Oak Ridge National Laboratory deserve special thanks for their generous hospitality and financial support. Without the use of the computational facilities at ORNL, some of the work in this dissertation would not be possible.

I would also like to express my appreciation to Althea Tate for her expert typing, not only of this dissertation but also of many other manuscripts. I am especially grateful to Sara Batter for her constant help and support over the years. Also, for a critical reading and correction of the manuscript, I would like to thank Prof. D. Ernst.

Finally, I would like to express my gratitude to our parents for their steady encouragement and support. Most of all, I wish to express my appreciation to my wife, Yeon-Hwa, and children, Dong-Seok and Ka-Eun, for their understanding and sacrifices during these difficult times.
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acknowledgements</td>
<td>ii</td>
</tr>
<tr>
<td>Table of Contents</td>
<td>iii</td>
</tr>
<tr>
<td>List of Figures</td>
<td>iv</td>
</tr>
<tr>
<td>List of Tables</td>
<td>v</td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. RELATIVISTIC NUCLEAR SYSTEM</td>
<td>28</td>
</tr>
<tr>
<td>A. MEAN FIELD THEORY</td>
<td>31</td>
</tr>
<tr>
<td>B. THE EQUATIONS OF MOTION OF THE MEAN FIELDS</td>
<td>37</td>
</tr>
<tr>
<td>C. PHYSICAL OBSERVABLES</td>
<td>43</td>
</tr>
<tr>
<td>III. RELATIVISTIC HARTREE CALCULATIONS FOR AXIALLY SYMMETRIC NUCLEI</td>
<td>46</td>
</tr>
<tr>
<td>A. EQUATIONS OF MOTION FOR AXIALLY SYMMETRIC TWO-DIMENSIONAL LATTICES</td>
<td>47</td>
</tr>
<tr>
<td>B. NUMERICAL METHODS AND PHYSICAL OBSERVABLES ON THE GRID</td>
<td>60</td>
</tr>
<tr>
<td>C. RESULTS</td>
<td>64</td>
</tr>
<tr>
<td>1. Spherical Nuclei</td>
<td>66</td>
</tr>
<tr>
<td>2. Axially Deformed Nuclei</td>
<td>73</td>
</tr>
<tr>
<td>IV. CONCLUSIONS</td>
<td>115</td>
</tr>
<tr>
<td>APPENDIX. RELATIVISTIC HARTREE-FOCK EQUATIONS FOR SPHERICAL SYSTEMS</td>
<td>117</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>122</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

| Figure 1. | Effect of Rho-Meson on Binding Energy Curves | 10 |
| Figure 2. | Comparison Between RH and DDHF for Charge Density | 13 |
| Figure 3. | Effect of Vacuum Fluctuation on Nuclear System | 15 |
| Figure 4. | Exchange Effect in Relativistic Nuclear System | 17 |
| Figure 5. | Correlation Effect in Relativistic Calculations | 19 |
| Figure 6. | Comparison Between Stochastic and RH Calculations | 23 |
| Figure 7. | Dispersion Relations for One-Dimensional Dirac Equation | 52 |
| Figure 8. | Energy Convergency Rate | 67 |
| Figure 9. | Grid Size Dependence of Binding Energy | 69 |
| Figure 10. | Baryon and Scalar Density of $^{12}$C | 82 |
| Figure 11. | Baryon and Scalar Density of $^{20}$Ne | 84 |
| Figure 12. | Baryon and Scalar Density of $^{24}$Mg | 86 |
| Figure 13. | Baryon and Scalar Density of $^{36}$Ar | 92 |
| Figure 14. | Baryon and Scalar Density of $^{38}$Ar | 94 |
| Figure 15. | Baryon and Scalar Density of $^{40}$Ar | 96 |
| Figure 16. | Baryon and Scalar Density of $^{40}$Ca | 98 |
| Figure 17. | Baryon and Scalar Density of $^{42}$Ca | 100 |
| Figure 18. | Baryon and Scalar Density of $^{44}$Ca | 102 |
| Figure 19. | Baryon and Scalar Density of $^{46}$Ca | 104 |
| Figure 20. | Baryon and Scalar Density of $^{48}$Ca | 106 |
| Figure 21. | Baryon and Scalar Density of $^{46}$Ti | 108 |
| Figure 22. | Baryon and Scalar Density of $^{48}$Ti | 110 |
| Figure 23. | Baryon and Scalar Density of $^{50}$Ti | 112 |
LIST OF TABLES

Table 1. Comparison Between RH and RHF.................................22
Table 2. Coupling Constants of RMF, RH, and RHF.........................26
Table 3. Meson Fields Included in Model Lagrangian Eq. (1)................34
Table 4. The Quantities Used in Eq. (53).................................56
Table 5. Parameter Sets Used in Axial Calculations.......................65
Table 6. Bulk Properties of $^{160}$O Obtained by Axial Calculations...71
Table 7. Single Particle Levels of $^{160}$O..............................72
Table 8. Bulk Properties of $^{40}$Ca and $^{48}$Ca........................74
Table 9. Single Particle Levels of $^{40}$Ca and $^{48}$Ca..................75
Table 10. Spin Mixing Probabilities of Spherical Nuclei..................77
Table 11. Bulk Properties of Axially Deformed Nuclei...................79
Table 12. Single Particle Levels of Axially Deformed Nuclei............80
Table 13. Comparison Between Relativistic and Nonrelativistic Calculations for Deformed Nuclei..............................89
I. INTRODUCTION

A fundamental problem in nuclear physics is understanding the many-body interaction in nuclei and nuclear collisions from first principles. Since the size of a nucleon is small compared to the mean distance between nucleons in a nucleus, and since the velocity of a nucleon is about one quarter of the speed of light, a nucleus is traditionally treated as a nonrelativistic many-nucleon system. Quantum mechanical models for this system can be constructed using a many-body Schrödinger equation. In the independent particle approximation, i.e., choosing the ground state of the system as a Slater determinant of single nucleon levels, this theory is reduced to a Hartree-Fock approximation [1,2], where each nucleon moves in a self-consistent potential arising from the interaction with all the other nucleons.

Since the nucleon-nucleon interaction is strong, the self-consistent Hartree-Fock potential cannot be approximated as the free space two-nucleon interaction. Here we need to use an effective interaction which also includes many-body effects. One of these interactions is the G-matrix, in which the interaction is adjusted to reproduce the properties of two- and three-nucleon systems [1]. However, it is difficult in practice to calculate these and to get quantitative agreement with experiment. Consequently, phenomenological [3] or semi-phenomenological [1] effective interactions have been adopted. Such interactions usually depend on the nuclear density and have a set of parameters adjusted to reproduce experimental data.
These density-dependent Hartree-Fock (DDHF) models have successfully described the shell structure of nuclei throughout the periodic table. However, there are still discrepancies and difficulties associated with the charge density and the spin-orbit interaction.

To remove the shortcoming of the traditional models, we need to reconsider the basic assumptions. Considering the nucleon as a relativistic Dirac particle, the spin degree of freedom is included in the relativistic theory naturally. Introducing meson fields, the finite range interactions between nucleons can be replaced with local interactions between the nucleons and mesons, as in the quantum electrodynamics (QED). Thus in this way, we can replace the traditional models by a relativistic quantum field theory in which nucleons and mesons play a fundamental role.

For intermediate energy collisions of nuclei, with the bombarding energy per nucleon comparable to the mass of the nucleon, it is necessary to consider relativistic effects. Both the target and projectile nuclei need to be described with a relativistic wavefunction. In this case, the kinematics should maintain the special relativistic conditions, such as the correct relation between energy and momentum. As the collision energy increases, relativistic effects become more important, and in very energetic heavy-ion collisions, the densities become several times higher than those observed in ordinary nuclei. Thus, the study of collisions between energetic heavy ions probes the equation of state at high densities and temperatures [4]. At high nuclear density, the velocity of a nucleon at the fermi momentum
approaches the speed of light. Therefore, it is essential to have a Lorentz covariant theory for the nuclear many-body problem.

It is also relevant to study relativistic effects in ordinary nuclei. Since the Fermi momentum of a nucleon at ordinary nuclear density is small (about $Mc/4$), the traditional approach assumes that relativistic effects will be small. However, the effective mass of nucleons inside a nucleus is approximately half of the free nucleon mass [5]. Thus, the nucleon velocity can become large even at normal density. Furthermore, there is growing evidence that the small effective nucleon potential arises from a cancellation between large attractive and repulsive potentials which are approximately several hundred MeV at ordinary densities.

One-boson exchange potential (OBEP) analyses of nucleon-nucleon scattering have revealed large and cancelling Lorentz scalar and vector potentials [6,7]. A relativistic treatment has made it possible to understand the detailed spin-dependent effects in nucleon-nucleus scattering [8]. The Lorentz transformation properties of these potentials lead naturally to various momentum, density, and spin dependences of the effective nucleon potential in a nucleus. The strong scalar and vector potentials are additive in the spin-orbit interaction, in opposition to in the binding energy. This additivity results naturally in an approximately correct spin-orbit splitting between the single nucleon levels.

A microscopic theory of nuclear many-body systems is generally a nonlocal theory if we consider only the nucleons as the degrees of
freedom. By introducing intermediating bosonic particles for the interactions between the nucleons (as a photon for the case of the electromagnetic interaction), these nonlocal interactions can be reduced to local interactions between the nucleons and these bosons. For the short-range properties of the N-N interaction, these intermediating bosons must be massive, i.e., mesons. Here the microscopic models for a nuclear system in the Hartree approximation become local. Furthermore, the causality of signals (retardation effect), which needs to be maintained in a relativistic theory, is easy to handle through the propagation of mesons.

Experimentally, significant pion production cross sections have been observed in light nucleus-nucleus collisions even at a laboratory energy per nucleon of 25 MeV [9]. This suggests that the dynamical pion degrees of freedom are important even in low energy collisions. An approximation of the relativistic nucleon-nucleon scattering in terms of meson exchanges results in the one-boson exchange potentials. Two-body analyses can obtain a very satisfactory description of N-N scattering phase shifts up to laboratory energies of 350 MeV with several different mesons [6,7]. These analyses show that the most important contributions come from the exchange of \( \pi, \sigma, \omega, \) and \( \rho \) mesons with respective spin, parity, and isospin quantum numbers, \((J^{\pi},T) = (0^{-},1), (0^{+},0), (1^{-},0), \) and \((1^{-},1)\). Other mesons are found to be less important.

The quadrupole moment of the deuteron requires a tensor force between the neutron and the proton. The existence of this force can be
explained via the one-pion exchange potential [5]. Furthermore, in explaining nuclear structure in terms of static two-nucleon potentials, there are discrepancies between the calculated and empirical properties of three-nucleon systems and nuclear matter. Genuine three-nucleon interactions, which cannot be obtained by a pairwise summation of the two-nucleon interaction, have been advanced to explain the observed central depression in the charge density of $^3$He [10]. Including the meson-meson interactions beside the nucleon-meson interactions, the genuine many-nucleon interactions appear naturally in the nuclear meson theory. A $\rho \pi \pi$ interaction with the mesons originating from three different nucleons in a nucleus is an example of a genuine three-body interaction.

It is well known that nucleons and mesons have a finite size. Thus, these are composed of more fundamental particles known as quarks with three colors. Since hadrons are colorless, these are considered as colorless compositions of quark and antiquark pairs (mesons) or of three quarks (baryons). In quantum chromodynamics (QCD), the interactions between quarks are mediated by the nonabelian gauge fields which are eight massless colored bosons (gluons) [11,12]. Since isolated quarks and gluons are not observed in the laboratory, it is an hypothesis that quarks and gluons are confined in a small region of space. For this hypothesis, it is assumed that color interactions are very strong at large distances and become weak at short distances (asymptotic freedom). Asymptotic freedom is required because of the finite size of hadrons. QCD is simple at short distances and becomes very
complicated at large distances.

Various phenomenological models, so-called bag models, have been developed to describe hadrons as the colorless confinements of quarks and gluons in a region of space [13,14]. Due to the strong forces between quarks at large distances, we can approximate a system of quarks having a low quark density as a system of colorless bags. For this approximation, the mean separation between quarks should be the order of the hadron size (confinement region). The interactions between bags can be interpreted as the residual color interaction. Since the mean distances (∼1.9 fm) between nucleons in a nucleus are larger than the nucleon size (electromagnetic radius ∼0.66 fm), it is hard to interpret the nucleon-nucleon interactions as the residual color interactions between nucleon bags. Introducing mesons with a radius of at least 0.3 fm, the hadron bags overlap, or at least touch each other. Therefore, there are residual interactions between quarks belonging in the nucleon bags and the meson bags. These residual color interactions can be identified as the interactions between nucleons and mesons. Furthermore, this argument implies that only the mesons which have masses up to about the nucleon mass have an important role in the interpretation of the nucleon-meson interactions as the origin of the nucleon-nucleon interactions of nuclear structure theory or low-energy collisions. Since the spatially separated bags do not have a color interaction between them, the nucleon-nucleon interaction is a short-range force. The weak long range tail of these interactions comes through the propagation of mesons from one nucleon to another. Thus,
nuclear theory with nucleon and meson degrees of freedom may provide a correct description of the many-nucleon system at low nucleon densities.

At extremely high density and temperature, nucleons themselves overlap each other. This means that the nuclei dissolve into a quark-gluon plasma. Since isolated quarks and gluons are not observed in the laboratory, this system may be considered as a large colorless bag or a system composed of larger bags than nucleon or mesons. The conditions and the nature of this transition are an interesting problem in physics. The relativistic study of the nuclear system with the nucleon and meson degrees of freedom permits an extrapolation from the observed properties of finite nuclei to nuclear matter under extreme conditions. This extrapolation would give some insight into the above transition.

With these arguments, it is reasonable to study nuclear many-body systems in the framework of a relativistic theory using nucleons and mesons as degrees of freedom. This study has been carried out by many authors [15-45] with various classes of approximations. A microscopic quantum description of this system must be Lorentz covariant and causal. Since the nucleon-nucleon interactions are replaced by the local interactions between nucleons and mesons, it should be based on a local theory. A theory describing such many-body systems is a relativistic quantum field theory. Furthermore, it should be able to be characterized by a finite set of parameters. This constraint of renormalizability is required to ensure unambiguous predictions for all
remaining physical quantities with these parameters (coupling constants and masses) determined by an appropriately chosen set of experimental data. Therefore, a relativistic quantum field theory for a many-body nucleon-meson system must be based on a renormalizable local Lorentz invariant Lagrangian density.

In contrast to quantum electrodynamics (QED), such a theory for a nuclear system must be a strong coupling theory. Thus, a perturbation technique, which is based on the expansion in terms of a small parameter as in QED, cannot be directly applied. However, self-consistent many-body techniques and selected summation of classes of Feynman diagrams have been employed in a relativistic description of nuclei. These result in Green's function techniques using Dyson's equation for summing selected diagrams. As discussed later in this chapter, these techniques are subject to arbitrary choices of the diagrams. Therefore, we also need to develop some other mechanisms which are not based on the ordinary perturbative concept. One of the examples is the relativistic mean field theory of Walecka.

To investigate the properties of high-density nuclear matter, a model relativistic many-body quantum field theory has been proposed by Walecka [15,16]. This model consists of a nucleon field coupled to a scalar isoscalar field, \( \sigma \), and a vector isoscalar field, \( \omega \). At high nucleon density, the sources of the scalar and vector fields become large. Thus, he assumed that the scalar and meson field operators can be replaced by their expectation values which can serve as classical fields in which the nucleons move. Furthermore, he also assumed that
the ground state of the nuclear matter can be approximated as a Slater
determinant of occupied nucleon states up to Fermi level without anti-
nucleon and mesons. This relativistic mean field theory (RMF) results
in a set of coupled self-consistent equations of nucleon and meson
fields, and can be solved exactly. Since this RMF theory is based on
classical meson fields, it includes only the direct interactions in
the effective equation of motion of the nucleon field, i.e., the
Hartree equation for the nucleon field. Serot [18] used this RMF
theory for a renormalizable SU(2)xU(1) Lagrangian in which there are
also a pseudoscalar isovector field, \( \pi \), and a vector isovector field,
\( \rho \). This calculation shows that the \( \rho \) meson should be included in the
relativistic study of nuclear matter (see Fig. 1), and that the pion
field has no effect on this RMF approximation since the source of the
pion field is zero by time-reversal invariance.

A relativistic Hartree approximation for finite nuclei with a
renormalizable SU(2)xU(1) Lagrangian density as in Ref. [18] was de-
developed by Horowitz and Serot [36] using Green's function techniques.
By neglecting the antinucleon in the theory, which naturally comes
through the Green's function, this theory for finite nuclei is essen-
tially the same as the RMF theory of Walecka for nuclear matter. By
determining the model parameters from the bulk properties of nuclei,
they examined the predictions of this relativistic Hartree approxima-
tion for closed-shell nuclei. These calculations show that the rela-
tivistic description of nuclei contains naturally a spin-orbit inter-
action of appropriate strength and the important noncentral, nonlocal,
Figure 1. Binding energy curves for infinite neutron matter with (dotted line, $g_p = 6.07$) and without (solid line) the $\rho$ meson field [18].
Binding Energy (MeV/A)

Effect of Rho Meson
and density-dependent effects incorporated in nonrelativistic calculations. However, the calculations for nuclei still yield unphysical oscillating structure in the interior of the nuclear charge distribution (Fig. 2).

Several classes of corrections to these relativistic theories have been calculated for both nuclear matter and finite nuclei. The effects of vacuum fluctuations and exchange terms for nuclear matter were studied by Chin [21,22] using the Green's function technique. The binding energy curves for nuclear and neutron matter show large effects due to vacuum fluctuations and exchange terms as shown in Figs. 3 and 4. Also other correlation effects have been shown to be large in nuclear matter by Brittan [23] as is shown in Fig. 5.

Using the one-boson exchange potentials (OBEP) for the nucleon-nucleon interaction, instead of using the meson degrees of freedom explicitly, the relativistic Hartree and Hartree-Fock approximations for finite nuclei have been derived by Miller and Green [37-39] and Brockmann [40]. This OBEP method corresponds to the relativistic theory of a nuclear system which has only the nucleon degrees of freedom interacting themselves through the OBEP. These calculations for spherical nuclei have established that a relativistic OBEP fit to the N-N scattering could provide a reasonable description of these nuclei, depending on which mesons were included and on precisely how the couplings were chosen. They also showed that the strong spin-orbit force results from the additive contributions of scalar and vector potentials, in contrast to the cancellation of these when contributing
Figure 2. Charge density distributions for $^{208}$Pb [36]. The dashed line is the relativistic Hartree result. The dotted line is the nonrelativistic density-dependent Hartree-Fock results. Solid line is the empirical distribution.
Figure 3. Binding energy curves for nuclear matter with (dotted line) and without (solid line) vacuum fluctuation effects [22].
Binding Energy (MeV/A)
Figure 4. Binding energy curves for neutron matter with and without vacuum fluctuation and exchange effects [22]. The solid curve is without the exchange term and the dotted curve is with exchange effects for relativistic mean field theory (no vacuum fluctuations). The dashed line is without exchange effects and the dot-dashed line is with exchange for the relativistic Hartree approximation which includes the vacuum fluctuation effects.
Binding Energy (MeV/A)

Neutron Matter
Figure 5. Binding energy curves of nuclear matter with (dashed line) and without (others) correlation effects [23]. The solid line is the mean field results without exchange effects and the dotted line is with exchange effects. In each case, the coupling constants are determined so that the minimum in the curve reproduces the saturation properties of nuclear matter.
to the binding energy. The exchange contribution to the binding energy is about 1.5 MeV per particle (Table 1) which is more than about 15 percent.

All of these calculations show that the relativistic theory describes the bulk properties of nuclear matter and finite spherical nuclei moderately well. They also show that the effects of vacuum fluctuations and exchange terms are of the same order of magnitude and are quite large. Furthermore, it also has been shown that the correlation effects of a higher order than vacuum fluctuation and exchange terms are even larger than the latter terms. This means that the prescriptions for selected summations of Feynman diagrams to give relativistic Hartree, Hartee-Fock, and Brueckner-Hartree-Fock approximations are subject to arbitrary choices. This arbitrariness leads to substantial uncertainties in the nuclear matter saturation curves as pointed out by Negele [46,47]. Serot, Koonin, and Negele [31] showed that the nuclear matter saturation curves of relativistic mean field calculations differ substantially from those of stochastic solutions using path integrals (Fig. 6). They consider a system which consists of nonrelativistic nucleons coupled to scalar and vector mesons in one spatial dimension. Since they used a nonrelativistic nucleon field, the results are quite similar to those obtained in the Hartree-Fock approximation. This means that we need to study the full quantum field theory more carefully in the as-yet-to-be-determined correct hierarchy of approximation schemes based on a quantitative evaluation of the above effects. Since the Hartree-Fock calculation is quite
Table 1. Comparison between relativistic Hartree and Hartree-Fock calculations [38]

<table>
<thead>
<tr>
<th></th>
<th>Binding Energy (MeV/A)</th>
<th>rms Charge Radii (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RH</td>
<td>RHF</td>
</tr>
<tr>
<td>$^{160}$O</td>
<td>7.35</td>
<td>8.90</td>
</tr>
<tr>
<td>$^{40}$Ca</td>
<td>8.25</td>
<td>9.68</td>
</tr>
<tr>
<td>$^{48}$Ca</td>
<td>8.55</td>
<td>10.29</td>
</tr>
</tbody>
</table>
Figure 6. Nuclear matter saturation curves comparing stochastic (dashed line) [31], relativistic mean field theory (solid line), and Hartree-Fock (dotted line) calculations.
Stochastic vs. Mean Field

\[ \text{Binding Energy (MeV/A)} \]

\[ \rho \text{ (fm}^{-1}\text{)} \]
similar to the stochastic solution, we should be able to employ the Hartree-Fock approximation without using the perturbative concept. This derivation has not been given. The aim of Chapter II is the systematic derivation of a relativistic Hartree-Fock approximation based on the mean field theory with quantized nucleon and meson fields.

On the other hand, most of the relativistic calculations for finite nuclei have been carried out for spherical systems. In such calculations, part of the quantum effects are included indirectly by fitting the Lagrangian parameters to the empirical properties of spherical nuclei or infinite nuclear matter. These calculations explain most of the bulk properties of spherical nuclei well. As we can see in Table 2, the parameter values differ by more than 10 percent between relativistic mean field, Hartree, or Hartree-Fock calculations, all resulting in the same bulk properties of nuclear matter.

However, the detailed structure of deformed nuclei provides a much wider range of experimental data to test relativistic nuclear shell models. Furthermore, these can test if the quantum effects in all the bulk properties of deformed nuclei can be included indirectly through appropriately fitting parameters in the same way as for spherical nuclei. The deformed nuclear shell model requires a representation in at least two spatial dimensions. For axial symmetry, we can eliminate the azimuthal angle dependence in the field equations and therefore only need to consider two spatial dimensions, namely the axial coordinate and the coordinate of the symmetry axis of the system. Developing the relativistic method for two-dimensional axially
Table 2. Model parameters to fit the binding energy and density 
\( (k_F^0 = 1.42 \text{ fm}^{-1}) \) for nuclear matter. The nucleon mass is 939 MeV, 
vector meson mass is 783 MeV, and scalar meson mass is 550 MeV. The 
models are relativistic mean field theory (RMFT), relativistic Hartree 
(RH) which include vacuum fluctuations, and relativistic Hartree-Fock 
(RHF) approximations. \( g_S \) is the scalar coupling and \( g_V \) is the vector 
coupling constants as defined in Eq. (1).

<table>
<thead>
<tr>
<th>Model</th>
<th>( g_S^2 )</th>
<th>( g_V^2 )</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMFT</td>
<td>91.64</td>
<td>136.2</td>
<td>[15]</td>
</tr>
<tr>
<td>RH</td>
<td>62.89</td>
<td>79.78</td>
<td>[22]</td>
</tr>
<tr>
<td>RHF</td>
<td>83.11</td>
<td>108.05</td>
<td>[25]</td>
</tr>
</tbody>
</table>

symmetric nuclei is the aim of this dissertation. A relativistic Hartree description of deformed nuclei will test if quantities such as total energy, radius, and intrinsic quadrupole moments can be reproduced with this model. If it is the case, the many-body effects can be included indirectly by fitting the parameters to empirical values as in the case for spherical nuclei, and thus we may not need to use the complexity of the full quantum field theory.

In Chapter II, a mean field theory will be applied to the nuclear system starting with a renormalizable relativistic Lagrangian constructed with nucleon and meson fields. Considering the nucleon and meson fields as quantum fields, a relativistic Hartree-Fock approximation will be derived. To facilitate the discussion in this chapter, the self-consistent relativistic Hartree-Fock single-nucleon equations are derived for spherical nuclei in the Appendix. The relativistic Hartree approximation for deformed nuclei will be discussed in Chapter III. Calculations using this approach for deformed nuclei will be presented. In the first section of this chapter, the relativistic expressions for the axially deformed nuclei are derived on a two-dimensional space lattice. An outline of the numerical solution is presented in the second section. In the third section, the results of calculations for spherical and axially deformed nuclei will be given. Finally, Chapter IV contains a brief discussion and suggestions for the possible improvements of the present work.
II. RELATIVISTIC NUCLEAR SYSTEM

We start from a renormalizable relativistic Lagrangian density constituted of a nucleon field, $\psi$, and a set of meson fields which will serve as the source of the nucleon-nucleon interaction. For the long-range attractive and short-range repulsive properties of this force, we need both scalar, $\phi$, and vector, $V_\mu$, meson fields. These may be associated with the isoscalar sigma and omega mesons [15]. For a more realistic description of mesonic degrees of freedom in nuclear systems, we must include the isovector pion, $\pi^\pm$, and rho meson, $b_\mu$, fields [18,36,40]. The simplest Lagrangian density that can be constructed from these fields together with an electromagnetic field, $A_\mu$, is

$$\mathcal{L} = \overline{\psi}(18-M)\psi + \mathcal{L}_{\text{ coupling}}$$

$$\mathcal{L}_{\text{ coupling}} = \frac{1}{2} (\partial_\mu \phi \phi^\mu - m_\phi^2 \phi^2) + g_s \overline{\psi} \phi \psi$$

$$- \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{2} m_\psi^2 \psi^{\mu\nu} - g_v \overline{\psi} V_\mu \psi$$

$$- \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e \overline{\psi} \gamma_\mu \frac{1}{2}(1 + \tau_3) A_\mu \psi$$

$$+ \frac{1}{2} (\partial_\mu \pi^\pm \partial_\nu \pi^\pm - m_\pi^2 \pi^\pm \pi^\pm) - i g_{\pi} \overline{\psi} Y_{5} \pi^\nu \pi^\mu$$

$$- \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} m_\rho^2 \rho^{\mu\nu} - g_{\rho} \overline{\psi} \gamma_\mu \rho^{\nu} \psi,$$

with

$$G_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,$$

$$\psi = (18-M)\psi.$$
and with the gauge conditions
\[ \begin{align*}
\partial_\mu V^\mu &= 0, \\
\partial_\mu A^\mu &= 0, \\
\partial_\mu b^\mu &= 0.
\end{align*} \] (5)

The first line is the Lagrangian for a free nucleon field. The last terms in the other lines are the nucleon-boson coupling terms, which are responsible for generating the nucleon-nucleon interaction. Here "boson fields" includes both the meson fields and the electromagnetic field in the above Lagrangian. The other terms are the free Lagrangian terms for the boson fields.

In the above Lagrangian, the arrow over the meson fields denotes that these fields are vector in isospin space and \( \tau_I \) are the associated 2x2 isospin matrices. The nucleon field \( \psi \) is a four-component Dirac spinor multiplied with the two-component isospinor of which the up-component is proton and the down-component is neutron. The Lorentz four-vectors and the various matrices are defined as in Ref. [68], that is

\[ V^\mu = (v_t, v_x, v_y, v_z) = g^{\mu\nu}v_\nu \]

with

\[ g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \]

\[ \partial_\mu = (\partial_t, \vec{\partial}) \]
\[ \gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix} \text{ and } \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \text{ for } i = 1, 2, 3, \]

\[ \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma_5 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}, \]

\[ \gamma = \gamma^\mu \gamma_\mu = \gamma^0 \gamma_t + \gamma_\cdot \vec{\gamma}, \]

where \( \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) is the 2x2 unit matrix and the 2x2 Pauli matrices \( \sigma^i \) are

\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

\( \gamma_t \) is the time derivative and \( \vec{\gamma} \) is the usual three-dimensional gradient.

For a more realistic theory, we may also need to consider meson-meson interactions such as the SU(2)xU(1) Lagrangian of Ref. [18]. In a relativistic Hartree approximation without any meson-meson interaction, the mean pion field is zero. Here the source term, which is an isoscalar density of the nucleon field, is zero by time reversal invariance. However, a pion field can exist in a relativistic Hartree-Fock approximation since the source is non-zero through the off-diagonal terms of the isoscalar density matrix. In a one-boson exchange potential approximation, which has no meson-meson interactions, numerical instability problems arise in relativistic Hartree-Fock calculations [38] due to a large effect of the pion field induced by the pion-nucleon couplings. In this calculation, the lower components of the single-nucleon wavefunction become larger and larger during the
Hartree-Fock iterations. To remove this problem, we may need some mechanism which can reduce the effect of the pion field. One possible mechanism would be including other interactions such as meson-meson interactions and derivative couplings. These also have problems such as pion condensation. Tezuka [19] has pointed out that pions, which are the least massive mesons with a mass of about 140 MeV, condense at normal nuclear density in a sigma model which has a sigma-pion interaction. However, for simplicity in deriving relativistic Hartree-Fock equations, we shall follow the conventional work in which these meson-meson interactions are ignored.

In section A, a brief review of mean field theory is presented. Introducing the creation and annihilation operators, a representation for the field operators is established. The ground state of a nucleus is defined in this representation. In section B, a semi-classical action is constructed as the expectation of the Lagrangian, Eq. (1), in this ground state. Using the variational method, the equations of motion for each mean field are derived in the mean field approximation in section B. Finally, in section C, expressions are derived for physical observables in terms of these mean fields.

A. MEAN FIELD THEORY

For a nuclear system which is described by the Lagrangian density of Eq. (1), we cannot evaluate the exact solution. If we use the path integral method [48-66], the effective action becomes fourth order in the nucleon field after integrating over the meson fields. This
resulting integral cannot be evaluated exactly. Furthermore, we cannot use an ordinary perturbation method either since the dimensionless coupling constants in Eq. (1) are of the order of 10 (see Tables 2 and 5 [15-45]). Therefore, we need to use an approximation which can manage the strong coupling aspects of the problem. A possible approximation is mean field theory [1,15].

In the mean field theory, a semi-classical action is defined by a matrix element of the Lagrangian between the chosen states of the many-body system. Through this matrix element calculation, the field operators are replaced with the corresponding semiclassical fields. Applying a variational principle on this action, we get the self-consistent equations of motion for each of these semi-classical fields. Physical observables may then be expressed in terms of these fields. Thus, the complexity of the full quantum field theory is greatly reduced in the mean field theory. Using normal ordering, the vacuum expectation values are removed. Since self-consistent interacting fields are used instead of free fields, there are no divergence problems. Furthermore, renormalization is carried out by simply adjusting the parameters, the masses and the coupling constants, to yield the experimental values of typical physical quantities.

Given the Lagrangian of Eq. (1), we will derive a relativistic Hartree-Fock approximation, of which the Fock term will provide the next order correction to a relativistic Hartree approximation. To evaluate the matrix element of the Lagrangian, we need to express the field operators in terms of the single-particle operators.
Conventionally, the nucleon field is considered as a quantized field, and the boson fields as classical. The nucleon field operator is decomposed into creation and annihilation operators in definite states. Due to the classical treatment, boson fields are c-number functions and the mean field theory reduces to a relativistic Hartree approximation. To go beyond this level of approximation, we need to consider the meson fields as quantized fields. However, this is more complicated than the classical treatment since the meson field operators depend on the nucleon field operators as sources. To solve for the meson fields in terms of the nucleon field, we consider the equations of motion for the boson field operators. Using the Heisenberg's equation of motion or equivalently the Euler-Lagrange equation for the Lagrangian density operator $\hat{\mathcal{L}}(x)$, we get

$$\left[\Box + m^2\right] \hat{\phi}^i(x) = g_1 \hat{\psi}(x) \Gamma_1 \hat{\psi}(x). \quad (6)$$

The index $i$ labels different boson fields, and the related quantities are defined in Table 3. Since the Laplace factor $[\Box + m^2]$ has no dependence on the field operators, we find the general solution of this operator equation as

$$\hat{\phi}^i(x) = \hat{\phi}_0^i(x) - \int d^4 x' D(x-x', m^2) g_1 \hat{\psi}(x') \Gamma_1 \hat{\psi}(x'), \quad (7)$$

where $\hat{\phi}_0^i$ is the solution of the homogeneous part of Eq. (6), and can be interpreted as free boson fields for our Lagrangian. The second term is a source dependent part of the equation since it depends on the nucleon field. Through Eq. (7), we can separate the boson field operators into a nucleon field dependent part and the free boson field.
Table 3. Meson fields which are included in the model Lagrangian of Eq. (1)

<table>
<thead>
<tr>
<th>Field Type</th>
<th>Field</th>
<th>$g_i$</th>
<th>$s_i$</th>
<th>$m_i$</th>
<th>$\Gamma_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar field</td>
<td>$\phi(x)$</td>
<td>$g_S$</td>
<td>$s_S$</td>
<td>$m_S$</td>
<td>$\Gamma_S$</td>
</tr>
<tr>
<td>Vector field</td>
<td>$V_\mu(x)$</td>
<td>$g_V$</td>
<td>$s_V$</td>
<td>$m_V$</td>
<td>$\gamma_\mu$</td>
</tr>
<tr>
<td>E.M. field</td>
<td>$A_\mu(x)$</td>
<td>$e$</td>
<td>$-s$</td>
<td>$0$</td>
<td>$\frac{1}{2}(\gamma_\mu + \gamma_5 \gamma_3)$</td>
</tr>
<tr>
<td>Pion field</td>
<td>$\pi^+(x)$</td>
<td>$-ig_\pi$</td>
<td>$s_\pi$</td>
<td>$m_\pi$</td>
<td>$\gamma_5 \gamma^+$</td>
</tr>
<tr>
<td>Rho meson</td>
<td>$b_\mu(x)$</td>
<td>$g_\rho$</td>
<td>$s_\rho$</td>
<td>$m_\rho$</td>
<td>$\gamma_\mu \gamma^+$</td>
</tr>
</tbody>
</table>
part. The Green's function $D$ is defined as

$$[\hat{\mathbf{\square}} + m_i^2]D(x-x',m_i) = -\delta(x-x'),$$

(8)

$$D(x-x',m_i) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x-x')}}{(k \cdot k - m_i^2)}$$

(9)

$$= \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} D(x-x',m_i),$$

where

$$m_i^2 = m_i^2 - \omega^2,$$

(10)

$$D(x-x',m_i) = D(x-x',\omega) = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot (x-x')}}{\left| k \right|^2 - m_i^2}$$

(11)

$$= -\frac{1}{4\pi} \frac{e^{-m_i \left| x-x' \right|}}{\left| x-x' \right|^2}.$$

To study the ground state properties of nuclei, we expand each field in a complete set of single-particle states. For the nucleon, these are the eigenstates of the single-particle Hamiltonian, and should be determined self-consistently through equations (24)-(26). The single-particle states of the meson fields are the eigenstates of the free field equation; i.e., the homogeneous part of Eq. (6). From these, we can define the vacuum state $|0\rangle$ in which all the particle and antiparticle levels are empty. The annihilation operators of the nucleon $|b_\alpha\rangle$ antinucleon $|d_\alpha\rangle$, and boson $|a_i^\alpha\rangle$ fields are defined as the operators which annihilate the vacuum. That is, $b_\alpha |0\rangle = 0,$

$d_\alpha |0\rangle = 0,$ and $a_i^\alpha |0\rangle = 0$ where $\alpha$ is the single-particle level index of each state. The index $i$ which labels different boson fields is
defined in Table 3. The creation operators \( (b^\dagger_\alpha, d^\dagger_\alpha, a^\dagger_\alpha) \) of these fields are defined as the hermitian conjugate of the corresponding annihilation operators. The creation and annihilation operators of boson fields commute with all the creation and annihilation operators except \([a^\dagger_\alpha a^{\dagger}_\beta] = \delta_{\alpha\beta} \delta_{ij}\). The creation and annihilation operators of the nucleon and antinucleon anticommute with each other except for \([b^\dagger_\alpha, b^\dagger_\beta] = \delta_{\alpha\beta} \) and \([d^\dagger_\alpha, d^\dagger_\beta] = \delta_{\alpha\beta}\). The field operators can be decomposed using the single-particle creation and annihilation operators as

\[\hat{\psi}(x) = \sum_{\alpha} \left[ \psi_\alpha^{(+)}(x)b_\alpha + \psi_\alpha^{(-)}(x)d_\alpha^\dagger \right],\]

\[\hat{\psi}(x) = \hat{\psi}^{\dagger}(x)\gamma_0 = \sum_{\alpha} \left[ \bar{\psi}_\alpha^{(+)}(x)b_\alpha^\dagger + \bar{\psi}_\alpha^{(-)}(x)d_\alpha \right],\]

for the nucleon field, and for the boson fields \(\phi^\dagger\), which are defined in Table 3,

\[\hat{\phi}^\dagger(x) = \hat{\phi}_0^\dagger(x) + \sum_{\alpha, \beta} \hat{\phi}_{\alpha\beta}^\dagger(x),\]

\[\hat{\phi}_0^\dagger(x) = \sum_{\alpha} \left[ f_\alpha^\dagger(x)a_\alpha + f_\alpha^{\dagger*}(x)a_\alpha^{\dagger} \right],\]

\[\hat{\phi}_{\alpha\beta}^\dagger(x) = - \int d^4 x' D(x-x', m_1) g_4 \left[ \psi_\alpha^{(+)}(x')b_\alpha^\dagger + \psi_\alpha^{(-)}(x')d_\alpha \right] \times \Gamma_1 \left[ \psi_\beta^{(+)}(x')b_\beta + \psi_\beta^{(-)}(x')d_\beta^\dagger \right].\]

The first term of Eq. (14) is the free boson field and the second term is the virtual boson induced through the boson-nucleon interaction. The functions \(f_\alpha^\dagger(x), \psi_\alpha^{(+)}(x), \) and \(\psi_\alpha^{(-)}(x)\) are the orthonormalized wave functions for each single-particle level.

In the relativistic calculations, the simplest ground state of a nucleus can be chosen as a Slater determinant \(\prod_{\alpha=1}^{A} b^\dagger_\alpha |0\rangle\) of the lowest A occupied single-nucleon levels. Here there are no
antinucleons and free bosons. Usually, the virtual boson fields, induced through the boson-nucleon interactions, are treated as classical fields. In this treatment, the induced boson fields are considered as independent fields. Thus the induced boson field does not carry the information about which nucleons are involved in the interaction through a particular boson exchange. To consider this information, we need to treat the induced boson fields as quantized fields as in Eq. (16).

In this section, we have expressed the field operators and the ground state of a nucleus in terms of the creation and annihilation operators of the interacting nucleon field and the free meson fields. The creation and annihilation operators need yet to be defined self-consistently through the equations of motion, which will be derived in the next section.

B. THE EQUATIONS OF MOTION OF THE MEAN FIELDS

Since the field operators and the ground state are expressed in terms of the single-particle operators, we can evaluate the semi-classical action for the Lagrangian of Eq. (1). Applying a variational principle on this action, the equations of motion of each field will be derived in this section. Employing quantized meson fields, these results will yield the next order correction to the relativistic Hartree approximation.

The semi-classical action $S$ is the expectation, for the ground state $|\phi_0\rangle$, of the normal ordered operator $\int d^4 x \hat{\mathcal{L}}(x)$. Normal
ordering is defined with respect to the vacuum state $|0\rangle$. For the free nucleon part of the Lagrangian, we need to find the expectation of $i\hat{\psi}(x)\hat{\psi}(x)$: between the $A$-nucleon Slater determinant,

$$|\phi_0\rangle = \prod_{\gamma=1}^A b_\gamma^\dagger |0\rangle.$$ From Eqs. (12) and (13),

$$i\hat{\psi}(x)\hat{\psi}(x): = \sum_{\alpha, \beta} \left[ \overline{\psi}_\alpha(x) b_\alpha^\dagger + \overline{\psi}_\alpha(x) d_\alpha \right] \left[ \psi_\beta(x) b_\beta + \psi_\beta(x) d_\beta^\dagger \right].$$

Since $|\phi_0\rangle$ is constructed only with $A$-nucleon levels in our case, $b_\alpha^\dagger |\phi_0\rangle = 0$ and $d_\alpha^\dagger |\phi_0\rangle = 0$ for an unoccupied level $\alpha$. For the occupied nucleon level $\alpha$, $b_\alpha^\dagger |\phi_0\rangle = \pm |\phi_{0-\alpha}\rangle$, where the new state $|\phi_{0-\alpha}\rangle$ is the Slater determinant of $A-1$ levels without the level $\alpha$ and the $\pm$ sign is determined through the anticommutation of $b_\alpha$ with $b_\gamma^\dagger$ to come next to $b_\alpha^\dagger$. Therefore

$$\frac{\langle \phi_0 | \hat{\psi}(x)\hat{\psi}(x): | \phi_0 \rangle}{\langle \phi_0 | \phi_0 \rangle} = \sum_{\alpha} \frac{\overline{\psi}_\alpha(x)\psi_\alpha(x)}{\overline{\psi}_\alpha(x)\psi_\alpha(x)},$$

where $\psi_\alpha(x)$ has been used for $\psi_\alpha^{(+))(x)}$ to simplify the notation because the antinucleons no longer appear in this expectation. The $A$ over the summation sign has been used to indicate that the sum $\alpha$ is limited to the $A$ occupied levels of the nucleon field.

Since there are no free bosons in the ground state, the free boson field part of Eq. (7) or (14) makes no contribution to the expectation of the Lagrangian. Thus for the boson field dependent
terms of the Lagrangian, we need to evaluate an expectation of the
form \( \hat{\psi}(x) \Gamma_1 \hat{\psi}(x') \Gamma_1 \hat{\psi}(x') \hat{\psi}(x) \) with normal ordering. This becomes,

\[
\frac{\langle \Phi_0 | \hat{\psi}(x) \Gamma_1 \hat{\psi}(x') \Gamma_1 \hat{\psi}(x') \hat{\psi}(x) | \Phi_0 \rangle}{\langle \Phi_0 | \Phi_0 \rangle}
\]

(18)

\[
= \sum_{\alpha, \beta} \left[ \overline{\psi}_\alpha(x) \Gamma_1 \overline{\psi}_\beta(x') \Gamma_1 \psi_\alpha(x) - \overline{\psi}_\alpha(x) \Gamma_1 \overline{\psi}_\alpha(x') \Gamma_1 \psi_\beta(x) \right],
\]

where the sums on \( \alpha \) and \( \beta \) are restricted to the occupied levels.

Since the terms with \( \alpha = \beta \) cancel each other, we do not need the condition of \( \alpha \neq \beta \) which is demanded by the Pauli principle. If we relate this result to the effective nucleon-nucleon interaction, the first term is the Hartree (direct) term, and the second term is the Fock (exchange) term. Because of Eq. (7) or (16), we can define the mean boson fields in the ground state as

\[
\phi_\alpha^1(x) = - \int \frac{d^4 x'}{m_1} D(x-x', m_1) e_1 \sum_{\alpha, \beta} = \sum_{\alpha, \beta} \overline{\psi}_\alpha(x) \Gamma_1 \psi_\beta(x),
\]

(19)

\[
\phi_{\alpha \beta}^i(x) = - \int \frac{d^4 x'}{m_1} D(x-x', m_1) e_i \rho_{\alpha \beta}^{i1}(x'),
\]

(20)

where the density matrices for each boson field are defined as

\[
\rho_{\alpha \beta}^{i1}(x') = \overline{\psi}_\alpha(x') \Gamma_1 \psi_\beta(x').
\]

(21)

Using Eqs. (18)-(20), in terms of the mean fields of nucleons and bosons, we get

\[
\frac{\langle \Phi_0 | \hat{\psi}^1(x) \hat{\psi}^1(x') \hat{\psi}_H^1(x) | \Phi_0 \rangle}{\langle \Phi_0 | \Phi_0 \rangle} = \phi_\alpha^1(x) \phi_{H1}^{i1}(x) - \sum_{\alpha, \beta} \phi_{\alpha \beta}^i(x) \phi_{\beta \alpha}^i(x),
\]

(22)
We should notice that the mean boson fields also depend on the nucleon level indices \( \alpha \) and \( \beta \).

Using Eqs. (17), (22), and (23), we get the expectation of the Lagrangian, Eq. (1), as

\[
\langle \phi_0 | \hat{\psi}(x) \Gamma_i \hat{\phi}^i(x) \hat{\psi}(x) | \phi_0 \rangle = \sum_{\alpha} \overline{\psi}_\alpha(x) \Gamma_i \hat{\phi}^i(x) \psi_\alpha(x)
\]

\[
= \sum_{\alpha} \overline{\psi}_\alpha(x) \Gamma_i \left[ \delta_{\alpha \beta} \phi_\beta(x) - F_{\beta \alpha}(x) \right] \psi_\beta(x).
\]

where \( s_i \) is defined in Table 3. Here the conditions of Eqs. (2)–(5) have been used to simplify the equation. If we eliminate the boson fields using Eqs. (19) and (20), this semi-classical Lagrangian becomes an effective Lagrangian in terms of nucleon mean fields. By the variation of the action \( S = \int d^4 x \langle \hat{\phi}(x) : \rangle \) with respect to each of the fields, the equations of motion are

\[
\sum_{\alpha, \beta} \left[ \overline{\psi}_\alpha(x) \Gamma_i \hat{\phi}^i(x) \psi_\beta(x) \right] = 0,
\]

\[
[ \square + m_i^2 ] \phi_i^0(x) = g_i \sum_{\alpha} \rho_{i \alpha}(x),
\]
with
\[ \phi_{\beta \alpha}^{1}(x) = \phi_{H}^{1}(x) \delta_{\alpha \beta} - \phi_{FB}^{1}(x). \] (27)

The density matrices are defined by Eq. (21). The sum \( \alpha \) and \( \beta \) are limited to the levels in the ground state \( |\psi_{0}> \) which are occupied by nucleons.

Since Eqs. (25) and (26) are linear equations for the meson fields, their solution may be expressed by
\[ \phi_{H}^{1}(x) = - \int d^{4} x' D(x-x', m) g_{1} \sum_{\alpha} \rho_{\alpha \alpha}^{1}(x'), \] (19)
\[ \phi_{FB}^{1}(x) = - \int d^{4} x' D(x-x', m) g_{1} \rho_{\alpha \beta}^{1}(x'). \] (20)

Inserting Eqs. (19) and (20) into Eq. (24), we get a nonlinear equation for a single nucleon state
\[ [(i\beta-M) + \sum_{i} \int d^{4} x' D_{i}(x-x', m) \sum_{\beta} \bar{\psi}_{\beta}(x') \Gamma_{i} \psi_{\beta}(x') \Gamma_{i}] \psi_{\alpha}(x) \] (28)
\[ - \sum_{i} \int d^{4} x' D_{i}(x-x', m) \Gamma_{i} \sum_{\beta} \bar{\psi}_{\beta}(x) \psi_{\beta}(x') \Gamma_{i} \psi_{\alpha}(x') = 0, \]
where
\[ D_{i}(x-x', m) = -s_{i}(g_{1})^{2} D(x-x', m). \] (29)

Equation (28) is the Dirac equation for single nucleon states in the Hartree-Fock approximation.

For states in a nucleus, \( \psi_{\alpha}(x) \) becomes \( \exp(-i\omega t)\psi_{\alpha}(x) \), and the time integral in Eq. (28) gives an energy conserving delta function. Equation (28) reduces to an eigenvalue problem for a time-independent Hartree-Fock Dirac equation for single nucleon states.
\[ \omega_a \psi_a(x) = \int d^3x' h(x,x') \psi_a(x') , \]

\[ h(x,x') = \delta(x-x')[(-i\alpha \gamma^0 + \gamma_0 \gamma^0) \]

\[ - \gamma_0 \sum \int d^3x'' D_1(x-x'',m) \bar{\psi}_\beta(x'') \Gamma_1 \psi_\beta(x') \Gamma_1 ]_{(30)} \]

\[ + \gamma_0 \sum \int [D_1(x-x'',\omega_\alpha - \omega_\beta) \Gamma_1 \psi_\beta(x') \bar{\psi}_\beta(x') \Gamma_1 ] \]

where \( \alpha = \gamma_0 \gamma^0 \) and the \( \omega_{\alpha} - \omega_{\beta} \) dependence of \( D_1 \) comes from the time integral \( t' \) in Eq. (28). This dependence shows that the meson propagator for the Fock term, which is nonlocal, must be retarded. This is the usual Hartree-Fock single-particle equation with general retarded nonlocal potentials. We can find the single-particle wave function \( \psi_a(x) \) by solving Eq. (30) self-consistently using the techniques discussed in the Appendix for spherical nuclei or in Chapter III for axially deformed nuclei. We assumed at the beginning a basis for the decomposition of field operators and the construction of the ground state \( \Phi_0 \). The single-particle levels of the self-consistent solution determine this basis.

We have obtained a relativistic Hartree-Fock approximation from the relativistic Lagrangian defined in Eq. (1). The mean field approximation has been used for the nucleon field, and the meson fields have been interpreted exactly in terms of the mean nucleon field. However, for the lowest order approximation, the meson fields may be considered as classical fields [15]. For classical meson fields, the expectation of the meson-nucleon interaction terms in the
Lagrangian become $\tilde{\phi}^i(x) \sum_{\alpha} \bar{\psi}_{\alpha}(x) \Gamma_{\alpha}^{\dagger} \psi_{\alpha}(x)$. Furthermore, $\tilde{\phi}^i$ in Eq. (24) has no dependence on the nucleon level index $\alpha$. In this lowest order approximation, the exchange (Fock) term, Eq. (20), does not appear in Eqs. (22) and (23). Due to the absence of the Fock term, the equation of motion for the nucleons becomes a relativistic Hartree Dirac equation instead of a Hartree-Fock equation. The classical meson fields are then given by

$$\tilde{\phi}^i(x) = -g_{\lambda} \int d^4 x' D(x-x',m_1) \frac{A}{\alpha} \rho_{\alpha}(x').$$

These approximations result in the relativistic mean field theory of Walecka [15].

C. PHYSICAL OBSERVABLES

We can calculate the value of physical quantities in the relativistic Hartree-Fock description of a nucleus, using Eqs. (19), (20), and (30). The baryon density in a nucleus becomes

$$\rho^{+}(x) = \sum_{\alpha} \bar{\psi}_{\alpha}(x) \gamma_0 \psi_{\alpha}(x).$$

(31)

The root mean square radius and the multipole moments of the nuclear density can be calculated from

$$R = \left[ \int d^3 x |^{+}_{x} |^2 \rho^{+}(x)/A \right]^{1/2},$$

(32)

$$Q_{\lambda m} = \int d^3 x |^{+}_{x} |^2 Y_{\lambda}^{m}(x) \rho^{+}(x),$$

(33)

where $A$ is the number of nucleons in a nucleus.

To find the total energy, let us consider the Hamiltonian of the
system defined by Eq. (1). Since this Lagrangian has no velocity-dependent interactions, the conjugate momentum of the nucleon and boson fields are $i\psi^\dagger$ and $s_i \partial_t \phi^i$. Therefore the Hamiltonian of this system is

$$H = \int d^3x \left[ \bar{\psi}(x) \gamma_0 i \partial_t \psi(x) + \sum_i s_i \partial_t \phi^i(x) \partial_t \phi^i(x) - \mathcal{L}(x) \right]. \quad (34)$$

Replacing the fields with the corresponding mean fields which we have found, and using the gauge conditions (5) and the equations of motion (24)-(26), the total energy becomes

$$E = \int d^3x \sum_{\alpha} \left[ \bar{\psi}_{\alpha}(x) \gamma_0 i \partial_t \psi_{\alpha}(x) 
+ \sum_i s_i \phi_H^i(x) \left\{ \frac{1}{2} \left[ \Box + m_i^2 \right] - \partial_t^2 \right\} \phi_H^i(x) 
- \sum_i s_i \sum_{\alpha,\beta} \phi_{F_{\alpha \beta}}^i(x) \left\{ \frac{1}{2} \left[ \Box + m_i^2 \right] - \partial_t^2 \right\} \phi_{F_{\alpha \beta}}^i(x) \right]. \quad (35)$$

Using Eqs. (8), (19), and (20) and $\psi_{\alpha}(x) = \exp[-i \omega_{\alpha} t] \psi_{\alpha}(x)$ for a static nucleus, we get

$$E = \sum_{\alpha} \omega_{\alpha} 
+ \frac{1}{2} \sum_{i,\alpha,\beta} \int d^3x \int d^3x' \left[ \bar{\psi}_{\alpha}(x) \Gamma_i^\dagger \psi_{\alpha}(x) \right. $$
$$\times D_i^\dagger(\mathbf{x} - \mathbf{x}', m) \bar{\psi}_{\beta}(x') \Gamma_i \psi_{\beta}(x') 
- \bar{\psi}_{\alpha}(x) \Gamma_i^\dagger \psi_{\alpha}(x) D_i(\mathbf{x} - \mathbf{x}', \omega_{\alpha} - \omega_{\beta}) \bar{\psi}_{\beta}(x') \Gamma_i \psi_{\beta}(x') \right] \quad (36)$$

The first term is the sum of the single nucleon energies. The second and the third terms are the direct (Hartree) and exchange (Fock) parts of the nucleon-nucleon interaction energy through meson exchange.
In this chapter, we have explicitly derived the relativistic Hartree-Fock description of nuclear systems defined with the Lagrangian of Eq. (1). Usually, meson fields are treated as classical fields in applying a mean field theory to this Lagrangian. This mean field approximation reduces to a relativistic Hartree approximation. Introducing quantized meson fields, we have shown that the exchange (Fock) terms are naturally included in a mean field approximation of the Lagrangian of Eq. (1). Since meson field operators are exactly expressed in terms of single nucleon operators, this Hartree-Fock approximation is the highest order approximation which we can obtain in a mean field approach. To go beyond this approximation, we would need to use approaches other than a mean field theory or use a more realistic ground state of a nucleus in which many-body effects can be considered. However, these new approaches should be able to reduce to the relativistic Hartree-Fock approximation as the first-order approximation.
As pointed out in Chapter II, the lowest approximation of the meson field theory for a nucleus is the relativistic Hartree approximation in which meson fields are treated as classical fields. By ignoring the vacuum fluctuation contributions to the nucleon field, this approximation is equivalent to a relativistic mean field theory [15]. Relativistic calculations for static finite nuclei have only been performed for spherically symmetric nuclei [35-40] using the model parameters determined from bulk properties of nuclear matter. In such calculations, part of the quantum effects are included indirectly by fitting the parameters of the Lagrangian to the empirical properties of nuclei. In the spherical calculations the lower components of the Dirac wavefunctions distinguish the scalar density from the baryon density and provide a density dependence for the central potential. Furthermore, spin-orbit splittings between single-particle levels are reproduced approximately, and consequently a relativistic nuclear shell model is established.

The detailed structure of deformed nuclei provides a much wider range of experimental data to test relativistic nuclear shell models and examine the foundation for the relativistic study of the collision problem. Quantities such as the intrinsic quadrupole moments of nuclei can be expected to have a different dependence on the relativistic effects than the bulk properties such as the total energy of the system. Through these quantities, we can also test the
relativistic Hartree approximation to see how well it can reproduce the properties of deformed systems. Deformed nuclei require a representation of the fields and single-particle wavefunctions in at least two spatial dimensions. If we assume axial symmetry, we can eliminate the azimuthal angle dependence in the field equations and therefore only need to consider two spatial dimensions, \( r \) and \( z \) in a cylindrical coordinate system.

In Section A of this chapter, the relativistic expressions for the axially symmetric nuclei are derived starting from the Lagrangian density given in Eq. (1) which is expanded to include nonlinear meson field \([20,35]\) terms through fourth order. First, the action of the system is discretized on the cylindrical two-dimensional grid. The variational method is applied on this discretized action to obtain the equations of motion for each of the fields. An outline of the method of solution is presented in Section B, together with the expressions on the space grid for physical quantities characterizing the system. Results for some spherically and axially symmetric nuclei obtained using this method are presented in Section C. For the case of spherically nuclei, a comparison is made with results of other calculations \([35-40]\).

A. EQUATIONS OF MOTION FOR AXIALLY SYMMETRIC TWO-DIMENSIONAL LATTICES

The Lagrangian density is taken to be

\[
\mathcal{L} = \mathcal{L}_{\text{lin}} - \frac{1}{3} \sum_1 \, g_2 \, \phi^3 - \frac{1}{4} \sum_1 \, g_3 \, \phi^4 ,
\]  

(37)
where \( \mathcal{L}_{\text{lin}} \) is the Lagrangian density with linear meson fields given by Eq. (1). The other terms represent the nonlinear self-interaction of meson fields [20,35]. These nonlinear terms were not included in the previous chapter. With these terms, Eq. (6) would be modified to become a nonlinear equation for the meson field operator, which is complicated to solve. However, in Hartree approximations, we treat the meson fields as classical. Therefore, we do not need to solve the operator Eq. (6). Instead, we need to solve a classical nonlinear equation for meson fields. This can be done in a simple way, as will be described later. We follow the same approach used in Chapter II, except here we impose explicitly the requirement of axial symmetry and we work in a lattice space.

For an axially symmetric system, the \( z \) component of total angular momentum, \( \mu \), is a good quantum number. The wave functions are thus eigenstates of \( J_z \) and hence contain a phase factor \( R_z(\varphi) = e^{-i(\mu-\sigma/2)\varphi} \) where \( \sigma/2 \) is the \( z \) component of the spin. The parity of the wave functions can be decomposed as a product of the \( r \) parity, which is defined as a reflection of \( x \) and \( y \) to \(-x\) and \(-y\), and the \( z \) parity, which is defined as the reflection of \( z \) to \(-z\). The \( r \) parity, which is equivalent to a 180 degree rotation about the \( z \)-axis, gives

\[
\Pi_r = R_z(\pi) = (-1)^{\mu-\sigma/2}
\]

For a nucleus with this symmetry, the \( z \) parity \( \Pi_z \), which is the reflection about the \( x-y \) plane, is given by

\[
\Pi_z = (-1)^{n_z},
\]

where \( n_z \) is the number of nodes in the \( z \) direction. For nuclei with good \( z \) parity, we need only consider positive \( \mu \) levels, since the state with \(-\mu\) is degenerate with the state \(+\mu\). Each
component of the Dirac spinor has a special relation to the other components under these parity operators. For axially symmetric nuclei, the relative r parities for each spinor component are (+, −, +, −). For a system with reflection symmetry about the x-y plane, the z parity is a good quantum number and the relative z parities of each component become (+, −, −, +). The relative total parities are thus (+, +, −, −). The system is time reversal invariant which further simplifies the problem. Invariance under the time reversal operator, \( T = i \gamma_1 \gamma_3 \) [68], imposes the conditions \( \bar{\psi}_a \gamma_a \psi = 0 \) and \( \bar{\psi}_a \gamma_5 \psi = 0 \). Here \( \psi_a \) is a single-nucleon wave function.

The single-nucleon state \( \psi_a \) in an axially symmetric nucleus thus becomes

\[
\psi(x) = e^{-i \omega t} \begin{pmatrix}
 e^{i(u-1/2)\varphi} U^{++}(r,z) \\
 e^{i(u+1/2)\varphi} U^{+-}(r,z) \\
 e^{i(u-1/2)\varphi} U^{-+}(r,z) \\
 e^{i(u+1/2)\varphi} U^{--}(r,z)
\end{pmatrix},
\]

where the index \( a \) has been omitted for simplicity. The gamma matrices are given explicitly, in cylindrical coordinates, as

\[
\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\]

and

\[
\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_r = \begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}, \quad \sigma_\varphi = \begin{pmatrix} 0 & -e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}.
\]

Using Eqs. (38) and (39), the densities in Eq. (21), which are the sources of the meson fields, become
where the index $i$ and the $\Gamma_i$ are defined in Table 3. The $\gamma_\mu$ in $\Gamma_i$, for axially symmetric nuclei, are

$$\gamma_0 = \gamma_0, \gamma_z = \gamma_3, \gamma_r = \gamma_1, \text{ and } \gamma_\varphi = \gamma_2.$$  

Note that Eqs. (40) and (41) have no azimuthal angle ($\varphi$) dependence.

For a static nucleus, $\rho = \sum \gamma_\alpha \psi_\alpha = 0$ and $\rho_5 = \sum \gamma_\alpha \gamma_5 \psi_\alpha = 0$ by time-reversal invariance. Since the densities $\rho_i$ are functions of $r$ and $z$ only, the meson fields also are independent of $\varphi$. Furthermore, the space components of the vector and rho meson fields become zero since $\rho = 0$. Thus, we need only consider the time component of the vector and rho meson fields. The pion field also does not contribute in the static Hartree approximation since the source of the pion is a pseudo-scalar, which is $\rho_5 = 0$. Since $\bar{\psi}_\alpha \gamma_1 \psi_\alpha = 0$ and $\bar{\psi}_\alpha \gamma_2 \psi_\alpha = 0$, the charged rho mesons do not contribute in this approximation.

To discretize the action of the Lagrangian, Eq. (37), we introduce the cylindrical two-dimensional grid as follows.

$$r_j = (j-1/2) \Delta r, \quad j = 1, 2, \ldots, N_r,$$

$$z_k = (k-1) \Delta z, \quad k = 1, 2, \ldots, N_z.$$  

The volume element is

$$\Delta V_j = (j-1/2) \Delta V,$$

$$\Delta V = 2\pi \Delta r \Delta z.$$  

The meson fields at grid points are defined as

$$\phi_i(j,k) = \sqrt{j-1/2} \phi_i(r_j, z_k).$$  

However, the nucleon field on the grid is necessarily more complicated due to the Fermion doubling problem [76]. To retain the
Hermiticity on the lattice, we need a symmetric definition for the
discretized spatial derivative. On the other hand, to eliminate the
Fermion doubling, we need to use adjacent points for the discretized
spatial derivative. For example, the dispersion relations of the free
one-dimensional Dirac equation [76] are
\[ \omega^2 - M^2 = \left[ 2 \sin \left( \frac{k \Delta x}{2} \right) \right]^2 \]
(45)
for \( (\partial_x \psi)(x_j) = \frac{1}{2 \Delta x} [\psi(x_{j+1}) - \psi(x_{j-1})] \).

\[ \omega^2 - M^2 = \left[ 2 \sin \left( \frac{k \Delta x}{2} \right) \right]^2 \]
(46)
for \( (\partial_x \psi)(x_j) = \frac{1}{\Delta x^2} [\psi(x_{j+1/2}) - \psi(x_{j-1/2})] \),

where \( k_x \) is the wavenumber. Both of these equations are hermitian
since a symmetric expression is used. As we can see in Fig. 7, the
first case has Fermion doubling problems (notice the degeneracy in the
spectrum) whereas the second case does not.

Thus, we eliminate the Fermion doubling problem without destroy­
ing the Hermiticity by defining each Dirac component of the nucleon
field on different mesh points as follows,
\[
\begin{align*}
P^+(j,k) &= \sqrt{J-1/2} \ U^+(r_j, z_k), \\
P^{-}(j+1/2,k+1/2) &= \sqrt{J} \ U^{-}(r_{j+1/2}, z_{k+1/2}), \\
P^+(j,k+1/2) &= \sqrt{J-1/2} \ U^+(r_j, z_{k+1/2}), \\
P^{-}(j+1/2,k) &= \sqrt{J} \ U^{-}(r_{j+1/2}, z_k).
\end{align*}
\]
(47)

Because the spinor components of the nucleon wave function depend on
the mesh points, the calculation of quantities that are a function of
the nucleon state on the grid are not obvious. We define the
Figure 7. Dispersion relations for the one-dimensional Dirac equation.

The dotted line is for Eq. (45) and the solid line is for Eq. (46).
DISPERSION RELATIONS

\[(\omega^2 - M^2)\Delta^2_x/4\]

\[k_x\Delta_x/\pi\]
following,

\[
\begin{align*}
[\kappa \nabla \mathcal{U}^n, \sigma] (r_{j^\prime} + 1/2, z_{k^\prime}) &= \frac{1}{\Delta r} \left[ \frac{\mathcal{P}^n_{\sigma} (j^\prime + 1, k^\prime)}{\sqrt{J^\prime + 1}} - \frac{\mathcal{P}^n_{\sigma} (j^\prime, k^\prime)}{\sqrt{J^\prime - 1}} \right], \\
[\kappa \nabla \mathcal{U}^n, \sigma] (r_{j^\prime}, z_{k^\prime} + 1/2) &= \frac{1}{\Delta z} \left[ \frac{\mathcal{P}^n_{\sigma} (j^\prime, k^\prime + 1)}{\sqrt{J^\prime + 1}} - \frac{\mathcal{P}^n_{\sigma} (j^\prime, k^\prime)}{\sqrt{J^\prime - 1}} \right], \\
[\mathcal{U}^n, \sigma] (r_{j^\prime} + 1/2, z_{k^\prime}) &= \frac{1}{2} \left[ \frac{\mathcal{P}^n_{\sigma} (j^\prime + 1, k^\prime)}{\sqrt{J^\prime + 1}} + \frac{\mathcal{P}^n_{\sigma} (j^\prime, k^\prime)}{\sqrt{J^\prime - 1}} \right], \\
[\mathcal{U}^n, \sigma] (r_{j^\prime}, z_{k^\prime} + 1/2) &= \frac{1}{2} \left[ \frac{\mathcal{P}^n_{\sigma} (j^\prime + 1, k^\prime) + \mathcal{P}^n_{\sigma} (j^\prime, k^\prime)}{\sqrt{J^\prime - 1}} \right],
\end{align*}
\]

where

\[
\begin{align*}
&j^\prime = j \quad \text{for } \sigma = +1, \\
&= j + 1/2 \quad \text{for } \sigma = -1, \\
&k^\prime = k \quad \text{for } \eta = \sigma, \\
&= k + 1/2 \quad \text{for } \eta = -\sigma.
\end{align*}
\]

Using Eqs. (44) and (47)-(51), the action on the grid becomes

\[
S = \int dt \ \Delta V \sum_{j, k} \sum_{\alpha} \sum_{\eta, \sigma} \left[ \mathcal{P}^n_{\alpha} (j^\prime, k^\prime) \mathcal{P}^n_{\alpha} (j^\prime, k^\prime) \right] + \frac{i}{\Delta r} \sqrt{J^\prime - 1/2} \mathcal{P}^n_{\alpha} (j^\prime, k^\prime) \left( \frac{\mathcal{P}^n_{\sigma} (j^\prime + 1/2, k^\prime)}{\sqrt{J^\prime + 1}} - \frac{\mathcal{P}^n_{\sigma} (j^\prime - 1/2, k^\prime)}{\sqrt{J^\prime - 1}} \right) + \frac{1}{\Delta z} \sqrt{J^\prime - 1/2} \mathcal{P}^n_{\alpha} (j^\prime, k^\prime) \left( \frac{\mathcal{P}^n_{\sigma} (j^\prime + 1/2, k^\prime) + \mathcal{P}^n_{\sigma} (j^\prime - 1/2, k^\prime)}{\sqrt{J^\prime}} \right)} \]

\[
+ \frac{1}{\Delta z} \left[ \mu_{\sigma} + \mu_{\eta}/2 \right] \mathcal{P}^n_{\alpha} (j^\prime, k^\prime) \left( \frac{\mathcal{P}^n_{\sigma} (j^\prime + 1/2, k^\prime)}{\sqrt{J^\prime + 1}} + \frac{\mathcal{P}^n_{\sigma} (j^\prime - 1/2, k^\prime)}{\sqrt{J^\prime - 1}} \right) + \frac{i}{\Delta z} \mathcal{P}^n_{\alpha} (j^\prime, k^\prime) \left( \mathcal{P}^n_{\sigma} (j^\prime, k^\prime + 1/2) - \mathcal{P}^n_{\sigma} (j^\prime, k^\prime - 1/2) \right)
\]
\[- \left( \sum_{i=1}^{n} s_i g_1 \eta_i t^i_{\alpha} \frac{\phi^i(j',k')}{\sqrt{j'-1/2}} \right) \rho^n_{\alpha} \sigma^*(j',k') \rho^n_{\alpha}(j',k') \]

\[+ \sum_{i=1}^{n} \left[ \frac{1}{2} s_i \left( \partial_t \phi^i(j,k) \right)^2 - \frac{1}{2} s_i m_1^2 \left( \phi^i(j,k) \right)^2 \right. \]

\[\left. - \frac{s_i}{2\Delta r} \left( \frac{\phi^i(j+1,k)}{\sqrt{j+1/2}} - \frac{\phi^i(j,k)}{\sqrt{j-1/2}} \right)^2 - \frac{s_i}{2\Delta z} \left( \frac{\phi^i(j+1,k)}{\sqrt{j+1/2}} - \frac{\phi^i(j,k)}{\sqrt{j-1/2}} \right)^2 \right. \]

\[\left. - \frac{1}{3} g_2 \left( \frac{\phi^i(j,k)}{\sqrt{j+1/2}} - \frac{\phi^i(j,k)}{\sqrt{j-1/2}} \right)^3 - \frac{1}{4} g_3 \left( \frac{\phi^i(j,k)}{\sqrt{j-1/2}} \right)^4 \right] \]

(53)

where \( j' \) and \( k' \) are the corresponding values of \( P^n_{\alpha} \sigma^*(j',k') \) in Eq. (52). The index \( i \) and \( s_i \) are defined in Table 3. The index \( i \), which should be summed, is restricted to the scalar field and time component of the vector, \( \rho \), and electromagnetic fields since the other components and the pion field vanish. The quantities \( t^i_{\alpha} \) and \( n^i \) are defined in Table 4. The meson fields \( \phi^i(j',k')/\sqrt{j'-1/2} \) are the interpolated values for the same mesh point as \( P^n_{\alpha} \sigma^*(j',k') \). This interpolation should be symmetric (such as Lagrange even-point interpolation) to maintain the Hermiticity of the field equations. For 2n-point interpolation

\[X(j',k') = X(j',k'),\]

\[X(j'+1/2,k') = \sum_{p=1}^{n} c_p [X(j'+p,k') + X(j'-p+1,k')],\]

\[X(j',k'+1/2) = \sum_{q=1}^{n} c_q [X(j',k'+q) + X(j',k'-q+1)],\]  

(54)
Table 4. The meson fields related quantities used in Eq. (53)

<table>
<thead>
<tr>
<th>$\phi^i$</th>
<th>$\phi$</th>
<th>$V_0$</th>
<th>$A_0$</th>
<th>$b_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau^i_\alpha$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\tau_3$</td>
</tr>
<tr>
<td>$\eta^i$</td>
<td>$\eta$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
\[ X(j'+1/2,k'+1/2) = \sum_{q=1}^{n} \sum_{p=1}^{n} c_q c_p \{ X(j'+p,k'+q) + X(j'-p+1,k'+q) + X(j'+p,k'-q+1) + X(j'-p+1,k'-q+1) \}, \]

where \( X(j',k') \) is the quantity to be interpolated, \( \phi^\dagger(j',k')/\sqrt{j'-1/2} \) for each meson field in Eq. (53), and where \( c_p \) are the interpolation coefficients.

Applying the variational principle to this discretized action, we get the equations of motion for single-nucleon states and for the meson fields. For the single-nucleon states, we obtain

\[
\omega \begin{pmatrix} P^+ \\ P^- \end{pmatrix} = \begin{pmatrix} M + W_+ & V \\ V^\dagger & -M + W_- \end{pmatrix} \begin{pmatrix} P^+ \\ P^- \end{pmatrix},
\]

(55)

where \( P^\pm = \begin{pmatrix} P^{++} \\ P^{+-} \end{pmatrix} \) and \( P^- = \begin{pmatrix} P^{-+} \\ P^{--} \end{pmatrix} \) are two-component vectors, and 2x2 matrices \( W_\pm \) and \( V \) are

\[
\omega_{\gamma,\gamma'}(j',k') = -\sum_i g_i g_i' \delta_{\gamma,\gamma'} \frac{1}{\sqrt{j'-1/2}} \delta_{\sigma,\sigma'},
\]

(56)

\[
(\mathbb{V}P^-)^\sigma(j',k') = Z_+^\sigma(j',k')P^-s^\sigma(j',k'+1/2) + Z_-^\sigma(j',k')P^-s^\sigma(j',k'-1/2) + R_+^\sigma(j',k')P^-s^\sigma(j'+1/2,k') + R_-^\sigma(j',k')P^-s^\sigma(j'-1/2,k'),
\]

(57)

\[
\begin{align*}
Z_+^\sigma(j',k') &= -\frac{\sigma}{N_z} \\
Z_-^\sigma(j',k') &= \frac{\sigma}{N_z} = Z_+^{\sigma^*}(j',k'-1/2),
\end{align*}
\]

(58)
\[ \begin{align*}
R_+^{\sigma}(j',k') &= -\frac{i}{\Delta r} \sqrt{\frac{j'-1/2}{j'}} - \frac{i\sigma}{2\Delta r} \frac{(u+\sigma/2)}{\sqrt{j'(j'-1/2)}} \\
R_-^{\sigma}(j',k') &= \frac{i}{\Delta r} \sqrt{\frac{j'-1/2}{j'-1}} - \frac{i\sigma}{2\Delta r} \frac{(u+\sigma/2)}{\sqrt{(j'-1/2)(j'-1)}} = R_+^{\sigma*(j'-1/2,k')}.
\end{align*} \]

\( \phi^i(j',k')/\sqrt{j'-1/2} \) in Eq. (56) are the interpolated values of the meson fields using Eq. (54). Equations (55)-(59) maintain the single-nucleon hamiltonian as Hermitian.

The equation of motion for the meson fields are

\[ m_i^2 \phi^i + H \phi^i + V \phi^i = S^i, \tag{60} \]

where

\[ [H \phi^i](j,k) = -\frac{1}{\Delta z} \left[ \phi^i(j,k-1) - 2\phi^i(j,k) + \phi^i(j,k+1) \right], \tag{61} \]

\[ [V \phi^i](j,k) = -\frac{1}{\Delta r} \left[ C_-(j,k) \phi^i(j-1,k) - 2\phi^i(j,k) + C_+(j,k) \phi^i(j+1,k) \right], \tag{62} \]

\[ C_+(j,k) = C_-(j+1,k) = \frac{1}{\sqrt{(j+1/2)(j-1/2)}}, \tag{63} \]

\[ S^i(j,k) = \sqrt{j-1/2} \rho_1^i(j,k) - \sqrt{j-1/2} s_1^i g_2^i \left( \frac{\phi^i(j,k)}{\sqrt{j-1/2}} \right)^2 + s_3^i \left( \frac{\phi^i(j,k)}{\sqrt{j-1/2}} \right)^3. \tag{64} \]
\[ \rho^i(j,k) = \left( \frac{1}{j-1/2} \right)^A \sum_\alpha \left\{ | P^+(j,k) |^2 \right. \]
\[ + \sum_{q=1}^n \sum_{p=1}^n c_c c_p \left[ | P^+_{\alpha}(j-1/2+p,k-1/2+q) |^2 + | P^+_{\alpha}(j+1/2-p,k-1/2+q) |^2 \right] \]
\[ + | P^+_{\alpha}(j-1/2+p,k+1/2-q) |^2 + | P^+_{\alpha}(j+1/2-p,k+1/2-q) |^2 \]
\[ - s_i \sum_{q=1}^n c_q \left[ | P^-_{\alpha}(j,k-1/2+q) |^2 + | P^-_{\alpha}(j,k+1/2-q) |^2 \right] \]
\[ - s_i \sum_{p=1}^n c_p \left[ | P^-_{\alpha}(j-1/2+p,k) |^2 + | P^-_{\alpha}(j+1/2-p,k) |^2 \right] \] \( \left. t^i_{\alpha} \right. \).

In Eq. (65), \( s_i \) and \( t^i_{\alpha} \) came from the factor of \( \Gamma_i \) in Eq. (40). This result for the densities demonstrates that the interpolation scheme here is the same as was used for the nucleon equations. This came about because we have used a symmetric interpolation, Eq. (54). In the expressions for the meson field equations, the nonlinear terms have been included into the source term \( S^i \). These nonlinear terms also can be grouped with the \( H \) and \( V \) terms in Eq. (60). The final solution will be the same in any case when the equations are solved self-consistently.

Summarizing this section, the discretized Dirac equation, Eq. (55), for the single-nucleon state and the discretized Helmholtz equation, Eq. (60), for meson fields together with Eqs. (56)-(59) and Eqs. (61)-(65) have been obtained for static axially symmetric nuclei. Using the different mesh representation for different components of the Dirac spinor, we have eliminated the Fermion doubling problem. Hermiticity has also been retained throughout. The meson field
equations are the same as in the nonrelativistic case [78]. The equa-
tion for the single-nucleon wave functions has the same general struc-
ture as in the nonrelativistic case [77-80], except it is a Dirac
four-component equation and a two-point equation for the derivatives.

B. NUMERICAL METHODS AND PHYSICAL OBSERVABLES ON THE GRID

The self-consistent coupled equations (55) and (60) can be solved
iteratively. For the discretized meson field equation, the n-th order
iteration of the equations

\[ [H + \Omega_{n+1}^{-1}\phi_{n+1}^{-1}] = [\Omega_{n+1}^{-1} - V]\phi_{n}^{-1} + S_{n}^{-1} \]  \hspace{1cm} (66)

\[ [V + \Omega_{n+1}^{-1}\phi_{n+1}^{-1}] = [\Omega_{n+1}^{-1} - H]\phi_{n+1/2}^{-1} + S_{n+1/2}^{-1} \]  \hspace{1cm} (67)

is used together with the Gaussian elimination methods [78] to solve
Eqs. (66) and (67). Here the densities \( \rho_1 \) in the source term \( S_1 \) are
calculated from the occupied nucleon states by Eq. (65) and the accel-
eration parameter \( \Omega_1 \) is chosen as

\[ \Omega_1 = \gamma_1 - (-1)^n \left[ \gamma_1^2 - \alpha_1 \beta_1 \right]^{1/2} \]  \hspace{1cm} (68)

where

\[ \alpha_1 = \left[ \frac{\pi}{2(Nz-1)\Delta z} \right]^2 + \frac{1}{2} m_1^2, \]

\[ \beta_1 = \left[ \frac{\pi}{\Delta r} \right]^2 + \frac{1}{2} m_1^2, \]

\[ \gamma_1 = \left[ \left( \alpha_1 \beta_1 \right)^{1/4} \left[ (\alpha_1 + \beta_1)/2 \right]^{1/2}. \]

For the discretized Dirac equation, we cannot directly use the
technique used in the nonrelativistic calculations for solving a
discretized equation [77-80] because the Dirac equation is a four-component equation that is linear in the first derivative. However, we can reduce Eq. (55) to an effective Schroedinger equation. By eliminating \( P^- \) in the equation for \( P^+ \) using
\[
P^- = [\omega + M - W_-]^{-1} \nabla^\dagger P^+
\]
we get an effective Schroedinger equation for \( P^+ \)
\[
\omega P^+ = (M + W_+ + \nabla[\omega + M - W_-]^{-1} \nabla^\dagger) P^+
\]
\[
= (M + W_+)^\dagger P^+ + [\omega + M - W_-]^{-1} \nabla^\dagger P^+
\]
\[
+ \{\nabla[\omega + M - W_-]^{-1}\} \nabla^\dagger P^+
\] (70)
where \( \nabla \) in the curly bracket denotes that the derivative acts only on the quantities in the same bracket. The \( W_+ \) term is a density dependent potential in the effective Schroedinger equation for \( P^+ \). The second term in Eq. (70) is the nonrelativistic kinetic energy with the effective mass of \( [\omega + M - W_-]/2 \), and the third term corresponds to a spin-orbit and other velocity-dependent potentials. Note that, in cylindrical coordinates, the first-order derivative \( \nabla \) can be realized as
\[
(\nabla P)^{\eta, \sigma} = \frac{\partial}{\partial z} \left( \frac{p^{\eta,-\sigma}}{r} \right) + \left[ \frac{\partial}{\partial r} - \frac{\sigma(\omega + \sigma/2)}{r} \right] \left( \frac{p^{\eta,-\sigma}}{r} \right).
\] (71)

We use the Damped Gradient Iteration method with Schmit orthogonalization [77-80] to solve the effective Schroedinger equation (70). For the \( n \)-th iteration, the damped gradient method gives
\[
P^+(n) = P^+(n) - \epsilon(1 + T/K_o)^{-1} \left[ h_{\text{eff}}(n) - \omega(n) \right] P^+(n),\] (72)
then using Eq. (69), we get \( P^-(n) \) as
Finally, we use the Schmit orthogonalization $\mathcal{O}$ for $\mathcal{P}(n)$ to get the new nucleon field $\mathcal{P}(n+1)$ as

$$\mathcal{P}(n+1) = \mathcal{O}[\mathcal{P}(n)].$$

In Eq. (72), $\hat{T}$ is the nonrelativistic kinetic energy operator in discrete form, $K_o$ fixes the kinetic energy damping scale, and $\epsilon$ is the imaginary time step [80].

The single-particle energy and relativistic kinetic energy of the nucleon can be calculated by

$$\omega_\alpha = \Delta V \sum_{j,k} P_\alpha^\dagger \begin{pmatrix} M + \hat{W}_+ & V \\ V^\dagger & -M + \hat{W}_- \end{pmatrix} P_\alpha,$$  \hspace{1cm} (75)

$$K_\alpha = \Delta V \sum_{j,k} P_\alpha^\dagger \begin{pmatrix} 0 & V \\ V^\dagger & 0 \end{pmatrix} P_\alpha.$$  \hspace{1cm} (76)

We define the spin mixing probabilities of each component of a Dirac spinor as

$$\mathcal{S}_\alpha^{\mathcal{N},\sigma} = \Delta V \sum_{j,k} \left| P_\alpha^{\mathcal{N},\sigma}(j',k') \right|^2$$ \hspace{1cm} (77)

For a spherical nucleus, the ratio of these probabilities, the spin-mixing ratio, is determined by the ratio of the corresponding Clebsch-Gordan coefficients of each component (see Eqs. (82) and (83) in the Appendix).

The total energy of the system becomes
\[
E = \sum_{\alpha} \omega_{\alpha} + \frac{1}{2} \Delta V \sum_{j,k} (j-1/2) \sum_i \left[ s_i g_i \frac{\Phi_{i}^{(j,k)}}{\sqrt{j-1/2}} \rho_{i}^{(j,k)} \right] \\
+ \frac{1}{8} g_2 \left\{ \frac{\Phi_{i}^{(j,k)}}{\sqrt{j-1/2}} \right\}^3 + \frac{1}{4} g_3 \left\{ \frac{\Phi_{i}^{(j,k)}}{\sqrt{j-1/2}} \right\}^4 .
\]

(78)

The baryon density on the grid is

\[
\rho_{B}^{(j,k)} = \sum_{\alpha} \left\{ \left| \frac{P_{\alpha}^{++}(j,k)}{\sqrt{j-1/2}} \right|^2 + \left| \frac{P_{\alpha}^{+-}(j+1/2,k+1/2)}{\sqrt{j}} \right|^2 + \left| \frac{P_{\alpha}^{-+}(j-1/2,k+1/2)}{\sqrt{j-1}} \right|^2 + \left| \frac{P_{\alpha}^{--}(j+1/2,k-1/2)}{\sqrt{j}} \right|^2 \right\}
\]

(79)

where the sum over \( \alpha \) is taken over all occupied single-nucleon levels for the nucleon density, and over all occupied proton or neutron levels for the proton or the neutron density.

The rms radii and spherical multipole moments are

\[
R_{\text{rms}} = \left[ \frac{1}{A} \Delta V \sum_{j,k} (j-1/2) \left\{ r_j^2 + (z_k - z_o)^2 \right\} \rho_{B}^{(j,k)} \right]^{1/2} ,
\]

(80)
\[ Q_{,m} = A V \sum_{j,k} (j-1/2)(r_j^2 + (z_k - z_o)^2)^{\xi/2} \chi_{j,k}^m(x) \rho_B(j,k), \]  
\[(81)\]

where \( A \) is the particle number and \( z_o \) is the center of the nucleus in the \( z \)-direction.

C. RESULTS

Results presented in this section have been obtained by solving the relativistic Hartree equations (55)-(65) for the Lagrangian parameters of Set I [22] in Table 5, which were originally fitted to nuclear matter. Deformed harmonic oscillator wave functions are used for the initial nucleon states. Four iterations (\( n=4 \) of Eqs. (66) and (67)) are used to solve for the meson fields. The parameter values of \( \varepsilon = 100 \text{ MeV}^{-1} \) and \( K_o = 40 \text{ MeV} \) are chosen for the damped gradient method, Eq. (72).

Lagrange four-point interpolation should be used in Eq. (54) in order to avoid numerical symmetry breaking between the \( r \) and \( z \) directions. For spherically symmetric nuclei, scalar quantities such as the density or the scalar and vector potentials on the \( z \) axis should be the same as they are on the \( r \) axis. For the case of two-point interpolation, this symmetry is broken due to the finite mesh size. This can be seen in the spin mixing ratios, Eq. (77), between the spin up and down components. They do not produce the correct values which are the square of the ratio of Clebsch-Gordan coefficients. We can reduce this symmetry breaking by either choosing a smaller mesh size or using a higher order interpolation scheme.
Table 5. Parameters of relativistic Lagrangian. The masses and $g_2$ are given in MeV and other coupling constants are dimensionless.

<table>
<thead>
<tr>
<th>Set</th>
<th>$M$</th>
<th>$m_S$</th>
<th>$m_V$</th>
<th>$m_P$</th>
<th>$g_S$</th>
<th>$g_V$</th>
<th>$g_P$</th>
<th>$g_2$</th>
<th>$g_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>938</td>
<td>550</td>
<td>783</td>
<td>770</td>
<td>10.3</td>
<td>12.6</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>II</td>
<td>938</td>
<td>520</td>
<td>783</td>
<td>770</td>
<td>10.47</td>
<td>13.78</td>
<td>8.076</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>III</td>
<td>938</td>
<td>500</td>
<td>783</td>
<td>770</td>
<td>9.4</td>
<td>13.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Total energy convergence, which is defined as the ratio of the absolute value of difference in energy between successive iterations to the energy in the last iteration, is shown in Fig. 8 for $^{160}$O and $^{12}$C. Here we note that the energy of the deformed $^{12}$C converges more slowly than $^{160}$O. The effect of the mesh size ($\Delta = \Delta_r = \Delta_z$) was checked by performing calculations for $^{160}$O and $^{12}$C with mesh sizes of 0.5, 0.25, and 0.137 fm. The dependences of binding energy per nucleon on the grid size is shown in Fig. 9. Also shown are the results from spherical calculations with the same interactions. The finite mesh size gives about 0.6 MeV per particle more binding with $\Delta = 0.5$ fm for $^{160}$O and about 0.8 MeV for $^{12}$C. In the spherical calculation, the energy has already converged to within about 0.1 MeV at the mesh size of 0.5 fm. This difference between the axial and spherical calculation comes from the difference in the calculation of the kinetic term. In the axial calculation, two-point derivatives are used whereas five-point derivatives were used in the spherical calculation. Other quantities depend much less on the mesh size for both $^{160}$O (spherical) and $^{12}$C (deformed) (see Tables 6, 7, and 11). Thus we choose to use a mesh size of 0.5 fm, together with an energy renormalization of 0.6 MeV per particle for spherical nuclei and 0.8 MeV for axially deformed nuclei. The $\Delta = \Delta_r = \Delta_z = 0.5$ fm mesh size is used in both the r and z directions.


To check the validity of the axially symmetric Hartree calculations, we have calculated the spherical nuclei $^{160}$O, $^{40}$Ca, and $^{48}$Ca,
Figure 8. Energy convergency rate which is defined as the ratio of the absolute value of the difference in total energy between successive iterations to the energy in the last iteration. The solid line is for $^{12}\text{C}$ and the dotted line is for $^{16}\text{O}$ obtained with axially symmetric calculations.
CONVERGENT RATE

Energy Convergence vs. Iteration Number

- $^{12}$C - "Ax
- $^{16}$O - "Ax
Figure 9. Grid size dependence of binding energy per nucleon. The solid line is for $^{12}\text{C}$ and the dotted line is for $^{16}\text{O}$ obtained with axially symmetric calculations. The dashed line is for $^{16}\text{O}$ obtained with spherically symmetric calculations.
GRID SIZE DEPENDENCE

- $^{12}\text{C} - \text{Axial}$
- $^{16}\text{O} - \text{Axial}$
- $^{16}\text{O} - \text{Sph}$

Binding Energy (MeV/A)

Grid Size (fm)
Table 6. Results of axially symmetric calculations for $^{160}$ with two different mesh sizes (Axi.1 and Axi.2) compared with the results of spherically symmetric calculations (Sph.). The spherical Hartree code of Ref. [35] was used to obtain the results in the column of Sph. All calculations use the same values for the parameters of Set I in Table 5. The numbers of proton and neutron levels included in the calculation are given for axial calculations. CM energy corrections [87,88] are not included in the binding energies. The corrections due to the finite nucleon size are not included in the radii.

<table>
<thead>
<tr>
<th></th>
<th>Axi.1</th>
<th>Axi.2</th>
<th>Sph.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_x$</td>
<td>$\Delta_x$ (fm)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>0.50</td>
<td>41</td>
<td>0.25</td>
</tr>
<tr>
<td>$N_z$</td>
<td>$\Delta_z$ (fm)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>0.50</td>
<td>42</td>
<td>0.25</td>
</tr>
<tr>
<td>No. of Prot. States</td>
<td></td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>No. of Neut. States</td>
<td></td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>Iteration No.</td>
<td></td>
<td></td>
<td>110</td>
</tr>
<tr>
<td>Energy Convergency ($\times 10^{-8}$)</td>
<td></td>
<td></td>
<td>3.52</td>
</tr>
</tbody>
</table>

|                  |       |       |      |
| Binding Energy (MeV) | -128.869 | -120.650 | -119.895 |
| B.E./A            |       |       | 8.054 | 7.541 | 7.493 |
| Total Sing. Part. E. |       |       | -463.929 | -453.087 | -453.255 |
| Meson Field Energy |       |       | 335.060 | 332.437 | 333.360 |
| Scalar Field E.   |       |       | 2339.710 | 2289.820 |
| Vector Field E.    |       |       | -1985.939 | -1938.864 |
| Coulomb Field E.   |       |       | -18.711  | -18.519  |
| Total Kinetic E.   |       |       | 862.945  | 844.470  | 832.416 |

<table>
<thead>
<tr>
<th>Radii (fm)</th>
<th>Proton</th>
<th>Neutron</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2.362</td>
<td>2.383</td>
<td>2.391</td>
</tr>
<tr>
<td></td>
<td>2.329</td>
<td>2.350</td>
<td>2.357</td>
</tr>
<tr>
<td></td>
<td>2.346</td>
<td>2.367</td>
<td>2.374</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$Q_{20}$ (fm$^2$)</th>
<th>Proton</th>
<th>Neutron</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0999</td>
<td>0.0409</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0.0973</td>
<td>0.0382</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0.1972</td>
<td>0.0791</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 7. Single-particle levels for the calculations of Table 6. The first row of each level is for single-particle energies and the second row is for single-particle kinetic energies defined by Eq. (76) in MeV.

<table>
<thead>
<tr>
<th>Level $\mu$</th>
<th>Proton</th>
<th>Neutron</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Axi.1</td>
<td>Axi.2</td>
</tr>
<tr>
<td>$1S_{1/2}$</td>
<td>-47.62</td>
<td>-46.96</td>
</tr>
<tr>
<td></td>
<td>36.88</td>
<td>36.68</td>
</tr>
<tr>
<td></td>
<td>55.45</td>
<td>54.10</td>
</tr>
<tr>
<td>$1P_{1/2}$</td>
<td>-23.38</td>
<td>-22.63</td>
</tr>
<tr>
<td></td>
<td>54.98</td>
<td>54.02</td>
</tr>
<tr>
<td></td>
<td>-12.01</td>
<td>-11.72</td>
</tr>
<tr>
<td></td>
<td>65.27</td>
<td>63.19</td>
</tr>
</tbody>
</table>
and these results are compared with those obtained from spherically symmetric calculations [35] with the same interaction. These results are summarized in Tables 6-10, and we note good agreement between the axial and the spherical calculations. As we can see in the tables, the discrepancy can be entirely associated with the finite mesh size. The results for $^{16}O$ with $\Delta = 0.25$ fm are given in Tables 6 and 7. These results are in very good agreement with the corresponding spherical calculations. If we renormalize the energy by 0.6 MeV per nucleon as noted in the previous discussion, the results of axial and spherical calculations have less than 0.1 MeV differences in binding energy per particle for all nuclei.

The spin mixing probabilities, Eq. (77), are given in Table 10. These values show that the axial calculation for spherical nuclei retains the spherical symmetry. In conclusion, the axial calculation with the energy renormalization gives good results for spherical nuclei.


Having established the validity of the axial relativistic Hartree method, it is applied to the axially deformed nuclei $^{12}C$, $^{20}Ne$, and $^{24}Mg$ using the same parameters as were used for spherical nuclei. The results are summarized in Tables 11 and 12 and Figs. 10-12. Experimental data and the results of nonrelativistic self-consistent calculations are summarized in Ref. [2]. The major discrepancy between the results of our method and the data is the values of the quadrupole moments. Especially for $^{12}C$, the solution is approximately spherical.
Table 8. Results of axially symmetric calculations for $^{40}$Ca and $^{48}$Ca compared with the results of spherically symmetric calculations using the same code as in Table 6.

<table>
<thead>
<tr>
<th></th>
<th>$^{40}$Ca</th>
<th></th>
<th>$^{48}$Ca</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_r$</td>
<td>25</td>
<td>25</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>$\Delta_r$ (fm)</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>$N_z$</td>
<td>26</td>
<td>26</td>
<td>26</td>
<td>26</td>
</tr>
<tr>
<td>$\Delta_z$ (fm)</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>No. of Prot. States</td>
<td>19</td>
<td>19</td>
<td>19</td>
<td>19</td>
</tr>
<tr>
<td>No. of Neut. States</td>
<td>19</td>
<td>19</td>
<td>19</td>
<td>19</td>
</tr>
<tr>
<td>Iteration No.</td>
<td>390</td>
<td>79</td>
<td>278</td>
<td>69</td>
</tr>
<tr>
<td>Energy Convergency ($\times 10^{-8}$)</td>
<td>1.61</td>
<td>0.073</td>
<td>1.22</td>
<td>0.086</td>
</tr>
<tr>
<td>Binding Energy (MeV)</td>
<td>-373.772</td>
<td>-358.204</td>
<td>-490.423</td>
<td>-466.678</td>
</tr>
<tr>
<td>Total Sing. Part. E.</td>
<td>-1205.922</td>
<td>-1187.967</td>
<td>-1585.654</td>
<td>-1554.965</td>
</tr>
<tr>
<td>Meson Field Energy</td>
<td>832.150</td>
<td>829.763</td>
<td>1095.231</td>
<td>1088.287</td>
</tr>
<tr>
<td>Scalar Field E.</td>
<td>6258.922</td>
<td>8026.295</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vector Field E.</td>
<td>-5339.147</td>
<td>-6844.128</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coulomb Field E.</td>
<td>-87.626</td>
<td>-66.936</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total Kinetic E.</td>
<td>2371.540</td>
<td>2312.270</td>
<td>3075.067</td>
<td>2987.887</td>
</tr>
<tr>
<td>Radii Proton (fm)</td>
<td>3.107</td>
<td>3.140</td>
<td>3.121</td>
<td>3.154</td>
</tr>
<tr>
<td>Neutron</td>
<td>3.048</td>
<td>3.080</td>
<td>3.247</td>
<td>3.290</td>
</tr>
<tr>
<td>Total</td>
<td>3.078</td>
<td>3.110</td>
<td>3.195</td>
<td>3.234</td>
</tr>
<tr>
<td>Q$_{20}$ Proton (fm$^2$)</td>
<td>0.1560</td>
<td>0.0354</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Neutron</td>
<td>0.1390</td>
<td>0</td>
<td>-0.0836</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>0.2950</td>
<td>0</td>
<td>-0.0482</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 9. Results of axially symmetric calculations for single-particle levels of \(^{40}\)Ca and \(^{48}\)Ca compared with the results of spherically symmetric calculations using the same code as in Table 6. The first row of each level is single-particle energy and the second row is single-particle kinetic energy in MeV.

<table>
<thead>
<tr>
<th>Level μ</th>
<th>Proton (^{40})Ca</th>
<th>Neutron (^{40})Ca</th>
<th>Proton (^{48})Ca</th>
<th>Neutron (^{48})Ca</th>
</tr>
</thead>
<tbody>
<tr>
<td>1S(^1/2) 1/2</td>
<td>-56.29</td>
<td>-55.95</td>
<td>-66.10</td>
<td>-65.72</td>
</tr>
<tr>
<td></td>
<td>26.92</td>
<td>26.89</td>
<td>27.93</td>
<td>27.91</td>
</tr>
<tr>
<td>1P(^3/2) 3/2</td>
<td>-38.95</td>
<td>-38.22</td>
<td>-48.08</td>
<td>-47.28</td>
</tr>
<tr>
<td></td>
<td>45.81</td>
<td>45.31</td>
<td>47.37</td>
<td>46.91</td>
</tr>
<tr>
<td></td>
<td>-38.84</td>
<td>-47.97</td>
<td>-44.82</td>
<td>-53.77</td>
</tr>
<tr>
<td></td>
<td>45.82</td>
<td>47.39</td>
<td>45.67</td>
<td>47.10</td>
</tr>
<tr>
<td></td>
<td>58.88</td>
<td>57.38</td>
<td>60.59</td>
<td>59.09</td>
</tr>
<tr>
<td>1D(^5/2) 5/2</td>
<td>-20.99</td>
<td>-20.13</td>
<td>-29.51</td>
<td>-28.56</td>
</tr>
<tr>
<td></td>
<td>62.04</td>
<td>60.44</td>
<td>64.28</td>
<td>62.71</td>
</tr>
<tr>
<td></td>
<td>-20.92</td>
<td>-29.43</td>
<td>-27.07</td>
<td>-35.46</td>
</tr>
<tr>
<td></td>
<td>61.80</td>
<td>64.05</td>
<td>63.01</td>
<td>65.03</td>
</tr>
<tr>
<td></td>
<td>-20.90</td>
<td>-29.42</td>
<td>-27.06</td>
<td>-35.46</td>
</tr>
<tr>
<td></td>
<td>61.81</td>
<td>64.07</td>
<td>63.03</td>
<td>65.06</td>
</tr>
</tbody>
</table>
Table 9. (continued)

<table>
<thead>
<tr>
<th></th>
<th>3/2</th>
<th>1/2</th>
<th>5/2</th>
<th>3/2</th>
<th>1/2</th>
<th>5/2</th>
<th>3/2</th>
<th>1/2</th>
<th>5/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D&lt;sub&gt;3/2&lt;/sub&gt;</td>
<td>3/2</td>
<td>9.77</td>
<td>9.66</td>
<td>18.21</td>
<td>18.00</td>
<td>16.66</td>
<td>16.31</td>
<td>25.18</td>
<td>24.72</td>
</tr>
<tr>
<td></td>
<td>1/2</td>
<td>9.84</td>
<td>18.28</td>
<td>16.60</td>
<td>25.11</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2S&lt;sub&gt;1/2&lt;/sub&gt;</td>
<td>1/2</td>
<td>8.80</td>
<td>9.22</td>
<td>17.24</td>
<td>17.59</td>
<td>12.86</td>
<td>13.33</td>
<td>21.44</td>
<td>21.82</td>
</tr>
<tr>
<td>1F&lt;sub&gt;7/2&lt;/sub&gt;</td>
<td>7/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-17.32</td>
<td>-16.24</td>
<td>79.13</td>
<td>76.32</td>
</tr>
<tr>
<td></td>
<td>5/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-17.20</td>
<td>78.69</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-17.25</td>
<td>78.59</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-17.32</td>
<td>76.38</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 10. Spin mixing probability Eq. (77) of axially symmetric calculations for spherical nuclei using the same code as in Table 6. The values in the last column (Sph.) are determined by the Clebsch-Gordan coefficient for the exact spherical solution. 1.C is the spin mixing probability for the spin-up upper component, 2.C is for the spin-down upper component, 3.C is for the spin-up lower component, and 4.C is for the spin-down lower component. μ is the third component of total angular momentum.

<table>
<thead>
<tr>
<th>Level</th>
<th>μ</th>
<th>Dirac Spinor Component</th>
<th>160 Proton</th>
<th>160 Neutron</th>
<th>40Ca Proton</th>
<th>40Ca Neutron</th>
<th>48Ca Proton</th>
<th>48Ca Neutron</th>
<th>Sph.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1S1/2</td>
<td>1/2</td>
<td>1.C</td>
<td>0.9854</td>
<td>0.9850</td>
<td>0.9885</td>
<td>0.9879</td>
<td>0.9886</td>
<td>0.9881</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.C</td>
<td>1.8E-6</td>
<td>1.8E-6</td>
<td>0.4E-6</td>
<td>0.4E-6</td>
<td>0.5E-6</td>
<td>0.6E-6</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.C</td>
<td>0.0049</td>
<td>0.0050</td>
<td>0.0039</td>
<td>0.0040</td>
<td>0.0038</td>
<td>0.0040</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.C</td>
<td>0.0097</td>
<td>0.0100</td>
<td>0.0077</td>
<td>0.0080</td>
<td>0.0076</td>
<td>0.0079</td>
<td></td>
</tr>
<tr>
<td>1P3/2</td>
<td>3/2</td>
<td>1.C</td>
<td>0.9805</td>
<td>0.9799</td>
<td>0.9820</td>
<td>0.9812</td>
<td>0.9811</td>
<td>0.9803</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.C</td>
<td>2.3E-6</td>
<td>2.4E-6</td>
<td>0.9E-6</td>
<td>0.9E-6</td>
<td>1.3E-6</td>
<td>1.4E-6</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.C</td>
<td>0.0039</td>
<td>0.0040</td>
<td>0.0036</td>
<td>0.0038</td>
<td>0.0038</td>
<td>0.0040</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.C</td>
<td>0.0156</td>
<td>0.0161</td>
<td>0.0144</td>
<td>0.0150</td>
<td>0.0151</td>
<td>0.0157</td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td></td>
<td>1.C</td>
<td>0.6534</td>
<td>0.6527</td>
<td>0.6500</td>
<td>0.6495</td>
<td>0.6364</td>
<td>0.6355</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.C</td>
<td>0.3273</td>
<td>0.3273</td>
<td>0.3319</td>
<td>0.3316</td>
<td>0.3447</td>
<td>0.3448</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.C</td>
<td>0.0078</td>
<td>0.0080</td>
<td>0.0073</td>
<td>0.0076</td>
<td>0.0076</td>
<td>0.0079</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.C</td>
<td>0.0115</td>
<td>0.0119</td>
<td>0.0108</td>
<td>0.0113</td>
<td>0.0113</td>
<td>0.0118</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.C/2.C</td>
<td>1.9961</td>
<td>1.9939</td>
<td>1.9584</td>
<td>1.9588</td>
<td>1.8463</td>
<td>1.8431</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2(4.C/3.C)</td>
<td>2.9724</td>
<td>2.9725</td>
<td>2.9655</td>
<td>2.9671</td>
<td>2.9758</td>
<td>2.9771</td>
<td>3</td>
</tr>
<tr>
<td>1P_{1/2}</td>
<td>1/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-----------</td>
<td>-----</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>1.C</strong></td>
<td>0.3233 0.3231 0.3280 0.3277 0.3411 0.3413</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>2.C</strong></td>
<td>0.6445 0.6434 0.6422 0.6417 0.6295 0.6287</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>3.C</strong></td>
<td>0.0322 0.0335 0.0298 0.0306 0.0294 0.0300</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>4.C</strong></td>
<td>3.3E-6 3.5E-6 1.8E-6 1.8E-6 3.6E-6 3.9E-6 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>2.C/1.C</strong></td>
<td>1.9935 1.9914 1.9579 1.9584 1.8453 1.8421 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 11. Same quantities as in Table 6 for the axially deformed nuclei

<table>
<thead>
<tr>
<th></th>
<th>$^{12}$C-1</th>
<th>$^{12}$C-2</th>
<th>$^{20}$Ne</th>
<th>$^{24}$Mg</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_r$ $A_r$ (fm)</td>
<td>21 0.50</td>
<td>41 0.25</td>
<td>21 0.50</td>
<td>21 0.50</td>
</tr>
<tr>
<td>$N_z$ $A_z$ (fm)</td>
<td>22 0.50</td>
<td>42 0.25</td>
<td>22 0.50</td>
<td>22 0.50</td>
</tr>
<tr>
<td>No. of Prot. States</td>
<td>8</td>
<td>8</td>
<td>14</td>
<td>15</td>
</tr>
<tr>
<td>No. of Neut. States</td>
<td>8</td>
<td>8</td>
<td>14</td>
<td>15</td>
</tr>
<tr>
<td>Iteration No.</td>
<td>442</td>
<td>405</td>
<td>222</td>
<td>132</td>
</tr>
<tr>
<td>Energy Convergency ($\times 10^{-8}$)</td>
<td>1.19</td>
<td>3.59</td>
<td>1.86</td>
<td>1.55</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$^{12}$C-1</th>
<th>$^{12}$C-2</th>
<th>$^{20}$Ne</th>
<th>$^{24}$Mg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binding Energy (MeV)</td>
<td>-94.100</td>
<td>-85.413</td>
<td>-166.432</td>
<td>-217.567</td>
</tr>
<tr>
<td>Total Sing. Part. E.</td>
<td>-388.590</td>
<td>-373.477</td>
<td>-609.222</td>
<td>-778.148</td>
</tr>
<tr>
<td>Meson Field Energy</td>
<td>294.490</td>
<td>288.063</td>
<td>442.790</td>
<td>560.581</td>
</tr>
<tr>
<td>Scalar Field E.</td>
<td>1968.224</td>
<td>1884.608</td>
<td>3131.355</td>
<td>3997.803</td>
</tr>
<tr>
<td>Vector Field E.</td>
<td>-1661.673</td>
<td>-1584.709</td>
<td>-2661.218</td>
<td>-3399.369</td>
</tr>
<tr>
<td>Total Kinetic E.</td>
<td>722.159</td>
<td>697.466</td>
<td>1150.414</td>
<td>1477.763</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Proton</th>
<th>Neutron</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radii (fm)</td>
<td>2.079</td>
<td>2.118</td>
<td>2.552</td>
</tr>
<tr>
<td></td>
<td>2.056</td>
<td>2.094</td>
<td>2.517</td>
</tr>
<tr>
<td></td>
<td>2.068</td>
<td>2.106</td>
<td>2.534</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Proton</th>
<th>Neutron</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{20}$ (fm²)</td>
<td>0.6754</td>
<td>0.1058</td>
<td>10.0566</td>
</tr>
<tr>
<td></td>
<td>0.6632</td>
<td>0.1039</td>
<td>9.7753</td>
</tr>
<tr>
<td></td>
<td>1.3385</td>
<td>0.2097</td>
<td>19.8319</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>25.6690</td>
</tr>
</tbody>
</table>
Table 12. Same quantities as in Table 10 for axially deformed nuclei. The S.P.E. is single-particle energy and the S.P.K.E. is single-particle kinetic energy which is defined by Eq. (76) in MeV. $\mu$ is the third component of total angular momentum, $\pi$ is the parity of the first component of Dirac wave function, and $\pi_z$ is the z parity for that component.

<table>
<thead>
<tr>
<th>$\mu^\pi$ $\pi_z$</th>
<th>Energies and Dirac Spinor Component</th>
<th>$^{12}$C-1 Proton Neutron</th>
<th>$^{20}$Ne Proton Neutron</th>
<th>$^{24}$Mg Proton Neutron</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S.P.E.</td>
<td>-53.26 -57.73</td>
<td>-51.69 -57.73</td>
<td>-56.37 -63.32</td>
</tr>
<tr>
<td></td>
<td>S.P.K.E.</td>
<td>53.07 54.02</td>
<td>36.13 36.98</td>
<td>36.54 37.39</td>
</tr>
<tr>
<td>1.C</td>
<td>0.9759 0.9754</td>
<td>0.9847 0.9842</td>
<td>0.9836 0.9831</td>
<td></td>
</tr>
<tr>
<td>2.C</td>
<td>3.8E-5 3.9E-5</td>
<td>0.0004 0.0004</td>
<td>0.0005 0.0005</td>
<td></td>
</tr>
<tr>
<td>3.C</td>
<td>0.0076 0.0078</td>
<td>0.0037 0.0038</td>
<td>0.0037 0.0038</td>
<td></td>
</tr>
<tr>
<td>4.C</td>
<td>0.0165 0.0168</td>
<td>0.0112 0.0115</td>
<td>0.0122 0.0125</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1/2$^-$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>S.P.E.</td>
<td>-19.64 -23.47</td>
<td>-32.81 -38.35</td>
<td>-40.27 -46.80</td>
</tr>
<tr>
<td></td>
<td>S.P.K.E.</td>
<td>62.86 64.55</td>
<td>55.65 57.05</td>
<td>58.94 60.39</td>
</tr>
<tr>
<td>1.C</td>
<td>0.7210 0.7205</td>
<td>0.9093 0.9095</td>
<td>0.9430 0.9436</td>
<td></td>
</tr>
<tr>
<td>2.C</td>
<td>0.2563 0.2561</td>
<td>0.0690 0.0682</td>
<td>0.0320 0.0305</td>
<td></td>
</tr>
<tr>
<td>3.C</td>
<td>0.0090 0.0093</td>
<td>0.0085 0.0088</td>
<td>0.0115 0.0119</td>
<td></td>
</tr>
<tr>
<td>4.C</td>
<td>0.0136 0.0141</td>
<td>0.0132 0.0136</td>
<td>0.0136 0.0140</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3/2$^+$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>S.P.E.</td>
<td>-18.19 -22.00</td>
<td>-25.46 -30.92</td>
<td>-29.23 -35.57</td>
</tr>
<tr>
<td></td>
<td>S.P.K.E.</td>
<td>62.42 64.16</td>
<td>55.94 57.50</td>
<td>56.65 58.25</td>
</tr>
<tr>
<td>1.C</td>
<td>0.9779 0.9772</td>
<td>0.9785 0.9792</td>
<td>0.9783 0.9776</td>
<td></td>
</tr>
<tr>
<td>2.C</td>
<td>2.1E-5 2.2E-5</td>
<td>0.0001 0.0002</td>
<td>0.0006 0.0006</td>
<td></td>
</tr>
<tr>
<td>3.C</td>
<td>0.0041 0.0043</td>
<td>0.0032 0.0033</td>
<td>0.0028 0.0029</td>
<td></td>
</tr>
<tr>
<td>4.C</td>
<td>0.0179 0.0185</td>
<td>0.0168 0.0174</td>
<td>0.0183 0.0190</td>
<td></td>
</tr>
<tr>
<td></td>
<td>S.P.E.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>---------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>1/2−</td>
<td>-15.28</td>
<td>-20.66</td>
<td>-19.45</td>
<td>-25.79</td>
</tr>
<tr>
<td></td>
<td>68.11</td>
<td>70.54</td>
<td>71.34</td>
<td>73.58</td>
</tr>
<tr>
<td>1.C</td>
<td>0.0656</td>
<td>0.0649</td>
<td>0.0375</td>
<td>0.0364</td>
</tr>
<tr>
<td>2.C</td>
<td>0.9006</td>
<td>0.9002</td>
<td>0.9265</td>
<td>0.9266</td>
</tr>
<tr>
<td>3.C</td>
<td>0.0329</td>
<td>0.0341</td>
<td>0.0347</td>
<td>0.0357</td>
</tr>
<tr>
<td>4.C</td>
<td>0.0009</td>
<td>0.0009</td>
<td>0.0013</td>
<td>0.0013</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>S.P.E.</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2+</td>
<td>-13.34</td>
<td>-18.37</td>
<td>-16.98</td>
<td>-22.95</td>
</tr>
<tr>
<td></td>
<td>67.52</td>
<td>69.79</td>
<td>70.63</td>
<td>72.94</td>
</tr>
<tr>
<td>1.C</td>
<td>0.8557</td>
<td>0.8556</td>
<td>0.7366</td>
<td>0.7386</td>
</tr>
<tr>
<td>2.C</td>
<td>0.1202</td>
<td>0.1194</td>
<td>0.2373</td>
<td>0.2342</td>
</tr>
<tr>
<td>3.C</td>
<td>0.0115</td>
<td>0.0119</td>
<td>0.0115</td>
<td>0.0120</td>
</tr>
<tr>
<td>4.C</td>
<td>0.0127</td>
<td>0.0132</td>
<td>0.0146</td>
<td>0.0152</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>S.P.E.</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3/2+</td>
<td>-13.25</td>
<td>-19.09</td>
<td></td>
</tr>
<tr>
<td></td>
<td>69.90</td>
<td>72.32</td>
<td></td>
</tr>
<tr>
<td>1.C</td>
<td>0.9363</td>
<td>0.9358</td>
<td></td>
</tr>
<tr>
<td>2.C</td>
<td>0.0390</td>
<td>0.0385</td>
<td></td>
</tr>
<tr>
<td>3.C</td>
<td>0.0069</td>
<td>0.0072</td>
<td></td>
</tr>
<tr>
<td>4.C</td>
<td>0.0178</td>
<td>0.0186</td>
<td></td>
</tr>
</tbody>
</table>
Figure 10. Baryon and scalar densities for $^{12}$C. The contours in the legend on the top right corner should be read from bottom to top as $1.4 \times 10^{-4} - 4.1 \times 10^{-4}$, $4.1 \times 10^{-4} - 1.2 \times 10^{-3}$, $1.2 \times 10^{-3} - 3.7 \times 10^{-3}$, $3.7 \times 10^{-3} - 1.1 \times 10^{-2}$, $1.1 \times 10^{-2} - 3.3 \times 10^{-2}$, $3.3 \times 10^{-2} - 0.1$, $0.1 - 0.15$, $0.15 - 0.2$, $0.2 - 0.25$, and those larger than 0.25, in units of fm$^{-3}$. 
BARYON DENSITY OF $^{12}\text{C}$

SCALAR DENSITY OF $^{12}\text{C}$
Figure 11. Same as in Fig. 10 except for $^{20}\text{Ne}$. 
BARYON DENSITY OF $^{20}\text{Ne}$

SCALAR DENSITY OF $^{20}\text{Ne}$
Figure 12. Same as in Fig. 10 except for $^{24}\text{Mg}$. 
BARYON DENSITY OF $^{24}\text{Mg}$

SCALAR DENSITY OF $^{24}\text{Mg}$
(empirical value is \( Q_{20} = -4.0 \, \text{fm}^2 \) [2]). The quadrupole moments of \(^{20}\text{Ne}\) and \(^{24}\text{Mg}\) also differ by more than a factor of 2 from the empirical values; empirical charge quadrupole moments are 19.2 and 22 \text{fm}^2 respectively [2]. This is contrary to the results of nonrelativistic calculations where the parameter set providing the best fit to the ground-state properties of spherical nuclei also yields a satisfactory description of nuclear deformations [81-85] (see Table 13 also).

The values of other observables for \(^{12}\text{C}\) are also found to be very close to those for the spherical nucleus. The spin mixing probabilities, Eq. (77), are close to the values for the spherical case. These are very different from the nonrelativistic results for \(^{12}\text{C}\) (see Tables 10 and 12). For example, the spin mixing ratio between spin down and up of the upper components for \( \mu^w = 1/2^- \) level is about 0.36, while it is 0.5 for the spherical \( 1\text{P}^{1/2}_3/2 \) level. On the other hand, this value is about 30 for the oblate solution of \(^{12}\text{C}\) in nonrelativistic calculations [83]. Similarly, it is found that the single-particle energy levels are more strongly bound than those obtained in nonrelativistic calculations [84,85]. Specifically, the \( 1/2^- \) level of \(^{12}\text{C}\) is more bound than the \( 3/2^- \) level, which is opposite to the results of Refs. [83] and [84].

These calculations were repeated using different initial wavefunctions for \(^{12}\text{C}\). In addition, the rho meson was included using the parameters of Set II in Table 5; these parameters have been used for the calculation of spherical nuclei [36]. Other parameter sets [35-38] which have been used for spherical calculations were also
Table 13. Axial Hartree results for the bulk properties of deformed nuclei. The first row for each nucleus is the results of relativistic axial Hartree calculations, and the second row is the results of nonrelativistic Hartree calculations with Skyrme-II forces using the axial code of Ref. [80].

<table>
<thead>
<tr>
<th></th>
<th>Binding Energy (MeV/A)</th>
<th>Radii (fm)</th>
<th>Quadrupole Moments (fm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Proton</td>
<td>Neutron</td>
</tr>
<tr>
<td>¹²C</td>
<td>-7.84</td>
<td>2.08</td>
<td>2.06</td>
</tr>
<tr>
<td></td>
<td>-7.07</td>
<td>2.48</td>
<td>2.46</td>
</tr>
<tr>
<td>²⁰Ne</td>
<td>-8.32</td>
<td>2.55</td>
<td>2.52</td>
</tr>
<tr>
<td></td>
<td>-8.32</td>
<td>2.83</td>
<td>2.81</td>
</tr>
<tr>
<td>²⁴Mg</td>
<td>-9.07</td>
<td>2.65</td>
<td>2.61</td>
</tr>
<tr>
<td></td>
<td>-8.39</td>
<td>3.00</td>
<td>2.94</td>
</tr>
<tr>
<td>³⁶Ar</td>
<td>-9.31</td>
<td>2.98</td>
<td>2.93</td>
</tr>
<tr>
<td>³⁸Ar</td>
<td>-9.48</td>
<td>3.00</td>
<td>3.02</td>
</tr>
<tr>
<td></td>
<td>-8.82</td>
<td>3.25</td>
<td>3.27</td>
</tr>
<tr>
<td>⁴⁰Ar</td>
<td>-9.63</td>
<td>3.01</td>
<td>3.09</td>
</tr>
<tr>
<td></td>
<td>-8.81</td>
<td>3.28</td>
<td>3.36</td>
</tr>
<tr>
<td>⁴⁰Ca</td>
<td>-9.34</td>
<td>3.11</td>
<td>3.05</td>
</tr>
<tr>
<td></td>
<td>-8.87</td>
<td>3.32</td>
<td>3.28</td>
</tr>
<tr>
<td>⁴²Ca</td>
<td>-9.51</td>
<td>3.10</td>
<td>3.10</td>
</tr>
<tr>
<td></td>
<td>-8.89</td>
<td>3.34</td>
<td>3.36</td>
</tr>
<tr>
<td></td>
<td>44Ca</td>
<td>46Ca</td>
<td>48Ca</td>
</tr>
<tr>
<td>-------</td>
<td>-------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-8.85</td>
<td>3.35</td>
<td>3.43</td>
</tr>
<tr>
<td></td>
<td>-9.95</td>
<td>3.11</td>
<td>3.20</td>
</tr>
<tr>
<td></td>
<td>-8.85</td>
<td>3.37</td>
<td>3.49</td>
</tr>
<tr>
<td></td>
<td>-10.22</td>
<td>3.12</td>
<td>3.25</td>
</tr>
<tr>
<td></td>
<td>-8.84</td>
<td>3.39</td>
<td>3.55</td>
</tr>
<tr>
<td></td>
<td>-8.75</td>
<td>3.45</td>
<td>3.46</td>
</tr>
<tr>
<td></td>
<td>-9.94</td>
<td>3.18</td>
<td>3.21</td>
</tr>
<tr>
<td></td>
<td>-8.78</td>
<td>3.46</td>
<td>3.52</td>
</tr>
<tr>
<td></td>
<td>-10.23</td>
<td>3.18</td>
<td>3.25</td>
</tr>
<tr>
<td></td>
<td>-8.85</td>
<td>3.47</td>
<td>3.57</td>
</tr>
</tbody>
</table>
tried. None of these substantially changed the final results which were discussed above.

Finally, we searched for values of the parameters which would give an oblate solution for $^{12}$C. For the parameter values of Set III in Table 5, the desired oblate solution for $^{12}$C was obtained. However, these parameter values do not provide a reasonable binding energy. In this case, the solution of $^{12}$C becomes unbound with a small positive binding energy. All other parameter sets which give the oblate solution for $^{12}$C also failed to provide reasonable binding energies. However, relativistic axial calculations for the Ca and Ti isotopes with the parameter Set I of Table 5 give reasonable quadrupole moments and binding energy. The results of these calculations together with the results for the Argon isotopes are summarized in Table 13 and Figs. 13-23. For comparison, the results of the nonrelativistic Hartree calculations with Skyrme-II forces using the axial code of Ref. [80] are included. The isotopes of Ca and Ti have a spherical surface and the nonzero quadrupole moments coming from an interior deformation of those nuclei (Figs. 13-23). For these isotopes, the results agree moderately well between relativistic and nonrelativistic calculations. For the Ar isotopes, the quadrupole moments disagree with the nonrelativistic calculations. The nonrelativistic calculations show that these isotopes have a two-centered density distribution [84]. Thus, the relativistic axial calculations work well for systems with spherical symmetry, but the results became worse as the deformation increases.
Figure 13. Same as in Fig. 10 except for $^{36}\text{Ar}$. 
BARYON DENSITY OF $^{36}$Ar

SCALAR DENSITY OF $^{36}$Ar
Figure 14. Same as in Fig. 10 except for $^{38}\text{Ar}$. 
BARYON DENSITY OF $^{38}$Ar

SCALAR DENSITY OF $^{38}$Ar
Figure 15. Same as in Fig. 10 except for $^{40}$Ar.
Figure 16. Same as in Fig. 10 except for $^{40}$Ca.
Figure 17. Same as in Fig. 10 except for $^{42}\text{Ca}$. 
BARYON DENSITY OF $^{42}$Ca

SCALAR DENSITY OF $^{42}$Ca
Figure 18. Same as in Fig. 10 except for $^{44}\text{Ca}$. 
BARYON DENSITY OF $^{44}\text{Ca}$

SCALAR DENSITY OF $^{44}\text{Ca}$
Figure 19. Same as in Fig. 10 except for $^{46}$Ca.
BARYON DENSITY OF $^{46}$Ca

SCALAR DENSITY OF $^{46}$Ca
Figure 20. Same as in Fig. 10 except for $^{48}\text{Ca}$. 
Figure 21. Same as in Fig. 10 except for $^{46}$Ti.
BARYON DENSITY OF $^{46}$Ti

SCALAR DENSITY OF $^{46}$Ti
Figure 22. Same as in Fig. 10 except for $^{48}$Ti.
BARYON DENSITY OF $^{48}_{\text{Ti}}$

Scalar Density of $^{48}_{\text{Ti}}$
Figure 23. Same as in Fig. 10 except for $^{50}$Ti.
These results can have the following interpretations: (1) The parameters in the Lagrangian need to depend on the energy or spatial variables to include the surface effects correctly. (2) We need to include some other interactions to compensate for the error coming from the assumption of classical meson fields. (3) Higher-order effects, rather than just the Hartree approximation, should be considered explicitly in order to include quantum effects correctly.

For spherical nuclei, the Lagrangian parameters are usually adjusted so as to obtain the bulk properties of the system without any consideration of single-particle energy levels. In the axial calculations of deformed nuclei, the single-particle levels seem to be too bound which is probably caused by the cancellation of the scalar and vector potentials in the nuclear surface. A basic assumption of the relativistic Hartree approximation is the classical treatment of the meson fields. For spherical nuclei, the error due to this assumption would be a function of just the radial variable. Therefore, this error can be compensated for by fitting the parameter values. However, for deformed nuclei, this error could depend on the angle variable also. Furthermore, for the system with a two-centered density distribution such as the Argon isotopes and for the highly oblate nuclei such as $^{12}$C, this assumption could not be applicable even in the central region of a nucleus.
IV. CONCLUSION

We have derived a relativistic Hartree-Fock approximation starting with the renormalizable Lagrangian density Eq. (1). Using the mean field theory for this Lagrangian, we obtained a Dirac equation, Eq. (24) or (30), for single-nucleon states, and Helmholtz equations, Eqs. (25) and (26), for the mean meson fields. Considering the meson fields as quantized fields, we obtained a Hartree-Fock approximation for this relativistic Lagrangian.

To check whether it is applicable to deformed nuclei as well as spherical nuclei, a method for carrying out relativistic Hartree calculations for axially deformed nuclei was presented. The validity of the method was checked by applying it to the structure of spherical nuclei. It was demonstrated that the results of axial Hartree calculations are in good agreement with the results of spherical Hartree calculations. These results for spherically symmetric nuclei include binding energies, radii, quadrupole moments, and spin mixing ratios. Furthermore, the axial relativistic Hartree calculations work moderately well for nuclei which have approximate spherical symmetry. However, the parameters fitted to describe spherical nuclei or infinite nuclear matter failed to reproduce properties of axially deformed nuclei which have either a highly oblate shape or two-centered density distributions. With the conventional parameters, the results were nearly those for a spherical potential. In particular, $^{12}$C was spherical instead of oblate in the relativistic Hartree calculation. The
single-particle wave functions were not changed substantially from the wave functions for a spherical potential, and the overall shape was similar to one in which the lowest spherical levels are filled. The parameters used were those which were adjusted to reproduce the bulk properties of the system, and have not been adjusted to reproduce individual single-particle energies.

From these results, we may conclude that the Lagrangian parameters for the relativistic Hartree approximation should be adjusted more carefully by including in the fitted data bulk properties of deformed nuclei. For this fitting, we may need to introduce space or energy dependence in the parameters. Another possibility is to include additional interactions to compensate for the higher-order effects which are neglected in this approximation, or to include higher-order effects explicitly. Further investigation of the relativistic method for deformed nuclei could be implemented by including other interactions or higher-order terms, i.e., the exchange term or the higher-order correlation effects. Nevertheless, given that there is a growing concern [46,47] about the validity of relativistic nuclear models, the origin of this failure of these models to produce qualitatively acceptable results for deformed nuclei should be further investigated.
APPENDIX: RELATIVISTIC HARTREE-FOCK EQUATIONS FOR SPHERICAL SYSTEMS

For a spherical system, i.e., doubly magic nuclei with well-closed shells, the single-particle Hamiltonian $h(x,x')$ in Eqs. (30) will be $\theta$ and $\phi$ independent. The single-particle state, which is an eigenstate of $h(x,x')$, will be of the form $[38,70]$:

$$
\psi_\alpha(x) = \frac{1}{r} \left( \begin{pmatrix} 1G(r) \chi^\mu_\kappa(\theta,\phi) \\ -F(r) \chi^\mu_{-\kappa}(\theta,\phi) \end{pmatrix} \right) \eta_t, \tag{82}
$$

where $G(r)$ and $F(r)$ are the upper and lower radial wave function components respectively, and are real functions of $r$. The $\eta_t$ is the eigenstate of isospin $\tau_3$, i.e., $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for protons and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for neutrons.

The $\chi^\mu_\kappa$ are functions of spin and angular variables,

$$
\chi^\mu_\kappa(\theta,\phi) = \begin{pmatrix} \langle \ell,1/2,\mu-1/2,1/2 | j,\mu \rangle \chi^{\mu-1/2}_\ell(\theta,\phi) \\ \langle \ell,1/2,\mu+1/2,-1/2 | j,\mu \rangle \chi^{\mu+1/2}_\ell(\theta,\phi) \end{pmatrix}, \tag{83}
$$

The $Y^m_\ell(\theta,\phi)$ is the usual spherical harmonic. The quantum numbers $\kappa$, $j$, $\ell$, $\bar{\ell}$, and $S$ are

$$
\begin{align*}
(\sigma \cdot \hat{l} + 1) \chi^\mu_\kappa &= -\kappa \chi^\mu_\kappa, \\
(\hat{j} \cdot \hat{l}) &= j^2 - \ell^2 - 3/4, \\
\hat{j}_3 \chi^\mu_\kappa &= \mu \chi^\mu_\kappa, \\
j &= |\kappa| - 1/2: \text{ total angular momentum,} \\
S &= \kappa/|\kappa|: \text{ sign of } \kappa. \tag{84}
\end{align*}
$$
\[ \ell = j + S/2 = |\kappa| + (S-1)/2: \] orbital angular momentum,
\[ \overline{\ell} = \ell - S. \]

The wave function \( \psi_\alpha^+(x) \) has the following properties when acted upon by Dirac matrices,
\[ -i\gamma_0 \gamma^+ \psi_\alpha^+(x) = \frac{1}{r} \begin{pmatrix} \left[ \partial_r F_\alpha (r) - \frac{k}{r} F_\alpha (r) \right] \chi^\mu_\alpha (\theta, \varphi) \\ \left[ \partial_r G_\alpha (r) + \frac{k}{r} G_\alpha (r) \right] \chi_-^\mu_\alpha (\theta, \varphi) \end{pmatrix} n_{r_\alpha}, \quad (85) \]
\[ \gamma_0 \psi_\alpha^+(x) = \psi_\alpha^+(x) \text{ with } F_\alpha = -F_\alpha, \]
\[ i\gamma_r \gamma^+ \psi_\alpha^+(x) = \psi_\alpha^+(x) \text{ with } G_\alpha + F_\alpha \text{ and } F_\alpha = -G_\alpha, \quad (86) \]
\[ \gamma_5 \psi_\alpha^+(x) = i\psi_\alpha^+(x) \text{ with } G_\alpha + F_\alpha, F_\alpha = -G_\alpha, \text{ and } \kappa_\alpha = -\kappa_\alpha. \]

The Green's function for the meson fields (Eq. (11)), in spherical coordinates, is
\[ D(x - x', m') = \sum_{p=0}^{\infty} \frac{1}{L} f_p(r, r', m') Y_p^q(\theta, \varphi) Y_q^q(\theta, \varphi), \quad (87) \]
where
\[ f_p(r, r', m') = n^k_p(m_r') j_p(m_r) \text{ for } m'^2 > 0 \]
\[ = |m'| n_p(|m'| r') j_p(|m'| r') \text{ for } m'^2 < 0. \quad (88) \]

The \( j_p \) and \( n_p \) are the modified spherical Bessel functions and the \( j_p \) and \( n_p \) the spherical Bessel functions. The symbols \( r_\prec \) and \( r_\succ \) are the lesser and greater respectively of the radial variables \( r \) and \( r' \).

Using Eqs. (82)-(88), the relativistic Hartree-Fock equation (Eq. (30)) becomes, after the sum over \( \gamma \) and \( x' \),
\[ \omega_\alpha \psi_\alpha(x) = \gamma_0 (-i \gamma \cdot \vec{v} + M - V_H) \psi_\alpha(x) \]
\[ - \sum \gamma_0 (V_F)_\alpha \frac{1}{r} \begin{pmatrix} iG_\gamma(r) x_{\alpha \gamma}^{\mu}(	heta, \varphi) \\ -F_\gamma(r) x_{\alpha \gamma}^{\mu}(	heta, \varphi) \end{pmatrix} n_{t_\alpha}, \]

where the sum over \( \gamma \) means sum over all the values of \( n_\gamma, \kappa_\gamma, \) and \( t_\gamma \) for occupied states. Since

\[ [G_\alpha(r)^2 + F_\alpha(r)^2] \frac{1}{r} \begin{pmatrix} iG_\gamma(r) x_{\alpha \gamma}^{\mu} \\ -F_\gamma(r) x_{\alpha \gamma}^{\mu} \end{pmatrix} n_{t_\alpha} = \begin{pmatrix} G_\gamma(r)G_\alpha(r) & -iG_\gamma(r)F_\alpha(r) \\ iF_\gamma(r)G_\alpha(r) & F_\gamma(r)F_\alpha(r) \end{pmatrix} \psi_\alpha(x), \]

the Fock potential can be changed to a local and diagonal potential by multiplying and dividing by \([G_\alpha(r)^2 + F_\alpha(r)^2] \). The relativistic Hartree-Fock single-particle equation will be reduced to the radial form

\[ \omega_\alpha \begin{pmatrix} iG_\alpha(r) \\ -F_\alpha(r) \end{pmatrix} = \gamma_0 (-i \gamma \cdot \vec{v} + M - V_H - V_{F_\alpha}) \begin{pmatrix} iG_\alpha(r) \\ -F_\alpha(r) \end{pmatrix}. \] \hspace{1cm} (89)

This is just the usual eigenvalue problem with a state-dependent potential. The potentials \( V_H \) and \( V_{F_\alpha} \) are, after neglecting the \( \theta \) and \( \varphi \) components of the \( V_\mu, A_\mu, \) and \( b_\mu \) fields

\[ \gamma_0 V_H = \sum \gamma \left[ (2J_\gamma + 1) \int dr' \left[ g_S^2 f_0(r, r', m_S) \gamma_0 [G_\gamma(r')^2 - F_\gamma(r')^2] \right. \right. \]
\[ - \left. \left. g_V^2 f_0(r, r', m_V) + e^2 f_0(r, r', 0) \frac{1}{2} (1+t_\gamma) \frac{1}{2} (1+t_\gamma) \right. \right. \]
\[ + g_\mu^2 f_0(r, r', m_\mu) t_\alpha t_\gamma \left. \right] \left. [G_\gamma(r')^2 + F_\gamma(r')^2] \right], \] \hspace{1cm} (90)
\[
\gamma_0 V_{\alpha} = - (2j_{\alpha} + 1)^{-1} \left[ G_{\alpha}(r)^2 + F_{\alpha}(r)^2 \right]^{-1}
\times \sum_{\gamma} \sum_{L} \int dr' \begin{pmatrix}
\left( z(\ell_{\gamma} l \gamma a_{\gamma} a_{\gamma} \alpha \alpha \frac{1}{2} L) \right) & 0 \\
0 & \frac{1}{2}(s_{\alpha} - s_{\gamma}) \left( z(\overline{\ell_{\gamma}} l \gamma \overline{a_{\gamma}} a_{\gamma} \alpha \alpha \overline{\frac{1}{2}} L) \right)
\end{pmatrix}
\]

\[
x[g_s^2 f_L(r, r', m') \gamma_{\gamma} t_{\gamma} a_{\gamma} \alpha \alpha \frac{1}{2} L \gamma(r') \gamma(r') G_{\gamma}(r') G_{\alpha}(r')

- \frac{1}{2}(s_{\alpha} - s_{\gamma}) \left( z(\overline{\ell_{\gamma}} l \gamma \overline{a_{\gamma}} a_{\gamma} \alpha \alpha \overline{\frac{1}{2}} L) \gamma(r') \gamma(r') \right)
\]

\[
\times \{ g_s^2 f_L(r, r', m') [2-n_{\gamma}^+] \pi_G(r, r', 0') \left( 1 + t_{\alpha} \right) \left( 1 + t_{\gamma} \right) n_{\gamma}^+ n_{\alpha}

+ g_{\rho}^2 f_L(r, r', m') [2-n_{\gamma}^+] \pi_G(r, r', 0') \left( 1 + t_{\alpha} \right) \left( 1 + t_{\gamma} \right) n_{\gamma}^+ n_{\alpha}
\]

\[
- \{ g_s^2 f_L(r, r', m') n_{\gamma}^+ n_{\gamma} \pi_G(r, r', 0') \left( 1 + t_{\alpha} \right) \left( 1 + t_{\gamma} \right) n_{\gamma}^+ n_{\alpha}

+ g_{\rho}^2 f_L(r, r', m') [2-n_{\gamma}^+] \pi_G(r, r', 0') \left( 1 + t_{\alpha} \right) \left( 1 + t_{\gamma} \right) n_{\gamma}^+ n_{\alpha}
\]

\[
+ g_{\rho}^2 f_L(r, r', m') [2-n_{\gamma}^+] \pi_G(r, r', 0') \left( 1 + t_{\alpha} \right) \left( 1 + t_{\gamma} \right) n_{\gamma}^+ n_{\alpha}
\]

\[
\times \left\{ \begin{pmatrix}
G_{\gamma}(r) G_{\alpha}(r) & -iG_{\gamma}(r) F_{\alpha}(r) \\
iF_{\gamma}(r) G_{\alpha}(r) & F_{\gamma}(r) F_{\alpha}(r)
\end{pmatrix} \right\}
\]

\[
(91)
\]
\[ + (2j_\alpha + 1)^{-1} \left[ G_\alpha(r)^2 + F_\alpha(r)^2 \right]^{-1} \]

\[
\times \sum_{\gamma L} \int dr' \begin{pmatrix}
\left( z(\bar{\gamma}, j_\gamma, l_\gamma', \alpha', \frac{1}{2} L) \right) & 0 \\
0 & 1/2(S_\alpha - S_\gamma) \times z(\bar{\gamma}, j_\gamma, l_\gamma', \alpha', \frac{1}{2} L)
\end{pmatrix}
\]

\[
\times g^2 \delta_L(r, r', m') (2 - \eta_\gamma \eta_\alpha) i \gamma \{ z(\bar{\gamma}, j_\gamma, l_\gamma', \alpha', \frac{1}{2} L) F_\gamma(r') G_\alpha(r') \}
\]

\[
\times \left( \begin{array}{cc}
G_\gamma(r) G_\alpha(r) & -iF_\gamma(r) F_\alpha(r) \\
G_\gamma(r) F_\alpha(r) & F_\gamma(r) F_\alpha(r)
\end{array} \right).
\]

The symbol \( z \) represents the \( z \)-coefficient of Blatt and Biedenharn [71-74]. The phase convention of the Clebsch-Gordan and \( z \)-coefficient is the same as in Messiah [74]. These equations ((89)–(91)) give the retardation correction of Miller with a factor of two correction [38, 39].
REFERENCES

[35] P.-G. Reinhard, M. Rufa, J. Maruhn, W. Greiner, and


