ABSTRACT

STUDY OF SPINOR SYMMETRIES IN NUCLEAR STRUCTURE

Serdar Kuyucak

Yale University, 1982

One of the possible spinor symmetries of the interacting boson-fermion model is studied. The interacting boson-fermion model is a direct extension of the interacting boson model of nuclear collective states, where the fermion degrees of freedom of the odd nucleon are coupled to the boson core. Since in this model nuclei are treated as a mixed system of bosons and fermions, in order to discuss its symmetries the ordinary concept of symmetry is extended to spinor symmetries.

In this work, we discuss the spinor symmetry, Spin(6), which arises when the bosons have SO(6) symmetry and the fermions occupy a single particle orbital with j=3/2 which is described by an SU(4) symmetry. The states are classified according to the irreducible representations of the group chain Spin(6)⊃Spin(5)⊃Spin(3)⊃Spin(2). An energy formula is derived assuming that the Hamiltonian can be written in terms of the Casimir operators of this group chain. The Spin(6) wave functions are constructed by expanding them into product of the boson SO(6) and fermion SU(4) wave functions. Using these wave functions we obtain closed expressions for electromagnetic (E0, M1, E2) transition rates, static moments and (one and two) nucleon transfer reaction intensities.

Comparison of the calculations with the experimental data on iridium and gold isotopes shows that the Spin(6) symmetry scheme forms a
reasonable zeroth order approximation in describing properties of these nuclei. Improvements over the results is possible by (i) incorporating other single particle orbitals \( (j=1/2, 5/2, 7/2) \) to the dominant \( j=3/2 \) orbital, (ii) mixing the boson SU(3) limit to the SO(6) limit. Although, in general, these effects must be dealt with numerical computations, the cases of mixing of the \( j=1/2 \) orbital and the boson SU(3) limit can be studied in perturbation theory. We also report the results of the perturbation calculations of the above effects.
STUDY OF SPINOR SYMMETRIES IN NUCLEAR STRUCTURE

A Dissertation

Presented to the Faculty of the Graduate School

of

Yale University

in Candidacy for the Degree of

Doctor of Philosophy

by

Serdar Kuyucak

May 1982
ACKNOWLEDGEMENTS

First, I would like to thank my thesis advisor, Professor Franco Iachello, for suggesting the problem considered here and for his help and guidance over the course of this research. I am indebted to Professor Jolie Cizewski for her careful reading of the thesis manuscript and thoughtful suggestions. She has also contributed to my understanding of the experimental situation in an essential way. I am thankful to Professor Allan Bromley for his initial influence in my study of nuclear physics and for the financial support during my graduate career.

I would like to express my appreciation to Pauline DiGioia and Sara Batter for their help and support over the years and to Sandy Sicignano for drawing the figures in this work.

Finally, I thank to my wife Zeynep for her moral support and for sharing the burden of my graduate study.
Table of Contents

I. Introduction 1

II. The Interacting Boson-Fermion Model and its Symmetries 8
   A. The Interacting Boson-Fermion Model 9
   B. Spinor Symmetries 13

III. The Spin(6) Limit of the Interacting Boson-Fermion Model 20
    A. Energy Spectra 21
    B. Wave Functions 27
    C. Electromagnetic Transition Rates 37
    D. Nucleon Transfer Intensities 49
    E. Examples of Spectra with Spin(6) Symmetry 58

IV. Symmetry Breaking in Perturbation 65
    A. Mixing of the $3s_{1/2}$ Orbital 66
    B. Mixing of the Boson SU(3) Limit 73

V. Spinor Symmetries in Molecular Spectra 77
    A. Spin(3) Limit 80

VI. Conclusions 84

Tables 86
Appendix A. Matrix Elements of Fermion Operators in the SU\(^{(F)}\)(4) basis

Appendix B. Matrix Elements of Boson Operators in the SO\(^{(B)}\)(6) basis

Appendix C. ODDA Parameters Needed to Produce the Spin(6) Limit

References
I. Introduction

Although nuclear physics is quite an old subject, today we are still lacking a detailed theory which can explain nuclear structure and reactions, starting from the free nucleon-nucleon interaction. This is, in part, due to the difficulty of the many-body problem posed by nuclei, which consists of many strongly interacting particles. The number of nucleons in medium mass and heavy nuclei (A=100-200) is neither small enough to allow exact calculations, nor so large that statistical techniques can be used. In addition, nuclear interactions are rather complex. We have, at present, a potential description based on exchange of various mesons. With the emerging quark structure of the nucleons and mesons, one may eventually hope to find a more fundamental way to calculate the potential governing the nucleon-nucleon interaction. But the preliminary calculations suggest that this will be a very difficult task, if at all feasible.

Given the formidability of the nuclear many-body problem, it is no surprise that most progress in nuclear physics has been made through phenomenological theories based on simple models. The first successful model, proposed in 1948 by Mayer [Ma 48] and Jensen and his collaborators [HJS 48], treated nucleons as moving independently in an average central potential. This model explained the magic numbers 2, 8, 20, 50, 82, 126 as a result of proton and neutron shell closures, hence the name "nuclear shell model". Soon after its initial success, experiments revealed that the independent particle shell model with no residual interaction could not be applied to all regions of the periodic...
table. For example, nuclei in the middle of the closed shells showed quadrupole moments and transition rates several orders of magnitude larger than those predicted by the independent particle shell model with no residual interaction. Obviously, only a very strong correlation in the motion of individual nucleons can produce such an effect. These data were interpreted by Rainwater [Ra 50] and Bohr [Bo 51] as being the result of the collective quadrupole motion of nucleons. Consequently, Bohr and Mottelson developed a collective model of nuclei in terms of the shape and deformations of the nuclear surface [Bo 52, BM 53]. Originally, this model applied to deformed nuclei in the middle of shells (rotational) and to spherical nuclei near the closed shells (vibrational). Later, many attempts were made in order to describe all collective motions throughout the periodic table in a single framework. In particular, Kumar and Baranger developed a unified geometrical description of collective motions, starting from a microscopic pairing-plus-quadrupole model [Ku 74]. However, it required large scale computations for each nucleus which limited its applicability as a practical model. Another line of approach has been through the boson expansion models, where one maps the fermion Hamiltonian into a boson Hamiltonian. This can be done in two different ways by mapping either states or operators. Belyaev and Zelevinsky proposed to map the operators in such a way that the commutation relations are preserved [BZ 62]. The most recent investigations based on this method have been carried out by Kishimoto and Tamura [KT 76, TWK 79], whose calculations show good agreement with the experimental data in many regions. The other method, proposed by Marumori et al [MYT 64], is to map the states in the Hilbert space and to define the operators in such a way that the
matrix elements are preserved by the mapping. Lee and Holzwart used this method in their description of anharmonic vibrational nuclei [LH 75]. Boson expansion models have also been employed on a phenomenological level, where simple boson Hamiltonians are parametrized without reference to the underlying fermion Hamiltonian. The most extensive calculations in this approach have been done by Greiner and collaborators who have diagonalized a generalized Bohr Hamiltonian by exploiting its U(5) symmetry [GG 71, HSM 80]. The common problem of the boson expansion models is that the boson Hamiltonian can not be restricted to two-body interactions. Thus, to get a good description of nuclei, one has to include higher order terms, which complicates the calculations and results in a large set of parameters in the case of the phenomenological models.

Collective models of odd-even nuclei have followed a parallel development in that a single particle degree of freedom was coupled to an even-even core described by a certain collective model, to obtain the corresponding model in odd-even nuclei. Examples are the particle-vibrator model [Bo 52], the Nilsson model [Ni 55], the particle-triaxial rotor model [Me 75], and the particle-γ-unstable rotor model [Le 76]. Each of these models suffer from the same shortcoming of the underlying even-even collective models; they are valid only in an appropriate mass region.

In recent years, an alternative unified description of collective states in even-even nuclei has been developed by Arima and Iachello [AI 75, 76, 78, 79]. In this model, nucleon pairs are treated as bosons which can occupy two levels, one with angular momentum L=0, called the s-boson, and one with angular momentum L=2, called the d-boson. Low-
lying collective states are assumed to be generated through the interaction of these s and d-bosons, hence the name "interacting boson model" (IBM) is given to this description. The physical basis of the IBM derives from a study of the effective nucleon-nucleon interaction, which shows that there is a large attractive interaction between identical nucleons in the state with total angular momentum L=0, a moderately attractive interaction in the state with L=2, and zero or repulsive interaction for states with larger L=4,6,etc. Thus, in a first approximation the fermion space is truncated to L=0 and L=2 pairs, which are subsequently replaced by s and d-bosons. The recognition of the quadrupole correlations had been the basis of the Bohr model. The new idea of the IBM has been the introduction of a scalar boson describing the pairing correlations, thus treating both correlations in a single framework. This feature of the IBM distinguishes it from the other boson models where only quadrupole boson variables have been employed. (One exception is the work of Janssen, Jolos, and Donau [JJD 74] whose Hamiltonian is equivalent to the IBM Hamiltonian.)

An important aspect of the IBM is that it has a definite group structure, the s-boson and five components of the d-boson span a six dimensional space, yielding U(6) as the group structure of the problem. This has two important implications. First, the totally symmetric irreducible representations of U(6) generate all low-lying collective states, thus allowing a unified description of the large variety of observed spectra. Secondly, in special limiting situations corresponding to dynamical symmetries, it is possible to find analytical solutions to most properties of a nucleus. Dynamical symmetries arise whenever the IBM Hamiltonian can be written in terms of invariant
(Casimir) operators of a complete chain of groups, starting with $U(6)$ and ending with $O(3)$. (This is required because angular momentum is a good quantum number.) There are three possible dynamical symmetries corresponding to the group chains

\[ U(6) \supset U(5) \supset O(5) \supset O(3) \supset O(2) \quad \text{I} \]

\[ U(6) \supset SU(3) \supset O(3) \supset O(2) \quad \text{II} \]

\[ U(6) \supset O(6) \supset O(5) \supset O(3) \supset O(2) \quad \text{III} \]

Each of the above cases was discussed in a series of papers [AI 76a, 78, 79] and reviewed in Ref. [AI 81]. Although real nuclei will never display an exact dynamical symmetry, analytical formulas obtained through them provide a semi-quantitative understanding of observed properties of nuclei without resorting to numerical calculations, which makes their study worthwhile.

The interacting boson model was extended to odd-even nuclei first in a straightforward manner, by directly coupling the fermion degree of freedom to the boson core described by the IBM Hamiltonian [AI 76b]. But, initially, not much progress was made because of the large number of parameters involved in the boson-fermion Hamiltonian. After the recognition of the special role played by the exchange term, and developments in the microscopic foundation of the IBM which allowed an estimate of parameters [IS 79], numerical calculations employing the interacting boson-fermion model (IBFM) started.

Since in the framework of the IBFM odd-A nuclei are treated as a
mixed system of bosons and fermions, its group structure is determined by the product of the boson and fermion groups. The dynamical symmetries of the IBM can be extended to the IBFM, provided boson and fermion systems both have dynamical symmetries, and there is an isomorphism between their Lie algebras. This extension is referred as a spinor symmetry \([la 80a, 81a]\). The smooth change of nuclear properties with the nucleon number suggests that a multiplet of nuclei (both even-even and odd-even), described by a certain dynamical symmetry, may belong to the same representation of a larger group. Since ordinary symmetries apply to systems of either bosons or fermions, in order to classify multiplets of nuclei a new type of symmetry is required. Symmetries applying to mixed systems of bosons and fermions have recently been considered in particle and gravitational physics and called supersymmetries \([CNS 75]\). Investigations so far indicate that in certain regions multiplets of nuclei can in fact be described by an approximate supersymmetry scheme based on the supergroup \(U(6/4)\) \([BBI 81]\). Thus, the first example of a supersymmetry in nature is observed in nuclear physics.

In the last few years, an algebraic model has been developed to describe the rotation-vibration spectra of molecules \([la 81b]\). This model has a similar structure to the IBM. Hence, the idea of spinor symmetries can also be applied to molecular spectra, extending its boson dynamical symmetries to mixed system of bosons and fermions.

In this thesis, the spinor symmetry associated with the \(O(6)\) limit of the IBM, which we will call \(Spin(6)\) symmetry, will be discussed in detail. In Chapter II, we will consider the IBFM and study the group structure of its \(Spin(6)\) limit. Consequences of the \(Spin(6)\) symmetry
are worked out in Chapter III. In particular, closed expressions for energies, electromagnetic transition rates and nucleon transfer intensities are derived, and compared to the available data on iridium and gold isotopes. In Chapter IV, several improvements over the Spin(6) symmetry scheme are discussed, including mixing of the $3s_{1/2}$ single particle orbital and of the boson SU(3) limit. Although not directly related to this work, the case of Spin(3), which might have possible applications in molecular spectra, is also discussed briefly in Chapter V.
II. The Interacting Boson-Fermion Model and Bose-Fermi Symmetries

In this chapter, we will introduce the interacting boson-fermion model, which forms the framework for the study of the collective states in odd-even nuclei. In order to discuss symmetries associated with this model we will generalize the ordinary concept of symmetry, which applies to either boson or fermion systems, to spinor symmetries which apply to mixed systems of bosons and fermions. In particular, we will study the group structure of the spinor symmetry associated with the \( \text{SO}(6) \) limit of the IBM, \( \text{Spin}(6) \), and show that this symmetry also emerges from the simplified boson-fermion Hamiltonian of Ref. [IS 79] which is used in practical applications.
A. The Interacting Boson-Fermion Model

Description of odd-even nuclei and two or more quasiparticle states in even-even nuclei requires, in addition to the collective degrees of freedom, also explicit introduction of the single particle degrees of freedom (Fig. 1). The collective degrees of freedom, as discussed in the introduction, are described by a set of \( N \) bosons with angular momenta \( J=0 \) (s-bosons) and \( J=2 \) (d-bosons). The single particle degrees of freedom are instead described by a set of \( M \) fermions, where \( M=1 \) for the one quasiparticle states in odd-even nuclei, \( M=2 \) for the two quasiparticle states in even-even nuclei, etc. If only the valence particles are considered as active, the angular momenta of the fermions are all those contained in a major valence shell. For example, for odd proton states in the shell 50–82 they are, \( j = 5/2, 7/2, 11/2, 3/2, 1/2 \) (Fig. 2). We introduce the creation \( (d_\mu^+, \mu = 0, \pm 1, \pm 2; s^+) \) and annihilation \( (d_\mu, s) \) operators for bosons, altogether denoted by \( b_\alpha^+ (\alpha = 1, \ldots, 6) \), \( b_\alpha \), and creation \( (a_{j\mu}^+, \nu = \pm j, \pm (j-1), \ldots, \pm 1/2) \) and annihilation \( (a_{j\mu}) \) operators for fermions, altogether denoted by \( a_{i}^+ (i = 1, \ldots, m) \), \( m = \Sigma (2j_i+1) \), \( a_i \). The Hamiltonian for the combined system of bosons and fermions, containing at most one and two-body interactions, can be written as

\[
H = H_B + H_F + V_{BF} \tag{2.1}
\]

where
\begin{align}
H_B &= \epsilon_s (s^+ s^-) + \epsilon_d (d^+ d^-) + \sum_{L=0,2,4} \frac{1}{2} (2L+1)^{1/2} \epsilon_L \left[ [d^+ x_d^+] (L) \times \{ \tilde{d}\tilde{x} \} (L) \right] (0) \\
&+ \frac{1}{\sqrt{2}} E \left[ [d^+ x_d^+] (2) \times \{ \tilde{d}\tilde{x} \} (2) + [d^+ x_s^+] (2) \times \{ \tilde{d}\tilde{x} \} (2) \right] (0) \\
&+ \frac{1}{2} \tilde{v}_o \left[ [d^+ x_s^+] (0) \times \{ \tilde{s}\tilde{x} \} (0) + [s^+ x_s^+] (0) \times \{ \tilde{d}\tilde{x} \} (0) \right] (0) \\
&+ u_2 \left[ [d^+ x_s^+] (2) \times \{ \tilde{d}\tilde{x} \} (2) \right] (0) + \frac{1}{2} u_0 \left[ [s^+ x_s^+] (0) \times \{ \tilde{s}\tilde{x} \} (0) \right] (0),
\end{align}

is the boson Hamiltonian of Refs. [AI 76,78,79],

\begin{align}
H_F &= \sum_{j\mu} \epsilon_j \left[ a_{j\mu}^+ a_{j\mu} \right] + \sum_{j,j',k,k',L} \frac{1}{2} u_{jj',kk',L} \left[ [a_{j\mu}^+ a_{j\mu}] (L) \times [a_{k\nu}^+ a_{k\nu}] (L) \right] (0),
\end{align}

is the fermion Hamiltonian and

\begin{align}
V_{BF} &= \sum_j \tilde{v}_j (0) \left[ [s^+ x_s^+] (0) \times [a_{j\mu}^+ a_{j\mu}] (0) \right] (0) + \sum_{jj'} \tilde{w}_{jj'} (2) \left[ [d^+ x_d^+ + d^+ x_s^+] (2) \times [a_{j\mu}^+ a_{j\mu}] (2) \right] (0) \\
&+ \sum_{j,j',L} \tilde{w}_{jj'} (L) \left[ [d^+ x_d^+] (L) \times [a_{j\mu}^+ a_{j\mu}] (L) \right] (0),
\end{align}

is the boson-fermion interaction. In Eqs. (2.2)-(2.4), \( \tilde{d}_\mu = (-)^\mu d_\mu \), \( \tilde{s}=s \), \( \tilde{a}_{j\mu} = (-)^{j-\mu} a_{j,-\mu} \) and the brackets denote tensor products.

For states with one fermion, \( M=1 \), only the one-body part of \( H_F \), the first term in Eq. (2.3), contributes.

Energy spectra of odd-even nuclei (or two or more quasiparticle states) can be calculated by diagonalizing the above Hamiltonian in an
appropriate basis. However, the number of parameters that must be specified for this calculation is too large to permit practical applications. This difficulty was overcome when it was recognized that only three terms were important in the boson-fermion interaction, a monopole, a quadrupole and an exchange term [IS 79, Ia 80b]. The simplified interaction can be written as

\[ V_{BF}' = \sum_j A_j \left[ M_B^{(0)} \times M_{Fj}^{(0)} \right] + \sum_{j,j',j''} \Gamma_{jj'} \left[ Q_B^{(2)} \times Q_{Fjj'}^{(2)} \right] + \sum_{j,j',j''} A_j^{(j''')} \left[ F_{j''}^{(j''')} \times F_{j''}^{(j''')} \right] (0) \]

\[ (2.5) \]

where

\[ M_B^{(0)} = [d^+ s d^+ s]^{(0)} \quad ; \quad M_{Fj}^{(0)} = [a_j^+ x d_j^+ x d_j]^{(0)} ; \]

\[ Q_B^{(2)} = [d^+ s + s^+ d_2^{(2)} + \chi [d^+ x d]^{(2)} ; \quad Q_{Fjj'}^{(2)} = [a_j^+ x a_{j'}^+]^{(2)} \]

\[ (2.6) \]

\[ F^{(j''')}_{j''} = [a_j^+ x d_j^{(j''')} ; \quad F^{(j''')}_{j''} = [d^+ x a_j^{(j''')} ; \]

Eq. (2.5) corresponds to a particular choice of the coefficients \( w_j^{(0)} \), \( w_{jj'}^{(2)} \) and \( w_{jj'}^{(L)} \) in Eq. (2.4). The symbol \( (:) \) in Eq. (2.6) denotes normal ordering of operators. Furthermore, an approximate microscopic theory of the IBFM allows one to estimate the angular momentum dependence of the parameters \( A_j \), \( \Gamma_{jj'} \), \( A_{jj}^{(j''')} \) [Sc 81, Ta 81].

In concluding this section we mention that, for practical calculations, one must also specify the number of bosons, \( N \), and
fermions, $M$. Since bosons are identified with correlated nuclon pairs, $N$ is given by the total number of proton ($\pi$) and neutron ($\nu$) pairs in the major valence shell. This number is counted from the nearest closed shell, i.e. it represents the number of active particle pairs from the beginning up to the middle of the shell and the number of active hole pairs from there on. Similarly, the $M$ unpaired fermions are taken as particle-like from the beginning up to the middle of the shell and as hole-like from there on. For example, in $^{191}_{77}$Ir$_{114}$, $N = N_{\pi} + N_{\nu} = 2 + 6 = 8$ and the odd proton is hole-like. Although, in principle, all calculations should be done in the proton-neutron formalism of the IBM-2 [AI 81], since we are interested in dynamical symmetries, we will prefer to use the IBM-1, where no distinction between proton and neutron bosons is made. Calculations indicate that this distinction affects only details of the results, but not their main features.
B. Spinor Symmetries

When the Hamiltonian of a system is written in terms of invariant operators of a symmetry group, dynamical symmetries arise and an analytical solution of the eigenvalue problem becomes possible. Until recently, the use of dynamical symmetries in physical problems has been restricted to either systems of bosons or fermions. For mixed systems of bosons and fermions, as encountered in the IBFM, the notion of dynamical symmetries must be extended. The new type of symmetry, called "spinor symmetry" [la 80a,81a], arises when the group structures of boson and fermion systems are similar so that their Lie algebras are isomorphic. Then, one can combine the boson and fermion group chains into a single group chain which, now, can be used to classify the mixed system of bosons and fermions.

Although we will not discuss in detail, we mention that Bose-Fermi symmetries can be further generalized. This possibility is suggested by the fact that the empirical parameters used in describing the adjacent even-even and odd-even nuclei appear to be approximately the same. Thus, both even and odd nuclei may belong to the same representation of a larger group. Since ordinary groups can accommodate either bosons or fermions only, a new algebraic construct is required for the classification of both bosonic and fermionic representations together. The algebraic machinery of this new construct, called superalgebras and supergroups, has recently been worked out in the context of particle and gravitational physics [CNS 75]. The corresponding symmetries, which now apply to mixed systems of bosons and fermions, have been called
supersymmetries. The implications of supersymmetries for nuclei in the framework of the IBFM has been studied in Ref. [BBI 81].

In order to introduce the concept of a spinor symmetry, we begin by studying the group structure of the interacting boson-fermion Hamiltonian, Eqs. (2.1)-(2.4). The boson part of $H$, Eq. (2.2), can be written in terms of the $36$ generators, $G_{\alpha \alpha'}^B = b_{\alpha}^* b_{\alpha'}$, of the unitary group $U^B(6)$. Similarly, the fermion part of $H$, Eq. (2.3), can be written in terms of the $m^2$ generators, $G_{ii'}^F = a_{i}^* a_{i'}$, of the unitary group $U^F(m)$. Here the superscripts $B$ and $F$ denote that the groups are constructed with boson and fermion operators respectively. Finally, the boson-fermion interaction, Eq. (2.4), consists of products of $G_{\alpha \alpha'}^B$ and $G_{ii'}^F$

$$
H_B = \sum_{\alpha \alpha'} \epsilon_{\alpha \alpha'} G_{\alpha \alpha'}^B + \sum_{\alpha \alpha' \beta \beta'} \frac{1}{2} u_{\alpha \alpha' \beta \beta'} G_{\alpha \alpha'}^B G_{\beta \beta'}^B,
$$

$$
H_F = \sum_{ii'} \eta_{ii'} G_{ii'}^F + \sum_{ii' kk'} \frac{1}{2} v_{ii' kk'} G_{ii'}^F G_{kk'}^F, \quad (2.7)
$$

$$
V_{BF} = \sum_{\alpha \alpha' ii'} w_{\alpha \alpha' ii'} G_{\alpha \alpha'}^B G_{ii'}^F,
$$

where the coefficients, $\epsilon_{\alpha \alpha'}$, $u_{\alpha \alpha' \beta \beta'}$, $\eta_{ii'}$, $v_{ii' kk'}$, $w_{\alpha \alpha' ii'}$, are appropriate combinations of the coefficients appearing in Eqs. (2.2)-(2.4). The only difference between the two forms, Eqs. (2.2)-(2.4) and (2.7) is that Racah's coupled tensor notation is used in the former, whereas an uncoupled notation is used in the latter.
From Eq. (2.7) it is clear that the group structure of $H$ is that of the direct product $U^B(6) \otimes U^F(m)$.

In general, this product group provides an appropriate basis in which the Hamiltonian (2.2) must be diagonalized numerically. However, under special situations, i.e., whenever the Hamiltonian does not mix the states, eigenvalue problem can be solved analytically. This occurs when the groups $U^B(6)$ and $U^F(m)$ have a common chain of subgroups. Then, one can construct subgroups of $U^B(6) \otimes U^F(m)$ in which the boson, $G^B$, and fermion, $G^F$, generators are combined together, and the Hamiltonian, written in terms of Casimir operators of a certain group chain, will be diagonal in the basis defined by that group chain.

Since the group $U^B(6)$ has three possible chains of subgroups, as mentioned in the introduction, there are three possible symmetries of this type. In this thesis, we will discuss the symmetry associated with the group chain

$$U^B(6) \supset SO^B(6) \supset SO^B(5) \supset SO^B(3) \supset SO^B(2). \quad (2.8)$$

Since the algebras of $SO^B(6)$, $SO^B(5)$, $SO^B(3)$ and $SO^B(2)$ are isomorphic to those of $SU^F(4)$, $Sp^F(4)$, $SU^F(2)$ and $SO^F(2)$ respectively, spinor symmetries are possible whenever $m$ is such that the fermions span an irreducible representation of $SU(4)$. The lowest dimensionalities of these representations are $m=4$ ($j=3/2$) and $m=20$ ($j=1/2, 3/2, 5/2, 7/2$). Here we consider the case $m=4$ which corresponds to quasiparticle states built on a $j=3/2$ level. Then the chain (2.8) and the fermion group chain
can be combined into

\[ U^{(B)}(6) \otimes U^{(F)}(4) \supseteq SO(B(6) \otimes SU^{(F)}(4) \supseteq Spin(6) \supseteq Spin(5) \supseteq Spin(3) \supseteq Spin(2) \],

where we have introduced the spinor groups \( [Gi\ 74] \)

\[
\begin{align*}
\text{Spin}(6) & \approx SU(4) \\
\text{Spin}(5) & \approx Sp(4) \\
\text{Spin}(3) & \approx SU(2) \\
\text{Spin}(2) & \approx SO(2)
\end{align*}
\]

In Racah's coupled tensor notation, the generators of the \( \text{Spin}(6) \) group can be written as

\[
\begin{align*}
G^{(1)}_\mu &= B^{(1)}_\mu - \frac{1}{\sqrt{2}} A^{(1)}_\mu \\
G^{(2)}_\mu &= B^{(2)}_\mu + A^{(2)}_\mu \\
G^{(3)}_\mu &= B^{(3)}_\mu + \frac{1}{\sqrt{2}} A^{(3)}_\mu
\end{align*}
\]

where

\[
B^{(J)}_\mu = (d^\dagger x \tilde{d})(J)_\mu, \quad J=1,3
\]
Here denotes the creation (annihilation) operator for fermions. In the same notation, generators of and those of are . As can be seen from Eq. (2.12), introduction of spinor groups amounts to taking combinations of boson and fermion multipole operators. For example, denoting the angular momentum of bosons by and of fermions by , the introduction of corresponds to the introduction of the total angular momentum operator

\[
\vec{J} = \vec{J}_B + \vec{J}_F
\]

Similarly, extension of this idea to and groups is equivalent to the introduction of the total quadrupole and octupole operators.

The dynamical symmetry associated with the group chain (2.10), which will be called symmetry, arises whenever the Hamiltonian can be written only in terms of the Casimir operators of the groups in this chain. Since the Casimir operators are given by scalar products of the group generators, must be of the form
which corresponds to a particular choice of parameters in Eq. (2.1). From Eq. (2.5), it is not at all clear that the simplified boson-fermion Hamiltonian of Ref. [IS 79] can also be put in the above form. In concluding this chapter we will show that this is indeed the case. For a single particle level with \( j=3/2 \), the boson-fermion interaction, Eq. (2.5) reduces to

\[
H = A_1 G^{(1)} G^{(1)} + A_2 G^{(2)} G^{(2)} + A_3 G^{(3)} G^{(3)} \tag{2.15}
\]

By a change of coupling transformation the last term can be written as

\[
V_{BF} = A \left[ [d^+ x d]^{(0)} \times [a^+ x a]^{(0)} \right]^{(0)} + \Gamma \left[ [d^+ x s + s^+ x d]^{(2)} \times [a^+ x a]^{(2)} \right]^{(0)}
\]

\[
+ A : [a^+ x d^{(2)}]^{(0)} \times [d^+ x a^{(2)}]^{(0)} : \ , \tag{2.16}
\]

In general, the sum over \( J \) in Eq. (2.17) would include \( J=1,2,3,4 \). However, for the case of a \( j=3/2 \) single particle level, the Wigner coefficient in Eq. (2.17) vanishes for \( J=4 \) and \( J=2 \). The vanishing of the \( J=4 \) coefficient is simply due to angular momentum algebra, whereas the vanishing of the \( J=2 \) coefficient is accidental and it is, to a large extent, the reason behind the closure of the Spin(6) algebra. The only surviving term in Eq. (2.17) which contains operators which are not
generators of $SO(B)(6)$ and $SU(F)(4)$ is then the $J=0$ term. However, this term is of the same form as the first term in Eq. (2.5). Inserting the appropriate value of the Wigner coefficient, one finds that the two terms cancel if $A=-A/\sqrt{5}$. Under these conditions, a Spin(6) symmetry also emerges from the boson-fermion Hamiltonian of Ref. [IS 79].
III. The Spin(6) Limit of the Interacting Boson-Fermion Model

In this chapter, consequences of the Spin(6) symmetry will be discussed. In particular, we will derive analytic formulas for energies, electromagnetic (E0, M1, E2) transition rates, static moments and intensities of one and two nucleon transfer reactions. Whenever possible, the underlying Spin(6) symmetry is exploited, which in turn reduces the calculational effort considerably. For example, in Section A, the energy formula is derived through the eigenvalues of the Casimir operators without any diagonalization. In Section B, wavefunctions are obtained again making use of the Casimir operators. The fact that transition operators can be written in terms of the group generators gives rise to selection rules, which greatly reduce the number of allowed transitions. We will make use of the selection rules in Sections C and D when calculating the electromagnetic transition rates and nucleon transfer intensities. A final consequence of the symmetry is relations between matrix elements of operators, which will be used in deriving the sum rules for the E2 transition rates.
- 21 -

A. Energy Spectra

One of the interesting aspects of dynamical symmetries is that the energy spectrum is obtained without any computational effort, such as solving a differential equation or a numerical diagonalization of the Hamiltonian. One only needs to classify the states according to the irreducible representations (ir. reps.) of the symmetry groups under consideration. Once a complete basis is constructed, energies are simply given by the eigenvalues of the Casimir operators of these symmetry groups whose solution for all classical groups is already known [PP 67, NR 77]. So we start with the labeling problem of the Spin(6) states. Construction of a complete basis amounts to finding all the physically allowed ir. reps. of the groups in the group chain (2.10). In general, the following labels are needed in order to uniquely characterize the states;

(a) The number of bosons, \([N]\), which labels the totally symmetric ir. reps. of \(U^B(6)\).

(b) The number of fermions, \([M]\), which labels the totally antisymmetric ir. reps. of \(U^F(4)\).

(c) A label, \(\Sigma\), which characterizes the totally symmetric ir. reps. of \(SO^B(6)\), where

\[
\Sigma = N, N-2, \ldots, 1 \quad \text{or} \quad 0 \quad ; \quad (N \text{ odd or even}) . \tag{3.1}
\]

(d) Three labels, \((\sigma_1, \sigma_2, \sigma_3)\), which characterize the ir. reps. of
Spin(6). The values of \((\sigma_1, \sigma_2, \sigma_3)\) contained in a given representation of \(U(B)^{(6)} \otimes U(F)^{(4)}\) are given in Ref. [BBI 81]. Here, we consider the cases of \(N\) bosons and \(M=0\) or \(M=1\) fermions which are relevant for the Spin(6) symmetry in even-even nuclei (zero quasiparticle states) and odd-even nuclei (one quasiparticle states). For these two cases the values of \((\sigma_1, \sigma_2, \sigma_3)\) are given by [Ia 80a]

i) \(N, M=0\) (zero quasiparticle states)

\[\sigma_3 = \sigma_2 = 0; \quad \sigma_1 = \Sigma = N, N-2, \ldots, 1 \text{ or } 0; \quad (N \text{ odd or even}) \quad . (3.2)\]

ii) \(N, M=1\) (one quasiparticle states)

\[\sigma_3 = \frac{1}{2}; \quad \sigma_2 = \frac{1}{2}; \quad \sigma_1 = N+ \frac{1}{2}, N- \frac{3}{2}, \ldots, \frac{3}{2} \text{ or } \frac{1}{2} \quad . (3.3)\]

\[\sigma_3 = -\frac{1}{2}; \quad \sigma_2 = \frac{1}{2}; \quad \sigma_1 = N- \frac{1}{2}, N- \frac{5}{2}, \ldots, \frac{3}{2} \text{ or } \frac{1}{2} \quad . (3.3)\]

Since the two representations, \((\sigma_1,\frac{1}{2},\frac{1}{2})\) and \((\sigma_1,\frac{1}{2},-\frac{1}{2})\) are complex conjugates of each other, we will not distinguish between the two and label them by \((\sigma_1, \sigma_2, |\sigma_3|)\). Then Eq. (3.3) is replaced by

\[\sigma_3 = \sigma_2 = \frac{1}{2}; \quad \sigma_1 = N+ \frac{1}{2}, N- \frac{1}{2}, N- \frac{3}{2}, \ldots, \frac{1}{2} \quad . (3.4)\]

(e) Two labels, \((\tau_1, \tau_2)\) which characterize the ir. reps. of Spin(5). The values of \((\tau_1, \tau_2)\) contained in each Spin(6) representation, \((\sigma_1, \sigma_2, \sigma_3)\) are given by

i) \(N, M=0\)

\[\tau_2 = 0; \quad \tau_1 = \sigma_1, \sigma_1-1, \ldots, 0 \quad . (3.5)\]
ii) $N, M=1$

\[ \tau_2 = \frac{1}{2}; \quad \tau_1 = \sigma_1, \sigma_1 - 1, \ldots, \frac{1}{2} \quad . \tag{3.6} \]

(f) A quantum number $v_\Delta$ which takes into account the fact that the step from $\text{Spin}(5)$ to $\text{Spin}(3)$ is not fully reducible, where

\[ v_\Delta = 0, 1, 2, \ldots \quad , \quad (N, M=0) \quad , \tag{3.7} \]

\[ v_\Delta = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \ldots \quad , \quad (N, M=1) \quad . \]

For $M=1$, integer values of $v_\Delta$ apply to the representations with $\sigma_3 = \frac{1}{2}$ and the half integer values to $\sigma_3 = -\frac{1}{2}$.

(g) The angular momentum, $J$, and its projection, $M_J$, which characterize the ir. reps. of $\text{Spin}(3)$ and $\text{Spin}(2)$ respectively. The values of $J$ contained in each $\text{Spin}(5)$ representation, $(\tau_1, \tau_2)$, are given by

i) $N, M=0$ [AI 79]

\[ J = [2\tau_1 - 6v_\Delta], [2\tau_1 - 6v_\Delta - 2], [2\tau_1 - 6v_\Delta - 3], \ldots , \tag{3.8} \]

\[ [\tau_1 - 3v_\Delta + 1], [\tau_1 - 3v_\Delta] \quad . \]

ii) $N, M=1$ [Ia 80a]

\[ J = [2(\tau_1 - \tau_2) - 6v_\Delta + \frac{3}{2}], [2(\tau_1 - \tau_2) - 6v_\Delta + \frac{1}{2}], \ldots , \tag{3.9} \]

\[ [(\tau_1 - \tau_2) - 3v_\Delta - \frac{1}{4} [1 - (-)^{2v_\Delta}] + \frac{3}{2}] \quad . \]

Angular momentum content of the lowest $\text{Spin}(5)$ representations, as given
by the rules (3.8) and (3.9) is shown in Table I. The notation used for
the representations of the spinor groups is that of Murnaghan [Mu 34].

Having classified the states uniquely in terms of the complete set
of labels, \([N], \{M\}, \Sigma, (\sigma_1, \sigma_2, \sigma_3), (\tau_1, \tau_2), \nu, J, M, J\) we can now
proceed with the diagonalization of the Spin(6) Hamiltonian, given by

\[ H = -A \mathcal{C}_6 + B \mathcal{C}_5 + C \mathcal{C}_3 \]  \hspace{1cm} (3.10)

where \(\mathcal{C}_6\), \(\mathcal{C}_5\) and \(\mathcal{C}_3\) are the quadratic Casimir operators Spin(6),
Spin(5) and Spin(3) respectively and can be written in terms of the
group generators as

\[ \mathcal{C}_6 = \frac{1}{4} \left[ G^{(1)}G^{(1)} + \frac{1}{2} G^{(2)}G^{(2)} + G^{(3)}G^{(3)} \right] \ , \]

\[ \mathcal{C}_5 = \frac{1}{6} \left[ G^{(1)}G^{(1)} + G^{(3)}G^{(3)} \right] \ , \]

\[ \mathcal{C}_3 = 10 G^{(1)}G^{(1)} \ . \]  \hspace{1cm} (3.11)

Here the operators, \(G^{(1)}\), \(G^{(2)}\) and \(G^{(3)}\) are the generators of the
Spin(6) group, defined in Eq. (2.12). The eigenvalues of the Casimir
operators, in the representations \((\sigma_1, \sigma_2, \sigma_3), (\tau_1, \tau_2), J\), are given by

\[ <\mathcal{C}_6> = \frac{1}{4} \left[ \sigma_1(\sigma_1 + 4) + \sigma_2(\sigma_2 + 2) + \sigma_3^2 \right] \ , \]

\[ <\mathcal{C}_5> = \frac{1}{6} \left[ \tau_1(\tau_1 + 3) + \tau_2(\tau_2 + 1) \right] \ , \]

\[ <\mathcal{C}_3> = [J(J + 1)] \ , \] \hspace{1cm} (3.12)
Thus, the Hamiltonian (3.10) which is diagonal in the group chain (2.10) has eigenvalues

\[ E(N, M, \Sigma, (\sigma_1, \sigma_2, \sigma_3), (\tau_1, \tau_2), \nu_\Delta, J, M_J) \]

\[ = -\frac{A}{4} [\sigma_1(\sigma_1 + 4) + \sigma_2(\sigma_2 + 2) + \sigma_3^2] + \frac{B}{6} [\tau_1(\tau_1 + 3) + \tau_2(\tau_2 + 1)] + C J(J+1). \]  

(3.13)

For \( M=0 \), Eq. (3.13) reduces to

\[ E(N, \Sigma = \sigma_1, \sigma_1, \tau_1, \nu_\Delta, J, M_J) = -\frac{A}{4} \sigma_1(\sigma_1 + 4) + \frac{B}{6} \tau_1(\tau_1 + 3) + C J(J+1), \]  

(3.14)

which is the same as the Eq. (3.7) of Ref. [AI 79] except for a constant term \((A/4) N(N+4)\).

In general, any combination of the linear and quadratic Casimir operators of \( U(B)(6) \), \( U(F)(4) \) and \( SO(B)(6) \) can be added to the Hamiltonian (3.10), since these are also diagonal in the group chain (2.10). The complete eigenvalue expression then becomes

\[ E(N, M, \Sigma, (\sigma_1, \sigma_2, \sigma_3), (\tau_1, \tau_2), \nu_\Delta, J, M_J) \]

\[ = E_0 + E_1 N + E_2 N^2 + E_3 M + E_4 M^2 + E_5 MN - \frac{A}{4} \Sigma(\Sigma+4) \]

\[ - \frac{A}{4} [\sigma_1(\sigma_1 + 4) + \sigma_2(\sigma_2 + 2) + \sigma_3^2] + \frac{B}{6} [\tau_1(\tau_1 + 3) + \tau_2(\tau_2 + 1)] + C J(J+1). \]  

(3.15)

Only the last four terms contribute to excitation energies, the rest being related to the binding energy of nucleus. The structure of the spectrum for even-even \((M=0)\) and odd-even \((M=1)\) nuclei are shown in
In concluding this section, we note the very specific relation among the boson-boson, boson-fermion and fermion-fermion interaction terms in the Hamiltonian \((3.10)\) which is dictated by the Spin(6) symmetry. This is in part due to the classification of states through the group chain \((2.10)\), which determines how the boson and fermion operators are combined, Eq. \((2.12)\), but mainly because the Hamiltonian is written in terms of the Casimir operators, Eqs. \((3.10)-(3.11)\). For example, in order to have the Spin(6) symmetry the quadrupole interaction term must be in the form

\[
g^{(2)} g^{(2)} = B^{(2)} B^{(2)} + 2 B^{(2)} A^{(2)} + A^{(2)} A^{(2)} ,
\]

which corresponds to boson-boson, boson-fermion and fermion-fermion interactions in the ratio 1:2:1, as schematically shown in Fig. 5.
In a group theoretical description of a physical system, energy eigenvalues are obtained without actually solving the eigenvalue problem. For all other properties one can still obtain analytic formulas, but now explicit expressions for the wave functions are required. A direct analytical solution of the eigenvalue problem often proves to be far from trivial and one has to resort to other means for construction of the wave functions. One common method is building up the excited states by repeated application of creation operators on the ground state of the system. This method is especially suitable if one is interested only in the low lying excitations of the system. Another method is to expand the wave functions into wave functions of another system whose complete solution is known. Using group theoretical techniques, one can obtain algebraic relations between the expansion coefficients, thus reducing the problem to solving a set of algebraic equations. This latter method will be used in the present problem.

For even-even nuclei in the SO(6) limit, the wave functions have already been constructed in Ref. [AI 79]. Since in the Spin(6) limit the odd fermion is assumed to sit in the fundamental spinor representation of SU(4), it is natural to expand the Spin(6) wave functions into wave functions of the product group \( \text{SO}^{(B)}(6) \otimes \text{SU}^{(F)}(4) \). In group theoretical terms, because of the isomorphism, \( \text{SO}(6) \approx \text{SU}(4) \approx \text{Spin}(6) \), this amounts to construction of spinor representations of SU(4) as Kronecker product of its tensor and fundamental spinor representations. In the same terminology, the
expansion coefficients correspond to the isoscalar factors [Wy 74] for the group chain SU(4) \supset Sp(4) \supset SU(2).

In this section we will first construct the expansion coefficients for the Spin(6) representation \( \sigma_1 = N + 1/2, \sigma_2 = \sigma_3 = 1/2 \), which is the lowest in energy and hence experimentally accessible. Next we will derive a formula connecting the expansion coefficients of \( \sigma_1 = N + 1/2, \sigma_2 = \sigma_3 = 1/2 \) and \( \sigma_1 = N - 1/2, \sigma_2 = \sigma_3 = 1/2 \) representations, thereby obtain the isoscalar factors of the next Spin(6) representation \( \sigma_1 = N - 1/2, \sigma_2 = \sigma_3 = 1/2 \). These will be of use in calculation of one nucleon transfer intensities. Finally, we will construct the expansion coefficients for coupling of two \( j = 3/2 \) fermions to the boson SO(6) core, which corresponds to two quasiparticle states in even-even nuclei. These factors are needed in the context of supersymmetries where nuclei with \( M = 0, 1, \) and \( 2 \) quasiparticles are considered as members of the same multiplet.

We start with the expansion

\[
| \left[ N + \frac{1}{2}, \tau_1, J \right] > = \sum \xi_{N', \tau', L', \tau_1} \left| \left[ N', \tau', L', \tau_1 \right] > \right| N, \tau_1, L > \times \left| \frac{1}{2}, \frac{1}{2}, \frac{3}{2} > \right., \quad (3.17)
\]

where \( \left[ N + \frac{1}{2}, \tau_1, J \right] > \) denotes the Spin(6) wave function

\[
\left[ N \right], M = 1, (\sigma_1 = N + \frac{1}{2}, \sigma_2 = \sigma_3 = \frac{1}{2}), (\tau_1, \tau_2 = \frac{1}{2}), J, M_J >,
\]

\( \left[ N, \tau', L' \right] > \) denotes the SO(6) wave function

\[
\left[ N \right], (\sigma_1 = N, \sigma_2 = \sigma_3 = 0), (\tau_1 = \tau', \tau_2 = 0), L', M' >
\]

and \( \left| \frac{1}{2}, \frac{1}{2}, \frac{3}{2} > \right. \) is the fundamental spinor representation of SU(4)

\[
\left| (\sigma_1 = \sigma_2 = \sigma_3 = \frac{1}{2}), (\tau_1 = \tau_2 = \frac{1}{2}), J = \frac{3}{2}, M >. \right.
\]
interested in reduced matrix elements of operators, we have not written magnetic quantum numbers in Eq. (3.17). The $\xi'$s are the expansion coefficients and the sum goes over $\tau'=\tau_1+\frac{1}{2}$, $L'=J+\frac{1}{2}$, $J+\frac{3}{2}$.

The technique used in such calculations consists of evaluating the matrix elements of the Casimir operators of the underlying symmetry group in both bases. These matrix elements will be diagonal in the basis of the symmetry group, but not in the expansion wave functions, thus resulting in the algebraic relations between the expansion coefficients. Since matrix elements of the quadrupole operator are known for $\text{SO}^B(6)$ wave functions, we consider the operator $G^{(2)} \cdot G^{(2)}$ which by virtue of Eqs. (2.12) and (3.11) can be written as

$$G^{(2)} \cdot G^{(2)} = A^{(2)} \cdot A^{(2)} + B^{(2)} \cdot B^{(2)} + 2 A^{(2)} \cdot B^{(2)}$$

(3.18)

$$= 4 C_6 - 6 C_5 .$$

Taking the matrix elements of this operator between Eq. (3.17) and the state $|N, \tau_1-\frac{1}{2}, L> \times |\frac{1}{2}, \frac{1}{2}, \frac{3}{2}>$, and making use of Eq. (3.12), we have

$$(N + \frac{1}{2}) (N + \frac{9}{2}) + \frac{3}{2} - \tau_1 (\tau_1 + 3) - \frac{3}{4} \xi^N, \tau_1 - \frac{1}{2}, L$$

$$= \sum_{\tau', L'} \left[ (N(N+4) - \tau'(\tau'+3) + \frac{5}{4}) \delta_{\tau_1-\frac{1}{2}, \tau'} \delta_{L,L'} \right]$$

(3.19)

$$-2 \sqrt{5} (-) \begin{pmatrix} L & \frac{3}{2} & J \\ \frac{3}{2} & L' & 2 \end{pmatrix} <N, \tau_1-\frac{1}{2}, L \parallel B^{(2)} \parallel N, \tau', L'> \xi^N, \tau', L' \left[ \begin{array}{c} N+\frac{1}{2}, \tau_1, J \end{array} \right] .$$
Rearranging terms we obtain

\[
\xi_{N, \tau_1 - \frac{1}{2}, L} = \frac{2\sqrt{5}}{(N-\tau_1 + \frac{1}{2}) L'} \sum_{J} \left\{ \begin{array}{c} L \frac{3}{2} J \\ \frac{3}{2} L' 2 \end{array} \right\} (-)^{J + L'} - \frac{3}{2} \right\}
\]

\[
\times <N, \tau_1 - \frac{1}{2}, L || B^{(2)} || N, \tau_1 + \frac{1}{2}, L'> \xi_{N, \tau_1 + \frac{1}{2}, L'}
\]

(3.20)

The matrix elements of the operator \( B^{(2)} \) between the the \( SO^B(6) \) states [AI 79] are collected in Appendix B. A second relation can be obtained by taking the matrix element of \( G^{(2)}, G^{(2)} \) between Eq. (3.17) and the state \( |N, \tau_1 + \frac{1}{2}, L> \times |\frac{1}{2}, \frac{1}{2}, \frac{3}{2}> \). This gives

\[
\xi_{N, \tau_1 + \frac{1}{2}, L} = \frac{2\sqrt{5}}{(N+\tau_1 + \frac{7}{2}) L'} \sum_{J} (-)^{J + L'} - \frac{3}{2} \right\}
\]

\[
\times <N, \tau_1 + \frac{1}{2}, L || B^{(2)} || N, \tau_1 - \frac{1}{2}, L'> \xi_{N, \tau_1 - \frac{1}{2}, L'}
\]

(3.21)

Eqs. (3.20)-(3.21) provide a set of eight linear homogenous equations in the eight isoscalar factors, and together with the normalization condition

\[
\sum_{\tau', L'} \left( \xi_{N+\frac{1}{2}, \tau_1, J} \right)^2 = 1
\]

(3.22)
The above system of equations can be further simplified by making use of Racah's factorization lemma [Wy 74], which states for the case considered here

\[
N, \tau', L' = N, \tau' \times \xi, L', \xi, N, \tau', L',
\]

(3.23)

where \(\xi, N, \tau', L'\) and \(\xi, \tau, J\) denote the isoscalar factors for the \(SU(4) \supset Sp(4)\) and \(Sp(4) \supset SU(2)\) group chains respectively. The former can be determined as

\[
\xi, N, \tau, L' = - \left[ \frac{N+\tau+\frac{7}{2}}{2N+4} \right]^{1/2}, \quad \xi, N, \tau+\frac{1}{2}, L' = \left[ \frac{N-\tau+\frac{1}{2}}{2N+4} \right]^{1/2}
\]

(3.24)

Using Eqs. (3.23)-(3.24) in (3.22) provides two separate normalization conditions

\[
\sum_{L'} \left( \xi, N, \tau', L', \xi, N, \tau, L' \right)^2 = \frac{N+\tau+\frac{7}{2}}{2N+4}, \quad \tau' = \tau_1 - \frac{1}{2}
\]

(3.25)

\[
\sum_{L'} \left( \xi, N, \tau', L', \xi, N, \tau+\frac{1}{2}, L' \right)^2 = \frac{N-\tau+\frac{1}{2}}{2N+4}, \quad \tau' = \tau_1 + \frac{1}{2}
\]

which reduces the system of eight equations into two systems of four equations.

As an example, we calculate the isoscalar factors for the ground
state band, |N+ 1/2, τ1, J=2τ1+ 1/2>. From the sum, τ' = τ1 ± 1/2, L' = J ± 1/2, J ± 3/2
only three SO(6) states contribute to the expansion (marked by x below)

τ' = τ1 ± 1/2. x 2τ1+1 x 2τ1-1 2τ1-2

SO(6) states

Using the appropriate values of the 6-j symbols and the reduced matrix elements of B(2), Appendix B, in Eq. (3.21) gives

\[
\xi_{N+ 1/2, \tau_1, 2\tau_1+ 1/2} = \left[ \frac{(N-\tau_1+ 1/2)(2\tau_1+ 2)(4\tau_1+ 3)}{(N+\tau_1+ 1/2)(2\tau_1+ 4)(4\tau_1+ 1)} \right]^{1/2} \xi_{N, \tau_1- 1/2, 2\tau_1- 1/2}
\]

(3.26)

\[
\xi_{N+ 1/2, \tau_1, 2\tau_1- 1/2} = \left[ \frac{4(N-\tau_1+ 1/2)(2\tau_1+ 2)(4\tau_1+ 3)}{(N+\tau_1+ 1/2)(2\tau_1+ 4)(4\tau_1+ 1)} \right]^{1/2} \xi_{N, \tau_1- 1/2, 2\tau_1- 1/2}
\]

Substituting above equations in the normalization condition (3.25) yields

\[
\xi_{N+ 1/2, \tau_1, 2\tau_1+ 1/2} = -\left[ \frac{N+\tau_1+ 7/2}{2N+4} \right]^{1/2} \xi_{N, \tau_1- 1/2, 2\tau_1- 1/2}
\]

(3.27)

Finally using Eq. (3.27) back in (3.21) gives for the remaining isoscalar factors

\[
\xi_{N+ 1/2, \tau_1, 2\tau_1+ 1/2} = \left[ \frac{(N-\tau_1+ 1/2)(2\tau_1+ 2)(4\tau_1+ 3)}{(2N+4)(2\tau_1+ 4)(4\tau_1+ 1)} \right]^{1/2}
\]

(3.28)

\[
\xi_{N+ 1/2, \tau_1, 2\tau_1- 1/2} = -\left[ \frac{(N-\tau_1+ 1/2)(4\tau_1- 2)}{(2N+4)(2\tau_1+ 4)(4\tau_1+ 1)} \right]^{1/2}
\]
In Table II we list a selected number of isoscalar factors obtained from Eqs. (3.20)-(3.21), to be used in the following sections in the calculation of matrix elements of various operators.

Isoscalar factors of the next relevant Spin(6) representation $\sigma_1=N-1/2$, $\sigma_2=\sigma_3=1/2$, can be obtained from orthogonality of the states. Consider the scalar product

$$<N-\frac{1}{2}, \tau_1, J | N+\frac{1}{2}, \tau_1, J> = \sum_{\tau L} N-\frac{1}{2}, \tau_1, J \sum_{\tau' L'} N+\frac{1}{2}, \tau_1, J \xi_{\tau L}^N, \tau, L \xi_{\tau' L'}^N, \tau', L' \delta_{\tau, \tau'} \delta_{L, L'},$$

(3.29)

which is zero, since different Spin(6) representations are orthogonal.

Using Racah's factorization lemma in Eq. (3.29) gives

$$\sum_{\tau} \xi_{\tau L}^N, \tau \xi_{N+\frac{1}{2}, \tau_1, J}^N, \tau \times \sum_{\tau' L'} \xi_{\tau' L'}^N, \tau', \tau = 0 \tag{3.30}$$

The second sum in Eq. (3.30) is equal to one by normalization of the Sp(4)$\supset$SU(2) factors. Together with the normalization condition

$$\sum_{\tau} \left( \frac{\xi_{N-\frac{1}{2}, \tau_1}^N, \tau}{\sum_{\tau} \xi_{N-\frac{1}{2}, \tau_1}^N, \tau} \right)^2 = 1 \tag{3.31}$$

Eq. (3.30) can be solved to yield

$$\xi_{N-\frac{1}{2}, \tau_1}^N, \tau = \left[ \frac{N-\tau_1 + \frac{1}{2}}{2N+4} \right]^{1/2}$$

$$\xi_{N+\frac{1}{2}, \tau_1}^N, \tau = \left[ \frac{N+\tau_1 + \frac{1}{2}}{2N+4} \right]^{1/2} \tag{3.32}$$
Combining Eqs. (3.24) and (3.32) gives the desired relationship between the isoscalar factors of the Spin(6) representations, $\sigma_1 = \frac{N+1}{2}$ and $\sigma_1 = \frac{N-1}{2}$.

\begin{align*}
\xi_{\frac{N}{2}, \frac{1}{2}, \frac{1}{2}, L} &= -\left[\frac{N-\frac{1}{2}}{N+\frac{1}{2}+\frac{7}{2}}\right]^{1/2} \xi_{\frac{N}{2}, \frac{1}{2}, \frac{1}{2}, L} \\
\xi_{\frac{N}{2}, \frac{1}{2}, \frac{1}{2}, L} &= \left[\frac{N+\frac{1}{2}+\frac{7}{2}}{N-\frac{1}{2}+\frac{1}{2}}\right]^{1/2} \xi_{\frac{N}{2}, \frac{1}{2}, \frac{1}{2}, L}
\end{align*}

(3.33)

Lastly, we construct the expansion coefficients for the two quasiparticle states in even-even nuclei described by an SO(6) symmetry. Both fermions are assumed to belong to the fundamental representation of SU(4), $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$. Product of two such representations is given by the Young Tableau

\[ \square \otimes \square = \square \otimes \square \]

which corresponds to

\[ [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}] \otimes [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}] = [1, 1, 1] \otimes [1, 0, 0] . \]

Since the fermions must be in the antisymmetric representations, only the representation $[1,0,0]$ is possible. Thus, the two quasiparticle states can be written as

\[ |\{2\}, \sigma_1 = 1, \tau, L \rangle = \frac{1}{2} \left| \frac{1}{2} \frac{1}{2} \frac{3}{2} \right\> \times \frac{1}{2} \left| \frac{1}{2} \frac{1}{2} \frac{3}{2} \right\> . \]
where \(|{2}, \sigma_1 = 1, \tau, L \rangle\) denotes the wave function

\(|N=0, M=2, (\sigma_1 = 1, \sigma_2 = \sigma_3 = 0), (\tau_1 = \tau, \tau_2 = 0), L, M \rangle\), and \(|\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rangle\) is the fundamental spinor representation of \(SU(4)\). The labels \(\tau\) and \(L\) take the values \(\tau=0, L=0\) and \(\tau=1, L=2\).

The two quasiparticle states in the \(SO(6)\) limit can be expanded into wave functions of the product group \(SO^B(6) \otimes SU^F(4)\), with the fermion state given by Eq. (3.34)

\[
\left| [N], \{2\}, N+1, \tau, L \right\rangle = \sum_{\tau', L'} \delta_{N+1, \tau, L} \left| \left[ [N], N, \tau', L' \right\rangle \times \left[ \{2\}, 1, \tau'', L'' \right\rangle \right.
\]

(3.35)

where the square brackets denote the boson number and curly brackets the fermion number, and the last three labels correspond to \(\sigma_1, \tau_1\) and \(J\) in order. The sum is over \(\tau = \tau, \tau \pm 1, \tau', L' = L, L \pm 1, L \pm 2\). The same technique, used in construction of \(\xi\) can be used here. So we take the matrix elements of \(G^B(2) \otimes G^F(2)\) between Eq. (3.35) and the state, \(|[N], N, \tau+1, L \rangle \times |\{2\}, 1, 1, 2 \rangle\). Making use of Eq. (3.12) gives

\[
\begin{align*}
\left[ (N+1)(N+5) - \tau(\tau+3) \right] & \delta_{N+1, \tau, L} \\
= \sum_{\tau', L'} & \left[ N(N+4) - \tau'((\tau'+3) + 1) \right] \delta_{\tau', \tau+1, L', L_1} \\
- 2 \sqrt{5} & (-)^{L+L'} \left\{ \begin{array}{ccc} L_1 & 2 & L \\ 0 & L' & 2 \end{array} \right\} \langle N, \tau+1, L_1 \parallel B^2(2) \parallel N, \tau', L' \rangle \delta_{L, L'} \delta_{L_1, L'} \delta_{N+1, \tau, L}
\end{align*}
\]

(3.36)
Rearranging and substituting value of the $6\text{-}j$ symbol we obtain

$$\xi_{N+1, \tau, L}^{N, \tau+1, L_1} = \frac{(-1)^{L+L_1+1}}{\sqrt{2L+1} (N+\tau+4)} \langle N, \tau+1, L_1 \parallel B(2) \parallel N, \tau, L \rangle \xi_{N+1, \tau, L}^{N, \tau, L}. \quad (3.37)$$

Similarly, for the state, $\{N, \tau, L \parallel \{2\}, 1, 1, 2\}$ we have

$$\xi_{N+1, \tau, L}^{N, \tau-1, L_1} = \frac{(-1)^{L+L_1+1}}{\sqrt{2L+1} (N-\tau+1)} \langle N, \tau-1, L_1 \parallel B(2) \parallel N, \tau, L \rangle \xi_{N+1, \tau, L}^{N, \tau, L}. \quad (3.38)$$

and for $\{N, \tau, L \parallel \{2\}, 1, 0, 0\}$

$$\xi_{N+1, \tau, L}^{N, \tau, L} = \frac{-1}{\sqrt{2L+1} N} \sum_{\tau'L'} \langle N, \tau, L \parallel B(2) \parallel N, \tau', L' \rangle \xi_{N+1, \tau, L}^{N, \tau', L'}. \quad (3.39)$$

Eqs. (3.37)-(3.39) provide 11 linear homogenous equations in the 11 isoscalar factors and together with the normalization condition, completely determine the isoscalar factors.

Again use of Racah's factorization lemma simplifies the solution.

Notice that

$$\xi_{N+1, \tau, L}^{N, \tau, L} = \xi_{N+1, \tau, L}^{N, \tau} \times \xi_{N+1, \tau, L}^{\tau, L} = \xi_{N+1, \tau}^{N, \tau}. \quad (3.40)$$

So one only needs the factor, $\xi_{N+1, \tau}^{N, \tau}$, which can be easily determined from the ground state band. All the rest of the factors are obtained by substituting $\xi_{N+1, \tau}^{N, \tau}$ in Eqs. (3.37)-(3.38). For future reference, we list in Table III the isoscalar factors for the lowest state band, $\{N, \{2\}, N+1, \tau, L=2\tau\}$. 
Whenever a system is described by a dynamical symmetry the operators corresponding to the observables of the system can be written in terms of the generators of the underlying symmetry group. Since generators of a group can connect only a limited number of its representations, this gives rise to selection rules in transitions, which in turn provide a quick test on the goodness of the proposed symmetry scheme. Further, if one assumes one-body boson operators, all transitions rates can be predicted with a single normalization parameter.

In this section, we will first calculate the electric quadrupole, $E_2$, transition rates and static moments, fully exploiting the Spin(6) symmetry. For the magnetic dipole, $M_1$, transition rates, however, one has to assume a symmetry breaking between the boson and fermion operators, because the corresponding group generator, $G^{(1)}$, is proportional to the angular momentum operator which is diagonal in the Spin(6) states. We conclude this section with derivation of isomer and isotope shifts which are related to the properties of the electric monopole, $E_0$, operator.
C.1. E2 Transitions

The most general form of a one-body E2 transition operator is given by

\[ T(E2) = T_B^{(E2)} + T_F^{(E2)} , \]  

(3.41)

where

\[ T_B^{(E2)} = \tilde{q}_2 \left[ d^+ \times s + s^+ \times \tilde{d} \right]^{(2)} + q'_2 \left[ d^+ \times \tilde{d} \right]^{(2)} , \]  

(3.42)

\[ T_F^{(E2)} = \sum_{jj} t_{jj}^{(2)} \left[ a_j^+ \times \tilde{a}_j \right]^{(2)} . \]

Assumption of SO(6) and SU(4) symmetries reduces the E2 operator, Eq. (3.41) to

\[ T(E2) = \tilde{q}_2 B^{(2)} + t^{(2)} A^{(2)} , \]  

(3.43)

with \( A^{(2)} \) and \( B^{(2)} \) defined in Eq. (2.13). Spin(6) limit is obtained if one further assumes that \( \tilde{q}_2 = t^{(2)} \), because then

\[ T(E2) = \tilde{q}_2 G^{(2)} , \]  

(3.44)

where \( G^{(2)} \) is a generator of the Spin(6) group. Since \( G^{(2)} \) can not
connect different Spin(6) representations, \( T^{(E2)} \) has the selection rules, \( \Delta \sigma_1 = \Delta \sigma_2 = \Delta \sigma_3 = 0 \). Further, \( G^{(2)} \) has tensorial character \((1, 0)\) with respect to Spin(5), so it must also satisfy, \( \Delta \tau_1 = 0, \pm 1, \Delta \tau_2 = 0 \). Comparing these selection rules with the corresponding ones in the even-even nuclei, \( \Delta \tau_1 = \pm 1, \Delta \tau_2 = 0 \), shows that quadrupole moments of even-even nuclei vanish in the SO(6) limit since they correspond to \( \Delta \tau_1 = 0 \), but they do not in odd-even nuclei because \( \Delta \tau_1 = 0 \) matrix elements are allowed in the Spin(6) limit.

Reduced matrix elements of \( G^{(2)} \) between states belonging to the Spin(6) representation \( \sigma_1 = N + 1/2, \sigma_2 = \sigma_3 = 1/2 \) can be evaluated by expanding the wave functions as in Section B

\[
\langle N + \frac{1}{2}, \tau_1, J \parallel G^{(2)} \parallel N + \frac{1}{2}, \tau_1', J' \rangle
\]

\[
= \sqrt{(2J + 1) (2J' + 1)} \sum_{\tau L} \xi_{N, \tau, L}^{N + \frac{1}{2}, \tau_1, J} \xi_{N', \tau', L'}^{N + \frac{1}{2}, \tau_1', J'} (-)^L + J' + \frac{3}{2} \]

\[
x \left[ \begin{array}{c}
L, J \parallel \frac{3}{2} \\
J', L', \frac{3}{2}
\end{array} \right] \langle N, \tau, L \parallel B^{(2)} \parallel N, \tau', L' \rangle + (-)^{J-J'} \left[ \begin{array}{c}
\frac{3}{2} \\
J', \frac{3}{2}
\end{array} \right]
\]

\[
x \langle \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \parallel A^{(2)} \parallel \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rangle \delta_{\tau, \tau'} \delta_{L, L'} .
\]

Inserting the appropriate values of the isoscalar factors, Table II, 6-j symbols and the reduced matrix elements of \( B^{(2)} \), Appendix B, we obtain all matrix elements of interest. From these we can derive \( B(E2) \) values using
Some $B(E2)$ values which have practical applications are listed in Table IV. In this table, we also list the matrix elements

$$Q(N+\frac{1}{2}, \tau_1, J) = \langle N+\frac{1}{2}, \tau_1, J \parallel T(E2) \parallel N+\frac{1}{2}, \tau_1, J \rangle,$$  

(3.47)

from which the quadrupole moments can be calculated using

$$Q_J = \sqrt{\frac{16\pi}{5}} \sqrt{\frac{J(2J-1)}{(J+1)(2J+1)(2J+3)}} Q(N+\frac{1}{2}, \tau_1, J).$$  

(3.48)

A summary of the $B(E2)$ values for the low lying levels, obtained from Table IV, is shown in Fig. 6.

Yet another consequence of the symmetry assumption is the relations between the matrix elements of operators. These relations can be stated as sum rules in transition rates and used as a consistency check on the calculations. As an illustration, we recall the Wigner-Eckart theorem which states for the $SO(3)\supset SO(2)$ group chain

$$\langle J', M' \parallel T^{(k)} \parallel J M \rangle = (-)^{J'-M'} \left\{ \begin{array}{cc} J' & k \ J \\ -M' & \kappa \ M \end{array} \right\} \langle J' \parallel T^{(k)} \parallel J \rangle.$$  

(3.49)

Eq. (3.49) can be used to obtain the familiar sum rule

$$\sum_{M'\kappa} \left| \langle J', M' \parallel T^{(k)} \parallel J M \rangle \right|^2 = \frac{1}{2J+1} \left| \langle J' \parallel T^{(k)} \parallel J \rangle \right|^2.$$  

(3.50)
Generalizing the Wigner-Eckart theorem to the group chain $\text{Spin}(5) \supset \text{Spin}(3) \supset \text{Spin}(2)$ gives

$$< N + \frac{1}{2}, \tau_1', \nu_\Delta', J', M', | T^{(k)}(\Lambda) | N + \frac{1}{2}, \tau_1, \nu_\Delta, J, M >$$

$$= \sum (-)^{J'-M'} \left( \begin{array}{ccc} J' & k & J \\ -M' & k & M \end{array} \right) < \nu_\Delta' \tau_1' J' | \Lambda k \tau J > < N + \frac{1}{2}, \tau_1', \nu_\Delta' | T(\Lambda) | N + \frac{1}{2}, \tau_1, \nu_\Delta >,$$

(3.51)

where $T(\Lambda)$ is a tensor operator which transforms under Spin(5) according to the representation $\Lambda = (\tau_1', \tau_2')$, and double barred matrix elements denote that it is reduced with respect to Spin(5). The quantities, $< \nu_\Delta \tau_1 J | \Lambda k \tau_1 J >$ are the coupling (Clebsch-Gordan) coefficients of the Spin(5) group and have the usual orthogonality relations

$$\sum_k < \nu_\Delta' \tau_1' J' | \Lambda k \tau_1 J >^* < \Lambda k \tau_1 J | \nu'' \tau_1'' J'' > = \delta_{\nu_\Delta' \nu_\Delta} \delta_{\tau_1' \tau_1''} \delta_{J' J''},$$

(3.52)

and

$$\sum_{\nu_\Delta} < \Lambda k \tau_1 J | \nu_\Delta' \tau_1' J' >^* < \nu_\Delta' \tau_1' J' | \Lambda k' \tau_1 J'' > = \delta_{k k'} \delta_{J J''}.$$

Using Eqs. (3.51)-(3.52), following sum rule can be obtained for the generator, $G^{(2)}$, which has the tensorial character $\Lambda = (1, 0)$ with respect to Spin(5)

$$\sum_{J M'} | < N + \frac{1}{2}, \tau_1', \nu_\Delta', J', M' | G^{(2)} | N + \frac{1}{2}, \tau_1, \nu_\Delta, J, M > |^2$$

$$= \frac{1}{2J+1} \sum_{\nu_\Delta} | < N + \frac{1}{2}, \tau_1', \nu_\Delta | G^{(2)} | N + \frac{1}{2}, \tau_1, \nu_\Delta > |^2.$$

(3.53)
Note that there is no sum over k, because \((1, 0)\) representation of Spin(5) contains only \(k=2\). Eq. (3.53) states that E2 transition rates summed over the final angular momentum states are independent of \(J\) and \(M\) quantum numbers. In particular, using the \(B(E2)\) values given in Table IV, we can deduce two practical sum rules from Eq. (3.53)

\[
\frac{1}{2J+1} \sum_J |\langle N+\frac{1}{2}, \tau_1+1, \nu'_\Delta, J' || G^{(2)} || N+\frac{1}{2}, \tau_1, \nu_\Delta, J \rangle|^2
\]

\[
= (N-\tau_1+\frac{1}{2}) (N+\tau_1+\frac{9}{2}) \frac{\tau_1+\frac{1}{2}}{2\tau_1+4}
\]

(3.54)

\[
\frac{1}{2J+1} \sum_J |\langle N+\frac{1}{2}, \tau_1, \nu'_\Delta, J' || G^{(2)} || N+\frac{1}{2}, \tau_1, \nu_\Delta, J \rangle|^2
\]

\[
= \left( \frac{2N+5}{2\tau_1+4} \right)^2 \frac{(4\tau_1+3)(3\tau_1-1)}{4\tau_1(4\tau_1-1)}
\]
C.2. M1 Transitions

Magnetic dipole transitions occur in odd-even nuclei with considerable strength, in contrast to even-even nuclei where they are largely retarded. A fact which points out the single particle character of M1 transitions. It is natural to expect this disparity between collective and single particle degrees of freedom to be reflected in the M1 transition operator in the form of a symmetry breaking between boson and fermion operators. In general a one-body M1 operator can be written as

\[ T^{(M1)} = T_B^{(M1)} + T_F^{(M1)} \quad , \]

where

\[ T_B^{(M1)} = q_1 \left[ d^+ \times d^-(1) \right] \quad , \]

\[ T_F^{(M1)} = \sum_{jj', jj'}^{(1)} \left[ a_{jj'}^+ \times a_{jj'}^- \right]^{(1)} \quad . \]

In particular, for a single particle with \( j = \frac{3}{2} \), \( T^{(M1)} \) becomes

\[ T^{(M1)} = q_1 B^{(1)} - \frac{t_1 A^{(1)}}{\sqrt{2}} \quad , \]

where the operators \( A^{(1)} \) and \( B^{(1)} \) are given by Eq. (2.13), and

\[ t_1 = -\sqrt{2} t^{(1)}_{3/2 3/2} \quad . \]
If, in addition $q_1 = t_1$, the $M1$ operator becomes a generator, $G^{(1)}$, of Spin(6). Since $G^{(1)}$ is proportional to the total angular momentum operator, $\vec{J} = \vec{J}_B + \vec{J}_F$, Eq. (2.12), it has only non-zero diagonal matrix elements and all $M1$ transitions are forbidden. Therefore we will consider Eq. (3.57) with $q_1 \neq t_1$.

The operator (3.57) has selection rules, $\Delta \sigma_1 = \Delta \sigma_2 = \Delta \sigma_3 = 0$ and $\Delta \tau_1 = 0, \pm 1$, $\Delta \tau_2 = 0$. Its non-vanishing matrix elements can be obtained by expanding the wave functions as in Section B.

\[
< N + \frac{1}{2}, \tau_1, J \parallel T^{(M1)} \parallel N + \frac{1}{2}, \tau_1', J'>
\]

\[
= \sqrt{(2J+1)(2J'+1)} \sum_{\tau,L} \xi^N, \tau, L, \xi^N, \tau', L' (-)^L + J' + \frac{1}{2} \xi^N, \tau + \frac{1}{2}, \tau_1, J \xi^N, \tau + \frac{1}{2}, \tau_1', J'
\]

\[
\times \left[ \begin{array}{c}
\text{LJ} \\
\text{J' L'}
\end{array} \right] \left[ \begin{array}{c}
\text{\frac{3}{2}} \\
\text{\frac{3}{2}}
\end{array} \right] < N, \tau, L \parallel q_1 B^{(1)} \parallel N, \tau', L'> - \frac{(-)^{J-J'}}{\sqrt{2}} \left[ \begin{array}{c}
\text{\frac{3}{2}} \\
\text{\frac{3}{2}}
\end{array} \right]
\]

\[
\times \left[ \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array} \right] \left[ \begin{array}{c}
\frac{1}{2} \\
\frac{3}{2}
\end{array} \right] \parallel \tau_1 A^{(1)} \parallel \frac{1}{2}, \frac{1}{2}, \frac{3}{2} > \delta_{\tau \tau'} \delta_{LL'}
\]

Inserting the appropriate values of the isoscalar factors, Table II, 6-\(j\) symbols and the reduced matrix elements of the operators $A^{(1)}$ and $B^{(1)}$, Appendix A-B, in Eq. (3.58) we obtain all matrix elements of interest. From these we can calculate the $B(M1)$ values using

\[
B(M1; J_1 \rightarrow J_f) = \frac{1}{2J_1 + 1} \left| < J_f \parallel T^{(M1)} \parallel J_1 > \right|^2.
\]
Some B(M1) values of practical interest are given in Table V. In this table, we also present the E2/M1 mixing ratios in terms of the quantity $	ilde{\lambda} = \tilde{q}_2/(q_1-t_1)$, and the matrix elements

$$\mu (N+\frac{1}{2}, \tau_1, J) = \langle N+\frac{1}{2}, \tau_1, J || T^{(M1)} || N+\frac{1}{2}, \tau_1, J \rangle \quad ,$$

(3.60)

from which the magnetic moments can be calculated using

$$\mu_J = \frac{J}{\sqrt{J(J+1)(2J+1)}} \mu (N+\frac{1}{2}, \tau_1, J) \quad .$$

(3.61)

A summary of the B(M1) values for the low lying levels, obtained from Table V, is shown in Fig. 7.
C.3. Isomer and Isotope Shifts

Isomer and isotope shifts are related to the properties of the $E_0$ operator

$$T(E_0) = \beta_0 [d^+ \times d](0) + \gamma_0 [s^+ \times s](0) + \sum_j t_j(0) [a_j^+ \times a_j](0). \quad (3.62)$$

For a single particle with $j=3/2$, Eq. (3.62) reduces to

$$T(E_0) = \tilde{\beta}_o n_d + \gamma_o N + t_o A(0), \quad (3.63)$$

where

$$\tilde{\beta}_o = \frac{\beta_o}{\sqrt{5}} - \gamma_o; \quad n_d = \sqrt{5} [d^+ \times d](0); \quad A(0) = [a^+ \times a](0); \quad t_o = t(0).$$

Mean square radius, $\langle r^2 \rangle$ is related to $T^{(E0)}$ by

$$\langle r^2 \rangle = \langle r^2 \rangle^c + \langle T^{(E0)} \rangle, \quad (3.64)$$

where $\langle r^2 \rangle^c$ is the mean square radius of the closed shell.

Substituting Eq. (3.63) in (3.64) gives

$$\langle r^2 \rangle = \langle r^2 \rangle^c + \tilde{\beta}_o \langle n_d \rangle + \gamma_o N + t_o \langle A(0) \rangle. \quad (3.65)$$

The expectation value of the operator $n_d$ can be evaluated using the
expansion of Section B. For states belonging to the Spin(6) representation $\sigma_1=\sigma_2=\sigma_3=1/2$ we have

$$<N+\frac{1}{2}, \tau_1, J, M | n_d | N+\frac{1}{2}, \tau_1, J, M>$$

$$= \sqrt{2J+1} \sum_{\tau L} (-)^{J+L+\frac{3}{2}} \left( \begin{array}{c} N, \tau, L \\ N+\frac{1}{2}, \tau_1, J \end{array} \right)^2 \left( \begin{array}{c} L J \frac{3}{2} \\ J L 0 \end{array} \right)<N, \tau, L | n_d | N, \tau, L>.$$ \hspace{1cm} (3.66)

Substituting the 6-j symbol and the expectation value of $n_d$ in the SO(6) basis, Appendix B, we obtain

$$= \sum_{\tau L} \left( \begin{array}{c} N, \tau, L \\ N+\frac{1}{2}, \tau_1, J \end{array} \right)^2 \frac{N(N-1)+\tau(\tau+3)}{2(N+1)}.$$ \hspace{1cm} (3.67)

The sum over L is done in Eq. (3.25), carrying over the $\tau$ sum results in

$$= \frac{N^2 + (\tau_1 - \frac{1}{2})(\tau_1 + \frac{7}{2})}{2N + 4}.$$ \hspace{1cm} (3.68)

Inserting Eq. (3.68) and the expectation value of $A^{(0)}$, Appendix A, in Eq. (3.65) gives

$$<r^2>_{N+1/2} = <r^2>^c + \tilde{c}_o \frac{N^2 + (\tau_1 - \frac{1}{2})(\tau_1 + \frac{7}{2})}{2N + 4} + \gamma_o N - \frac{t_o}{2}.$$ \hspace{1cm} (3.69)

From this expression we can calculate the isomer shift, which is defined as the difference in $<r^2>$ between the first excited state and the ground state.
\[ \delta \langle r^2 \rangle = \tilde{\beta}_o \left( \langle n_d \lambda_{\tau_1} = \frac{3}{2} \rangle - \langle n_d \lambda_{\tau_1} = \frac{1}{2} \rangle \right) = \tilde{\beta}_o \frac{5}{2N+4} \quad . \quad (3.70) \]

This should be compared with the isomer shift in the adjacent even-even nucleus [AI 79]

\[ \delta \langle r^2 \rangle = \tilde{\beta}_o \frac{2}{N+1} \quad . \quad (3.71) \]

Similarly one can compute the isotope shift, which is defined as the difference in \( \langle r^2 \rangle \) between isotopes in their ground states

\[ \Delta \langle r^2 \rangle = \langle r^2 \rangle^{(N+3/2)} - \langle r^2 \rangle^{(N+1/2)} = \tilde{\beta}_o \frac{N^2 + 5N + 2}{2(N+2)(N+3)} + \gamma_o \quad . \quad (3.72) \]

to be compared with the isotope shift in even-even nuclei

\[ \Delta \langle r^2 \rangle = \langle r^2 \rangle^{(N+1)} - \langle r^2 \rangle^{(N)} = \tilde{\beta}_o \frac{N(N+3)}{2(N+1)(N+2)} + \gamma_o \quad . \quad (3.73) \]

Finally we derive the odd-even shift, defined as

\[ \Delta \langle r^2 \rangle = \langle r^2 \rangle^{(N+1/2)} - \langle r^2 \rangle^{(N)} \quad , \quad (3.74) \]

is given by

\[ \Delta \langle r^2 \rangle = \tilde{\beta}_o \frac{N}{(N+1)(N+2)} - \frac{\tau_o}{2} \quad . \quad (3.75) \]
D. Nucleon Transfer Intensities

In addition to electromagnetic transition rates, it is of interest to calculate intensities of transfer reactions. We begin with a discussion of two-nucleon transfer reactions. Since two-nucleon transfer is interpreted as a boson transfer in the IBM, same operator as for even-even nuclei can be used in calculations for odd-even nuclei. The L=0 transfer operator for two protons (π) or two neutrons (ν) can be written as in Eq. (7.1) of Ref. [Al 79]

\[
\hat{P}^{(0)}_{π} = \alpha_π \left( \frac{N_π + 1}{N+1} \right)^{1/2} s^+ \left( \Omega_π - \frac{N_π}{N} - \frac{N_π}{N} n_d \right)^{1/2},
\]

\[
\hat{P}^{(0)}_{ν} = \alpha_ν \left( \frac{N_ν + 1}{N+1} \right)^{1/2} s^+ \left( \Omega_ν - \frac{N_ν}{N} - \frac{N_ν}{N} n_d \right)^{1/2},
\]

and their hermitian conjugates for the inverse reactions. Above, \(n_π(ν)\) denotes number of proton (neutron) bosons, and \(Ω_π(ν)\) is the proton (neutron) pair degeneracy of the major shell. In order to calculate two-nucleon transfer intensities, matrix elements of the operators \(s^+\) (s) are needed. These operators have tensorial characters \((1, 0, 0)\) and \((0, 0)\) with respect to Spin(6) and Spin(5) respectively. Therefore they give rise to selection rules \(δσ_1=0, ±1, δσ_2=δσ_3=0\), and \(δτ_1=δτ_2=0\). The operators in Eq. (3.76) satisfy above selection rules approximately because of the square root factors containing \(n_d\). Nevertheless, this gives rise to only a small breaking of these rules (for the states with degeneracy in quantum number) as can be seen by expanding the square
root factor in powers of $n_d$.

For transitions from the ground state of an odd-even nucleus we evaluate the matrix elements of $s^\dagger$ using the expansion of Section B

$$
<[N+1], N' + \frac{1}{2}, \frac{1}{2}, \frac{3}{2} || s^\dagger || [N], N + \frac{1}{2}, \frac{1}{2}, \frac{3}{2}>
$$

$$
= -4 \sum_{N''} \left[ \xi_{N''} \xi_{N'} \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \xi_{N} \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \right] \left[ \begin{array}{ccc} 0 & 3 & 3 \\ 3 & 2 & 2 \\ 2 & 2 & 2 \end{array} \right] 

\left( [N+1], N'', 0, 0 \right) \left( [N], N', N, 0, 0 \right) 

\left( [N+1], N', 1, 2 \right) \left( [N], N', 1, 2 \right) 

\left( [N+1], N', N+1, N+2 \right) \left( [N], N, N+3 \right)
$$

(3.77)

substituting the appropriate values of the isoscalar factors, Table II, 6-j symbols, and matrix elements of $s^\dagger$, Appendix B, we obtain

$$
= \left[ \frac{2(N+1)(N+5)}{N+3} \right]^{1/2} \delta_{N', N+1} - \frac{2}{N+2} \left[ \frac{2}{N+3} \right]^{1/2} \delta_{N', N} + \frac{2N(N+4)}{N+2} \delta_{N', N-1}
$$

(3.78)

Intensity of transfer reactions is defined as

$$
I([N], [M], \sigma_1, \tau_1, J \rightarrow [N'], [M'], \sigma_1', \tau_1', J')
$$

(3.79)

$$
= \frac{1}{2J_1+1} \left| <[N'], [M'], \sigma_1', \tau_1', J' || P_\pm || [N], [M], \sigma_1, \tau_1, J> \right|^2
$$

Use of Eq. (3.78) in (3.79) yields the following intensities for two-nucleon transfer to the ground state of an odd-even nucleus.
\[ I([N],N+\frac{1}{2},\frac{1}{2},\frac{3}{2} \rightarrow [N+1],N+\frac{3}{2},\frac{1}{2},\frac{3}{2}) = a^2 \left(\frac{N+5}{v_{N+1}}\right) \left(\frac{\Omega_{N+2} - N}{2(N+2)}\right) \]

\[ I([N],N+\frac{1}{2},\frac{1}{2},\frac{3}{2} \rightarrow [N+1],N+\frac{1}{2},\frac{1}{2},\frac{3}{2}) = a^2 \left(\frac{2(N+1)}{2(N+1)(N+2)^2(N+3)}\right) \left(\frac{\Omega_{N+2} - N}{2(N+2)}\right) \]

\[ I([N],N+\frac{1}{2},\frac{1}{2},\frac{3}{2} \rightarrow [N+1],N-\frac{1}{2},\frac{1}{2},\frac{3}{2}) = a^2 \left(\frac{(N+1)N(N+4)}{2(N+1)(N+2)^2}\right) \left(\frac{\Omega_{N+2} - N}{2(N+2)}\right) \]

These should be compared with the intensities in the adjacent even-even nucleus \([\text{Al 79}]\]

\[ I([N],N,0,0 \rightarrow [N+1],N+1,0,0) = a^2 \left(\frac{N+4}{2(N+2)}\right) \left(\frac{\Omega_{N+2} - N}{2(N+2)}\right) \]

\[ I([N],N,0,0 \rightarrow [N+1],N-1,0,0) = a^2 \left(\frac{(N+1)N(N+3)}{2(N+1)^2(N+2)}\right) \left(\frac{\Omega_{N+2} - N}{2(N+2)}\right) \]

Which shows that to a good approximation \((1/N^2)\), presence of the fermion degree of freedom does not affect the two-nucleon transfer intensities, in consistence with the boson picture of the process.

Since now, wave functions of both even-even and odd-even nuclei are known, it is also possible to calculate one nucleon transfer reactions between adjacent nuclei. These also provide an important test for the more general supersymmetry scheme where nucleon transfers are considered as transitions between members of a multiplet. In contrast to two-nucleon transfer which is bosonic in character, one nucleon transfer involves both boson and fermion degrees of freedom, which makes the transfer operator very complex. An approximate form of the transfer
operator is \([V_i 81]\)

\[
P^+(i) + \mu = \zeta_j a^+_{j\mu} + \sum_j \zeta_j' [d^+ s^-(2) x a^+_{j\mu}] 
\]

\[
+ \theta_j [s^+ x a^+_{j\mu}] + \sum_j \theta_j' [d^+ x a^+_{j\mu}] 
\]

\[
P^-(i) - \mu = (p^+(i) + \mu)^+ .
\] (3.83)

where \(P^+_\mu\) denotes the addition operator and \(P^-\) the subtraction operator. When only one single particle level with \(j=3/2\) is considered, Eq. (3.82) simplifies to

\[
P^+(i) + \mu = \zeta a^+_{i\mu} + \zeta' [d^+ s^-(2) x a^+_{i\mu}] + \theta [s^+ x a] + \theta' [d^+ x a] 
\]

\[
(3.84)
\]

where \(\zeta = \zeta_2\), \(\zeta' = \zeta_2\), \(\theta = \theta_2\), \(\theta' = \theta_2\). The operators in Eq. (3.84) have tensorial characters \([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]\) and \([\frac{3}{2}, \frac{1}{2}, \frac{1}{2}]\) with respect to Spin(6), thus their matrix elements obey the selection rules

\[
\Delta \sigma_1 = \pm \frac{1}{2}, \quad \Delta \sigma_2 = \Delta \sigma_3 = 0 .
\]

With respect to Spin(5) they have only tensorial character \((\frac{1}{2}, \frac{1}{2})\), because \(j=3/2\) is not contained in \((\frac{3}{2}, \frac{1}{2})\). Hence we have the additional selection rules \(\Delta \tau_1 = \pm \frac{1}{2}, \Delta \tau_2 = 0\) (Fig. 8).

Although the operators in Eq. (3.84) effectively add a fermion, they contribute to different reactions. The first two terms contribute to reactions from an even-even nucleus with \(N\) bosons to an odd-even nucleus with \(N\) bosons and one fermion, whereas the last two terms contribute to reactions from an odd-even nucleus with \(N\) bosons and one fermion to an even-even nucleus with \(N+1\) bosons. Thus, for reactions of the first type, \(N, M=0 \Rightarrow N, M=1\), the appropriate operators are
\[ \hat{p}_+^+ = \zeta a^+ + \zeta' \left[ (d^+ \times s)^{(2)} \times a^+ \right] \frac{3}{2} \] ; \[ \hat{p}_- = \hat{p}_+^+ . \] (3.85)

For transitions from the ground state of an even-even nucleus one needs the following matrix element

\[ <[N], \{1\}, N'+\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \| \hat{p}_+ \| [N], N, 0, 0 > \]

\[ = -2 \zeta \frac{N, 0, 0}{N'+\frac{1}{2}, \frac{1}{2}, \frac{3}{2}} \left\{ \begin{array}{ccc} \frac{3}{2} & 0 \\ \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{array} \right\} \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \| a^+ \| 0, 0, 0 > \]

\[ +4 \zeta' \frac{N, 0, 0}{N'+\frac{1}{2}, \frac{1}{2}, \frac{3}{2}} \left\{ \begin{array}{ccc} 2 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{array} \right\} \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \| a^+ \| 0, 0, 0 > \]

\[ \times <[N], N, 1, 2 \| (d^+ \times s)^{(2)} \| [N], N, 0, 0 > . \] (3.86)

where \( N'=N, \ N-1 \). Substituting the isoscalar factors, 6-\( j \) and 9-\( j \) symbols, and matrix elements of operators we obtain

\[ = 2 \left[ \frac{N+4}{2N+4} \right]^{1/2} \left( \zeta - \frac{N}{\sqrt{5}} \zeta' \right) \delta_{N',N} - 2 \left[ \frac{N}{2N+4} \right]^{1/2} \left( \zeta + \frac{N+4}{\sqrt{5}} \zeta' \right) \delta_{N',N-1} . \] (3.87)

For simplicity, in calculation of intensities we will consider only matrix elements of the operator \( a^+ \). Using the definition, Eq. (3.79), the following intensities are obtained from Eq. (3.87)

\[ I([N], \{0\}, N, 0, 0 \to [N], \{1\}, N'\frac{1}{2}, \frac{1}{2}, \frac{3}{2} ) = 4\zeta^2 \frac{N+4}{2N+4} , \] (3.88)

\[ I([N], \{0\}, N, 0, 0 \to [N], \{1\}, N'-\frac{1}{2}, \frac{1}{2}, \frac{3}{2} ) = 4\zeta^2 \frac{N}{2N+4} . \]
For the inverse reaction one has

\[
\langle \mathcal{N}, \mathcal{N}, \tau, L \parallel p_- \parallel \mathcal{N}, \{1\}, \mathcal{N}^+, \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rangle
\]

\[
= -2 \zeta \sqrt{2L+1} \sum_{\tau', L'} \left\{ \begin{array}{ccc} 0 & L & L \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{array} \right\} \langle 0, 0, 0 \parallel p_- \parallel \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rangle \delta_{\tau, \tau'} \delta_{L, L'}
\]

\[
+ 4 \zeta' \sqrt{2L+1} \sum_{\tau', L'} \left\{ \begin{array}{ccc} \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{array} \right\} \langle 0, 0, 0 \parallel p_- \parallel \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rangle
\]

\[
\times \langle \mathcal{N}, \mathcal{N}, \tau, L \parallel (d^+ \times s)^{(2)} \parallel \mathcal{N}, \mathcal{N}, \tau', L' \rangle \quad , \quad (3.89)
\]

\[
= -2 \left[ \frac{N+4}{2N+4} \right]^{1/2} \left( \zeta - \frac{N}{\sqrt{5}} \zeta' \right) \delta_{\tau, 0} \delta_{L, 0} + 2 \left[ \frac{N}{2N+4} \right]^{1/2} \left( \zeta - \frac{N+4}{\sqrt{5}} \zeta' \right) \delta_{\tau, 1} \delta_{L, 2} .
\]

\[
(3.90)
\]

Again restricting the transfer operator to the first term, \( a^\dagger \), we obtain the following intensities for one nucleon transfer from odd-even to even-even nucleus

\[
I([\mathcal{N}], \{1\}, N^+, \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rightarrow [\mathcal{N}], \{0\}, N, 0, 0) = \zeta^2 \frac{N+4}{2N+4} \quad ,
\]

\[
(3.91)
\]

\[
I([\mathcal{N}], \{1\}, N^+, \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rightarrow [\mathcal{N}], \{0\}, N, 1, 2) = \zeta^2 \frac{N}{2N+4} \quad .
\]

\[
(3.91)
\]

For transfer reactions of the second type, \( N^+, M=0 \rightarrow N, M=1 \), the appropriate operators are

\[
\tilde{p}_+ = \theta \left[ s\dagger \times a \right] \left( \frac{3}{2} \right) + \theta' \left[ d\dagger \times a \right] \left( \frac{3}{2} \right) ; \quad \tilde{p}_- = \tilde{p}_+ \quad .
\]

\[
(3.92)
\]
If $\theta' = -\sqrt{5}\theta$, the above operator has tensorial character \( \left\{ \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \right\} \) with respect to Spin(6), thus its matrix elements satisfy the restricted selection rules $\Delta \sigma_1 = \pm \frac{1}{2}$, $\Delta \sigma_2 = \Delta \sigma_3 = 0$. We will consider the transitions from the ground state of an even-even nucleus via the operator, Eq. (3.92) with $\theta' = -\sqrt{5}\theta$

\[
< [N], \{1\}, N' + \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \| P_+ \| [N+1], N+1, 0, 0 >
\]

\[
= 4 \theta \sum_{\tau \ell L} \xi_{\tau, \ell, L} \left[ \begin{array}{ccc} L & 3 & 3 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{array} \right] \left[ \begin{array}{ccc} 0 & 0 & 0 \\ \ell' & \ell & \ell \end{array} \right] \left[ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{array} \right] \left[ \begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]
\]

\[
\times < [N], N, \tau, \ell, L \| b_{\ell} \| [N+1], N+1, 0, 0 > \delta_{N',N} \quad , (3.93)
\]

where $b_0 = s$, $b_2 = -\sqrt{5} d$, and $N' = N - 1$ transition is forbidden by the above selection rule. Substituting the isoscalar factors, 9-j symbols and matrix elements of $b_{\ell}$ one obtains

\[
= 2 \theta \sqrt{N+1} \delta_{N',N} \quad , (3.94)
\]

which yields the intensities

\[
I([N+1], \{0\}, N+1, 0, 0 \rightarrow [N], \{1\}, N+\frac{1}{2}, \frac{1}{2}, \frac{3}{2}) = 4\theta^2 (N+1) \quad , (3.95)
\]

\[
I([N+1], \{0\}, N+1, 0, 0 \rightarrow [N], \{1\}, N-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}) = 0 \quad .
\]
Similarly, for the inverse reaction one has

\[
\langle [N+1], N+1, \tau, L \parallel \tilde{P}_+ \parallel [N], \{1\}, N+ \frac{1}{2} \parallel \frac{1}{2} \parallel \frac{3}{2} \rangle
\]

\[
= 4 \theta \sqrt{2L+1} \sum_{\tau', \tau''} \xi_{N, \tau', L'} \left( \begin{array}{ccc} L & 0 & L \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{array} \right) \langle 0, 0, 0 \parallel \tilde{a}_L \parallel \frac{1}{2} \parallel \frac{3}{2} \rangle 
\]

\[
\times \langle [N+1], N+1, \tau, L \parallel b_{L''}^+ \parallel [N], N, \tau', L' \rangle , \quad (3.96)
\]

\[
= -2 \theta \sqrt{N+1} \delta_{\tau,0} \delta_{L,0} + 2 \theta \sqrt{N+5} \delta_{\tau,1} \delta_{L,2} , \quad (3.97)
\]

with the following intensities for transitions from odd-even to even-even nucleus

\[
I([N], \{1\}, N+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rightarrow [N+1], \{0\}, N+1, 0, 0) = \theta^2 (N+1) , \quad (3.98)
\]

\[
I([N], \{1\}, N+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rightarrow [N+1], \{0\}, N+1, 1, 2) = \theta^2 (N+5) .
\]

We conclude this section with a calculation of one nucleon transfer from an odd-even nucleus to two-quasiparticle states of an even-even nucleus, via the operator, Eq. (3.92). Two-quasiparticle states arise in the supersymmetric classification of nuclei, hence following results will be useful in the context of supersymmetries [BBI 81]. For transitions from the ground state of an odd-even nucleus to two-quasiparticle states of the adjacent even-even nucleus one has
\[
\langle [N-1], \{2\}, N, \tau, L \parallel \vec{P}_- \parallel [N], \{1\}, N+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rangle \\
= 4 \sqrt{5} \theta \sum_{\tau'' L''} \xi_{N-1, \tau_1, L_1} \xi_{N, \tau', L'} \left( \begin{array}{ccc} L & L & L \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{array} \right) \\
\times \langle \{1\}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \parallel a^\dagger \parallel \{2\}, 1, \tau_2, L_2 \rangle \langle [N-1], N-1, \tau_1, L_1 \parallel \vec{B}''_L \parallel [N], N, \tau', L' \rangle \\
(3.99)
\]

substituting the appropriate values of the isoscalar factors from Tables II-III, 9-j symbols and matrix elements of \( b_L \) we obtain

\[
=- 2 \theta \left[ \frac{N(N+4)}{N+2} \right]^{1/2} \delta_{\tau, 0} \delta_{L, 0} + 2 \theta \left[ \frac{N^2}{N+2} \right]^{1/2} \delta_{\tau, 1} \delta_{L, 2} \\
(3.100)
\]

which yields the following intensities

\[
I([N], \{1\}, N+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rightarrow [N-1], \{2\}, N, 0, 0) = \theta^2 \frac{N(N+4)}{N+2} \\
(3.101)
\]

\[
I([N], \{1\}, N+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rightarrow [N-1], \{2\}, N, 1, 2) = \theta^2 \frac{N^2}{N+2} 
\]
E. Examples of Spectra with Spin(6) Symmetry

In order to have spectra with Spin(6) symmetry, three conditions must be met: (1) the adjacent even-even nucleus displays an O(6) symmetry, (2) the odd nucleon occupies a single particle level with \( j = 3/2 \) and (3) the low-lying states are mainly built on this level. There are, at present, three regions where the O(6) symmetry seems to be appropriate. Two were discussed in Ref. [AI 79], the Os-Pt region and the Xe-Ba region. The third, Kr-Sr region, has been recently discussed by Kaup and Gelberg [KG 79]. These three regions are shown in Fig. 9. Inspection of the single particle level structure then shows that a Spin(6) symmetry could occur for:

(a) Odd-proton nuclei in the Os-Pt region with the odd-proton occupying the \( 2d_{3/2} \) level.

(b) Odd-neutron nuclei in the Xe-Ba region with the odd-neutron occupying the same \( 2d_{3/2} \) level.

(c) Odd-proton nuclei in the Kr-Sr region with the odd-proton occupying the \( 2p_{3/2} \) level.

A difficulty with a simple treatment of odd-even nuclei in terms of the Spin(6) symmetry is that the single particle levels with \( j = 3/2 \) mix with the other neighboring single particle levels. In particular, in the regions (a), (b) and (c), the single particle levels \( 3s_{1/2} \) (a, b) and \( 2p_{1/2} \) (c) play an important role. Therefore, before considering a nucleus as an example of the Spin(6) symmetry, one should check carefully whether the criterion (3) is fulfilled or not.
In this section, we shall study the extent to which the iridium and gold isotopes in the region (a) can be described by the Spin(6) symmetry. Spectroscopic factors for these nuclei are currently being investigated by various groups. The most complete data is available for $^{193}\text{Ir}$ [CB 81, IA 81], which we list in Table VI. A study of Table VI reveals that spectroscopic strengths of the low-lying levels are consistent with the Spin(6) predictions. Of special importance is the $s_{1/2}$ strength whose more than 75% lies above 800 KeV. The only exception in this case is the state at 559 KeV which carries a considerable fraction of the $2d_{5/2}$ strength. This state is presumably the first level of a structure build on top of the $2d_{5/2}$ single particle level. In Ref. [CB 81] spectroscopic factors for the other iridium isotopes are also investigated with the conclusion that $^{195,197}\text{Ir}$ can not be considered as examples of the Spin(6) symmetry. In gold isotopes, data for spectroscopic strengths are available only for $^{195,197,199}\text{Au}$ [MP 78]. Analysis of these data rules out $^{197,199}\text{Au}$ as possible candidates of the Spin(6) symmetry. (Although $^{197}\text{Au}$ has been suggested, mostly on the basis of the energy spectra, as a good example of the Spin(6) symmetry [Ve 81], we will not consider it here because criterion (3) is not fulfilled.) Little data is known on the lighter gold isotopes to make a direct check of criterion (3). However, from a study of other properties it appears that $^{191,193}\text{Au}$ can be described by the Spin(6) symmetry [Wo 81, 82]. Thus, in the following, we will limit our discussion to the nuclei, $^{191,193}\text{Ir}$ and
(1) Energy levels.

In Figs. 10, 11, 12 and 13 we show a comparison between the experimental energy levels of the nuclei $^{191,193}\text{Ir}$ and $^{191,193}\text{Au}$ and the theoretical levels calculated using Eq. (3.15). Although deviations occur, the structure of the spectrum appears to be well described by the Spin(6) symmetry. In order to display the observed deviations more quantitatively, we present in Table VII a compilation of the experimental and theoretical energies, together with their deviation,

$$\Delta = E^{\text{th}}_\text{e} - E^{\exp}_\text{e}.$$  

The average deviation, 

$$\phi = \frac{\sum_1 |E^{\text{th}}_\text{e} - E^{\exp}_\text{e}|}{\sum_1 E^{\exp}_\text{e}},$$

appears to be $\phi \approx 20\%$. All states shown in Figs. 10, 11, 12 and 13 and Table VII belong to the Spin(6) representation $(N+\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ which is the lowest in energy. Only two parameters, $(B/6)$ and $C$, are needed to describe these states. The experimental information on states belonging to the representations $(\sigma, \frac{1}{2}, \frac{1}{2})$ with $\sigma < N+\frac{1}{2}$ is less complete. Nevertheless it is of interest to study the available information, especially that related to $3/2^+$ states which can be detected by $(p,t)$ reactions [SL 78, LS 78]. Excitation energies must be now calculated using the full formula, Eq. (3.15). Denoting by $E^{*}_{3/2}$ the excitation energies of $3/2^+$ states, one has

$$E^{*}_{3/2} (\Sigma, \sigma, \tau) = \left(\frac{A_1}{4}\right) \left[ N(N+4) - \Sigma (\Sigma+4) \right] + \left(\frac{A}{4}\right) \left[ (N+\frac{1}{2})(N+\frac{9}{2}) - \sigma (\sigma+4) \right]$$

$$+ \left(\frac{B}{6}\right) \left[ \tau (\tau + 3) - \frac{7}{4} \right].$$

(3.102)
This must be compared with a similar expression for the excitation energies of $0^+$ states in even-even nuclei

$$E_0^* (\Sigma = \sigma_1, \tau_1, \bar{\tau}_1) = \left(\frac{A_1 + A}{4}\right)[N(N + 4) - \sigma_1(\sigma_1 + 4)] + \left(\frac{B}{6}\right)[\tau_1(\tau_1 + 3)].$$

(3.103)

The three lowest $0^+$ states in even-even nuclei have quantum numbers $\sigma_1 = N, \tau_1 = 0$ (the ground state); $\sigma_1 = N, \tau_1 = 3; \sigma_1 = N - 2, \tau_1 = 0$. In $^{192}$Pt the excited $0^+$ states are known to occur at 1195 KeV and 1617 KeV. In addition, in $^{192}$Pt another $0^+$ state occurs at 1542 KeV. This state is presumably the two quasiparticle, $(2d_3/2)^2$, state ($M=2$) which for the purposes of the present study will be neglected. The excitation energy, 1617 KeV, allows one to estimate the constant $(A_1 + A)/4 \approx 45$ KeV. When going to odd-even nuclei, each representation is expected to split into two parts with $\sigma_1 = \Sigma \pm \frac{1}{2}$. This is actually observed in $^{191}$Ir, Fig. 14. The splitting of the representations $\sigma_1 = \Sigma \pm \frac{1}{2}$ is observed in $^{191}$Ir to be 538 KeV. This allows one to estimate the constant $(A/4) \approx 27$ KeV. From this value one can predict where the representation with $\sigma_1 = N - \frac{3}{2}$ should lie, $E^* = 1674$ KeV. Experimentally, this state appears to be observed at 1596 KeV, Fig. 14. It is interesting to note that an additional $3/2^+$ state is also observed in $^{191}$Ir. This state is presumably the three quasiparticle state, $(2d_{3/2})^3$, ($M=3$) and it is the corresponding state to the 1542 KeV, $0^+$ state in $^{192}$Pt. In conclusion, it appears that not only states of the Spin(6) representation ($N+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$), but also states belonging to other representations are well described by the Spin(6) symmetry.
(2) Electromagnetic transition rates

Both E2 and M1 electromagnetic transition rates have been measured in $^{191,193}$Ir for some of the low-lying states. In Table VIII we present a comparison between the experimental and theoretical B(E2) values. These were calculated using the results of Table IV. If one assumes that the E2 operator is given by Eq. (3.44), there is only one overall constant $q^2 = +0.139$ eb. Again, it appears that B(E2) values are reasonably well described by the Spin(6) symmetry. The agreement is better for $^{193}$Ir than for $^{191}$Ir. E2 transition rates in $^{191,193}$Au are currently being studied by Wood [Wo 81, 82]. Preliminary results on branching ratios indicate that the Spin(6) selection rules are well satisfied. It is interesting to note that the same constant, $q^2 = +0.139$ eb fits the observed transition rates both in $^{191}$Ir and $^{193}$Ir. Furthermore, this value also fits the observed rates in the Os-Pt-Au and Hg isotopes, Fig. 12 of Ref. [BBI 81]. This suggests that the entire region of nuclei may be describable by a symmetry larger than Spin(6). As discussed in Ref. [BBI 81], this larger symmetry appears to be the supersymmetry U(6/4).

A comparison between experimental and theoretical B(M1) values is shown in Table IX. If one assumes that the M1 operator is given by Eq. (3.57), M1 transitions are determined in terms of only one overall constant, $q_1 - t_1$. The values of $q_1$ and $t_1$ can be estimated as follows. The constant $q_1$ is related to the magnetic moment of the $2_1^+$ state in the adjacent nucleus by

$$\mu_{2_1^+} = \sqrt{\frac{2}{5}} q_1.$$

(3.104)
Assuming $\mu_{2+} = 0.8 \mu_N$, this gives $q_1 = +1.27 \mu_N$. The constant $t_1$ is related to the magnetic moment of a $d_{3/2}$ single particle state by

$$
\nu_{3/2}^{(s.p.)} = \frac{3}{2 \sqrt{10}} t_1 \quad .
$$

(3.105)

Assuming the free proton values, $g_1 = 1.0, g_s = 5.58$, yields $\nu_{3/2}^{(s.p.)} = +0.13 \mu_N$ and therefore $t_1 = +0.27 \mu_N$. The values shown in Table IX are calculated using $(q_1 - t_1) = 1 \mu_N$. Although the qualitative features of the experimental results are reproduced by the Spin(6) formulas, major discrepancies occur. These large deviations, not observed for energies and $E2$ transitions, reflect the fact that $M1$ transitions are dominated, to a large extent, by the single particle part of the $M1$ operator and thus are very sensitive to admixtures of other single particle levels. The effects of these admixtures will be discussed in the next chapter.

(3) Electric and magnetic moments.

Electric quadrupole, $Q$, and magnetic dipole, $\mu$, moments can also be calculated using the formulas in Tables IV and V. For some of the low-lying states, they are given in Table X. No further parameters are needed here in addition to $q_2, q_1$ and $t_1$. These are taken as in the previous subsection. The comparison in Table X shows once more that both quadrupole and dipole moments are qualitatively described by the Spin(6) symmetry. The calculated dipole moment of the $1/2^+_1$ state is smaller than the experimental value which again points out to admixtures
of the $3s_{1/2}$ orbital in this level. However, the observed values
(0.470 $\mu_N$ in $^{191}$Ir and 0.540 $\mu_N$ in $^{193}$Ir) clearly indicate that
this level is mainly built on the $2d_{3/2}$ (and not on the $3s_{1/2}$)
orbital, since the single particle magnetic moment of a proton in the
$3s_{1/2}$ orbital is $\mu^{(s.p.)}_{1/2} = 2.79 \mu_N$.

(4) Transfer reactions.

Both one and two-nucleon transfer reactions in the Os-Pt region
have been studied. The results of one-nucleon transfer reactions to and
from $^{193}$Ir are shown in Table XI. Of the four possible transfer
reactions, three appear to be consistent with the predictions of the
Spin$(6)$ symmetry using the operators Eq. (3.84). The fourth reaction,
$^{193}$Ir $\rightarrow$ $^{194}$Pt, shows a large violation of the $\Delta I = \pm \frac{1}{2}$ rule. This
point will be further discussed in the next chapter when we consider the
modifications arising from a more general form of the transfer operator.

Two-nucleon transfer reactions on odd-even nuclei have recently
been studied by Cizewski et al. [CB 81]. They appear to be consistent
with Eqs. (3.80)-(3.81), as shown in Table XII.

Summarizing the results discussed above, we conclude that the
Spin$(6)$ symmetry provides a reasonable zeroth order description of both
$^{191,193}$Ir and $^{191,193}$Au, although discrepancies remain in a detailed
comparison between calculated and experimental results, especially for
M1 transitions and one-nucleon transfer intensities. In the next
chapter, we will investigate the possible improvements over the Spin$(6)$
symmetry in the light of the present discussion.
IV. Symmetry Breaking in Perturbation

Although the main properties of some odd-even nuclei are reasonably well described by the Spin(6) symmetry, several deviations occur. There are three possible sources of breaking of the Spin(6) symmetry. First, other single particle levels are admixed into the dominant j=3/2 single particle level. Second, the boson part of the Hamiltonian may not be well described by a $SO^B (6)$ symmetry. And third, the boson-fermion interaction may not be of the form imposed by the Spin(6) symmetry, Fig. 3. In general, these effects must be studied numerically. However, the case in which the symmetry breaking is caused by the presence of a j=1/2 orbital can be treated in perturbation theory. This will form the topic of Section A. As can be seen from Fig. 2, this situation occurs in the Ir-Au region because of the small energy difference between the $2d_{3/2}$ and $3s_{1/2}$ proton orbitals and hence is of considerable experimental interest.

Another effect which can be studied in perturbation theory is the mixing of the SU(3) limit of the IBM in the boson $O(6)$ limit. The Os-Pt region has been treated in the framework of the IBM and interpreted as initiating an $O(6) \rightarrow SU(3)$ transition [CC 78]. Thus, it is expected that including mixing of the SU(3) limit will describe the boson core of the Ir-Au nuclei better and, in particular, will modify those properties related to the quadrupole operator, since this operator is different in the two limits.
A. Mixing of the $j=1/2$ Orbital

So far we have discussed the spinor symmetry associated with the boson group $S_0^B(6)$ and fermion group $SU^F(4)$, where fermions occupy a $j=3/2$ orbital. However, there is another, simpler, spinor symmetry involving fermions in a $j=1/2$ orbital. Since the symmetry group of $j=1/2$ fermions is $U_F(2)$, the matching of the groups is

\[
U^B(6) \supseteq S_0^B(6) \supseteq S_0^B(5) \supseteq S_0^B(3) \supseteq S_0^B(2) \\
\approx \approx \approx \\
U^F(2) \supseteq SU^F(2) \supseteq SO^F(2)
\]

which can be combined into

\[
U^B(6) \otimes U^F(2) \supseteq S_0^B(6) \otimes SU^F(2) \supseteq S_0^B(5) \otimes SU^F(2) \supseteq \text{Spin}(3) \supseteq \text{Spin}(2)
\]

Here the boson and fermion groups meet at the level of $\text{Spin}(3)$, so in classifying the states the same quantum numbers $\sigma_1, \tau_1, v_\Delta$ as in Ref. [AI 79] are used. The quantum number $J$ is given by Eq. (3.8) for systems with no fermions, $M=0$, and by

\[
J = L + \frac{1}{2} \quad ; \quad L = 0
\]

\[
J = \frac{1}{2} \quad ; \quad L = 0
\]

for systems with one fermion, $M=1$. The Hamiltonian, which is diagonal
in the chain (4.2), can be written in terms of the Casimir operators of
the groups in (4.2). The corresponding energy formula for a system with
N bosons and M fermions, following Eq. (3.15), is given by

\[ E(N, M, \sigma, \tau, \nu^\Delta, J, M_J) = E_0^1 + E_1^1 N + E_2^1 N^2 + E_3^1 M + E_4^1 M^2 + E_5^1 M N \]

\[ - \left( \frac{A_1 + A_4}{4} \right) \sigma(\sigma+4) + \left( \frac{B}{6} \right) \tau(\tau+3) + CJ (J + 1) \]  

(4.4)

Only the last three terms contribute to the excitation spectrum, which
is shown in Fig. 15 for the case M=1.

When both \( j=3/2 \) and \( j=1/2 \) orbitals are present, one must admix the
level structure of Fig. 4 with that of Fig. 15. This can be done
numerically using the program ODDA [Sc 79]. However, if the mixing is
small, its effects can be studied in perturbation theory. With the
fermion creation (annihilation) operators \( a_{1/2}^\dagger, a_{3/2}^\dagger \),
a \( a_{1/2} \), one can only form the tensors \( [a_{1/2}^\dagger \times a_{3/2}^\dagger]^k \),
their hermitian conjugates, to couple the \( j=3/2 \) and \( j=1/2 \) systems.
Since the angular momentum operator must be diagonal, the term with \( k=1 \)
can not contribute, and only the term with \( k=2 \) survives. Thus, the
coupling Hamiltonian can be written as

\[ H_C = \beta \left[ (d^\dagger x_s + s^\dagger x_d)^{(2)} \cdot (a_{1/2}^\dagger \times a_{3/2}^\dagger - a_{3/2}^\dagger \times a_{1/2}^\dagger)^{(2)} \right] \]  

(4.5)

where \( \beta \) is a measure of the strength of the interaction. The Spin(3)
wave functions are simply given by the product of the SO(6)(B) wave
functions with the fermion state \( J=1/2, M \). So the matrix elements
of $H_c$ between the Spin(6) and Spin(3) states can be easily evaluated using the results of Chapter III, Section B

$$< N + \frac{1}{2}, \tau, J, M | H_c | N, \tau', J, M >$$

$$= \beta (-)^{J+L'+1} \frac{3}{2} \sum_{\tau L} \xi^{N, \tau, L} \left\{ \frac{L}{2} \frac{3}{2} \frac{J}{2} \right\} \sqrt{5}$$

$$\times <N, \tau, L \parallel (d^\dagger \times s + s^\dagger \times d)^{(2)} \parallel N, \tau', L'> .$$

Results obtained from Eq. (4.6) for the low-lying states are shown in Table XIII.

In perturbation theory, the states can be written as

$$| \phi > = \left(1 - \gamma_\phi^2\right)^{1/2} | \phi >_{3/2} + \gamma_\phi | \phi >_{1/2} ,$$

where the subscripts 3/2 and 1/2 denote states built on the $j=3/2$ and $j=1/2$ orbitals, and $\gamma_\phi$ is given by

$$\gamma_\phi = \frac{< \phi | H_c | \phi >}{E_{\phi_{1/2}} - E_{\phi_{3/2}}} .$$

The energy denominators can be evaluated using Eqs. (3.15) and (4.4).

Placing the lowest state of the $3s_{1/2}$ configuration at an energy $\tilde{\Lambda}$ above the lowest state of the $2d_{3/2}$ configuration, one obtains

$$E_{\phi_{1/2}} (N, \tau, J) - E_{\phi_{3/2}} (N + \frac{1}{2}, \tau, J) = \tilde{\Lambda} + \frac{R}{6} \left( \tau + 3 \right) (\tau + 3) + \frac{7}{4} + 3C .$$

(4.9)
Energy denominators for the lowest states are shown in Table XIV. The energy shift in second order perturbation theory is given by

\[ \Delta \phi = \langle \gamma_\phi \rangle^2 \left[ E_{\phi \frac{1}{2}} - E_{\phi \frac{3}{2}} \right]. \]  

(4.10)

which can be evaluated using Tables XIII and XIV. We note that, in general, one should consider mixing of all states of a given angular momentum in Eq. (4.7). However, because of the energy denominator, only the lowest member of each angular momentum appreciably contributes to the mixing.

One can also calculate the changes in other properties due to the mixing of the $3s_{\frac{1}{2}}$ orbital. The most general one-body $E2$ transition operator is given by Eqs. (3.41) and (3.42). For the case $j=1/2, 3/2$ and $q_2'=0$, this becomes

\[ T^{(E2)} = q_2 \left[ d^\dagger s^\dagger s + s^\dagger d \right]^{(2)} + t_2 \left[ a_{3/2}^\dagger x a_{3/2} \right]^{(2)} + t'_2 \left[ a_{1/2}^\dagger x a_{3/2} - a_{3/2}^\dagger x a_{1/2} \right]^{(2)}, \]  

(4.11)

where we have introduced $t_2' = \frac{t_2^{(2)}}{\sqrt{\frac{1}{2}} \frac{3}{2}}$. For $M1$ transitions, the appropriate operator is given by Eqs. (3.55) and (3.56). Introducing $t_1' = -\sqrt{\frac{3}{2}} t_1^{(1)}$, this becomes

\[ T^{(M1)} = q_1 \left[ d^\dagger x d \right]^{(1)} - \frac{t_1^{(1)}}{\sqrt{\frac{3}{2}}} \left[ a_{3/2}^\dagger x a_{3/2} \right]^{(1)} - \frac{t'_1^{(1)}}{\sqrt{\frac{3}{2}}} \left[ a_{1/2}^\dagger x a_{1/2} \right]^{(1)}. \]  

(4.12)
The M1 operator does not contain a part with $t^{(1)}_{\frac{1}{2} \frac{3}{2}}$ because there is no single particle M1 matrix element between a $d_{\frac{3}{2}}$ and a $s_{\frac{1}{2}}$ orbital.

An inspection of Table XIV shows that mixing of the $3s_{\frac{1}{2}}$ orbital affects most the states $\frac{1}{2}_{1}$, $\frac{5}{2}_{2}$, $\frac{3}{2}_{2}$, etc., since the energy denominator for these states is small. As an example, we discuss the changes in the properties of the $\frac{1}{2}_{1}$ state introduced by the mixing of the $3s_{\frac{1}{2}}$ orbital. The reduced E2 matrix element for the transition $\frac{1}{2}_{1} \rightarrow \frac{3}{2}_{1}$ which before the mixing was given by

\begin{equation}
< \frac{1}{2}_{1} \parallel T^{(E2)} \parallel \frac{3}{2}_{1} > = + q_{2} \left( \frac{2}{5} \right)^{1/2} [N(N+5)]^{1/2},
\end{equation}

now becomes

\begin{equation}
< \frac{1}{2}_{1} \parallel T^{(E2)} \parallel \frac{3}{2}_{1} > = q_{2} \left( \frac{2}{5} \right)^{1/2} [N(N+5)]^{1/2} \sqrt{1-\gamma_{3/2}^{2}} \sqrt{1-\gamma_{1/2}^{2}}
\end{equation}

\begin{equation}
\quad \quad + \left[ \frac{2(N+4)}{N+5} \right]^{1/2} \gamma_{3/2} \gamma_{1/2} + \frac{t_{2}'}{q_{2}} \left[ \frac{1}{4N(2N+4)} \right]^{1/2} \sqrt{1-\gamma_{1/2}^{2}} \gamma_{3/2}
\end{equation}

\begin{equation}
\quad \quad - \frac{t_{2}'}{q_{2}} \left[ \frac{25(N+4)}{2N(N+5)(2N+4)} \right]^{1/2} \sqrt{1-\gamma_{3/2}^{2}} \gamma_{1/2}
\end{equation}

Similarly the reduced M1 matrix element becomes

\begin{equation}
< \frac{1}{2}_{1} \parallel T^{(M1)} \parallel \frac{3}{2}_{1} > = \sqrt{1-\gamma_{3/2}^{2}} \sqrt{1-\gamma_{1/2}^{2}} (q_{1} - t_{1}) \left[ \frac{3N(N+5)}{5(2N+4)} \right]^{1/2}
\end{equation}

and magnetic moment of the $\frac{1}{2}_{1}$ state is changed to
\[ \mu_1^{1/2} = \frac{1}{10^{1/2}} \left( q_1 - \frac{t_1}{2} \right) \left( 1 - \gamma_1^{2/2} \right) + \left( \frac{5}{2} \right)^{1/2} t_1' \gamma_1^{2/2} \] 

(4.16)

In order to estimate the effects of the perturbation we need to specify the various parameters in the above equations. Although this requires a full microscopic theory, for the purpose of showing the effects of the mixing, it is sufficient to make reasonable estimates of these parameters. As can be seen from Fig. 2, the \( 3s_{1/2} \) orbital is 350 KeV above the \( 2d_{3/2} \) orbital. This places the structure built on the \( 3s_{1/2} \) orbital at an energy \( \Delta = 700 \text{ KeV} \) above that built on the \( 2d_{3/2} \) orbital, the extra 350 KeV coming from the interaction of the odd fermion with the bosons. Choosing as a strength of the coupling \( \beta = 35 \text{ KeV} \) gives \( \gamma_{1/2}^{1/2} = 0.334 \) and \( \gamma_{3/2}^{1/2} = 0.135 \). For the electromagnetic properties, we simply take \( t_2' = t_2 = +0.139 \text{ eb} \) and \( t_1' = 5.58 \mu_N \), which is the appropriate spin g-factor \( (g_s) \) for a proton in a \( s_{1/2} \) orbital. In Table XV we list the results of the calculations obtained with the above parameters. A study of Table XV and results that can be obtained in a similar way for all other levels, shows that the mixing of the \( 3s_{1/2} \) orbital removes only part of the discrepancies between the Spin(6) calculations and the experiment. In some cases, as for the magnetic moment of the \( 1/2_1 \) state, the perturbed Spin(6) results disagree with experiment more than the unperturbed results. The fact that the experimental result is larger than the unperturbed calculation but smaller than the perturbed one
indicates that the value $t'_1 = 5.58 \mu_N$ used in the perturbed calculation is too large. It has been found in many nuclei that the free spin $g$-factors must be renormalized due to polarization of the closed shells. In this case, a quenching of $g_s = 6 (g_s)^{\text{free}}$ (which is the value used in other collective models) would reproduce the observed magnetic moment, $\mu_{1/2}$.

In this section, we have discussed the improvements over the Spin(6) symmetry resulting from the mixing of the $3s_{1/2}$ orbital. However, for a very detailed comparison, one should include all single particle orbitals in the major valence shell ($2d_{3/2}$, $3s_{1/2}$, $2d_{5/2}$, $1g_{7/2}$ in this case). This can be done numerically using the computer code ODDA [Sc 79].
B. Mixing of the Boson SU(3) Limit

An accurate description of the boson core in odd-even nuclei is essential for detailed comparisons with experiment, since the boson degrees of freedom dominate in most nuclear properties. The even-even nuclei in the Os-Pt region have provided the first examples of the $O(6)$ symmetry of the IBM [AI 79]. Although the $O(6)$ symmetry produces many of the characteristic features of the nuclei in this region, discrepancies remain in energy spectra and E2 transition rates (away from $^{196}$Pt). Casten and Cizewski have interpreted this region as initiating an $O(6) \rightarrow SU(3)$ transition and obtained very good agreement with experiment for the $B(E2)$ values and E2 branching ratios [CC 78]. The energy spectra, however, can be improved only by explicitly introducing the proton and neutron degrees of freedom [BD 80]. Since we are interested in perturbations of the Spin(6) symmetry, we will neglect the distinction between the proton and neutron bosons and concentrate on the modifications introduced by the mixing of the boson SU(3) limit.

The SU(3) limit is obtained when the boson Hamiltonian (2.2) can be written in the form

$$H_B' = k \tilde{Q} \tilde{Q} + C \tilde{L} \tilde{L},$$

(4.17)

where $\tilde{Q}$ is the quadrupole operator of SU(3) and is given by

$$\tilde{Q} = [d^+ \times \tilde{s} + s^+ \times \tilde{d}]^{(2)} - \frac{\sqrt{7}}{2} [d^+ \times \tilde{d}]^{(2)}.\quad (4.18)$$
The $O(6) \rightarrow SU(3)$ transition near the $O(6)$ region can be studied by introducing $H'_B$, Eq. (4.17), as a perturbation Hamiltonian. (The second term in Eq (4.17) is diagonal in the $O(6)$ basis and, therefore, will be omitted.) The energy shift in first order perturbation theory is given by

$$\Delta E(N+ \frac{1}{2}, \tau_1, J) = \langle N+ \frac{1}{2}, \tau_1, J | \kappa \bar{Q} \bar{Q} | N+ \frac{1}{2}, \tau_1, J \rangle.$$  

(4.19)

Expanding the Spin(6) wave functions as in Chapter III, Section B, this becomes

$$\sum_{\tau \tau' L} \sum_{N+ \frac{1}{2}, \tau_1, J} \xi_{N, \tau, L} \xi_{N, \tau', L'} \langle N, \tau, L | \bar{Q} \bar{Q} | N, \tau', L \rangle.$$  

(4.20)

Eq. (4.20) can be evaluated using the isoscalar factors, Table II, and the matrix elements of $\bar{Q}$ in the $O(6)$ basis, Appendix B. We shall not proceed, however, with this evaluation, since, on the basis of the previous studies [CC 78, BD 80], no improvement in energies is expected. As an illustration, we list in Table XVI perturbed energies obtained for $\kappa=1$ KeV using the computer code ODDA.

Instead we discuss the modifications to other properties that arise from the change in the boson quadrupole operator. To keep the calculations simple, we shall still use the Spin(6) wave functions. The boson part of the $E2$ transition operator now takes the form given in Eq. (3.42). Matrix elements of the first term in Eq. (3.42) (together with the fermion part) have already been calculated in Eq. (3.45) and listed in Table IV. Matrix elements of the term, $(d^\dagger x d)^{(2)}$, can be
calculated in a similar way. We give in Table XVII some modified B(E2) values of practical interest.

Another property which is affected from the change in the quadrupole operator is the one-nucleon transfer reaction. The transfer operator in Eq (3.85) takes the form

\[ \tilde{P}_+ = \zeta a^+ + \zeta' \left[ \left( (d^+ x s + s^+ x d) (2) + \chi (d^+ x \tilde{d}) (2) \right) \times a^+ \right]^{(3/2)} ; \quad \tilde{P}_- = (\tilde{P}_+)^\dagger. \]

Matrix elements of the operators in Eq. (4.21) can be evaluated as in Section D of Chapter III. Intensities for one-nucleon transfer from odd-even to even-even nuclei, Eq. (3.88), are changed to

\[ I(N, 0, 0 \rightarrow N+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2}) = 4\zeta^2 \frac{N+4}{2N+4} \frac{1}{(N+1)^2} , \]

\[ I(N, 0, 0 \rightarrow N+ \frac{1}{2}, \frac{5}{2}, \frac{3}{2}) = 2\zeta^2 \chi^2 \frac{N(N-1)(N+4)(N+5)(N+6)}{7(2N+4)(N+1)^2} , \]

\[ I(N, 0, 0 \rightarrow N- \frac{1}{2}, \frac{1}{2}, \frac{3}{2}) = 4\zeta^2 \frac{N}{2N+4} \frac{1}{(N+3)^2} . \]

Similarly, intensities for the inverse reaction, Eq. (3.91), are changed to

\[ I(N+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rightarrow N, 0, 0) = \zeta^2 \frac{N+4}{2N+4} \frac{1}{(N+1)^2} , \]

\[ I(N+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rightarrow N, 1, 2) = \zeta^2 \frac{N}{2N+4} \frac{1}{(N+5)^2} , \]

\[ I(N+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rightarrow N, 2, 2) = \zeta^2 \chi^2 \frac{N(N-1)(N+5)(N+4)^2}{14(2N+4)(N+1)^2} . \]
\[ I(N+\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rightarrow N, 3, 0) = \zeta^2 \chi^2 \frac{N(N-1)(N-2)(N+5)(N+6)}{42(2N+4)(N+1)^2}, \]

\[ I(N+\frac{1}{2}, \frac{1}{2}, \frac{-3}{2} \rightarrow N, 3, 3) = \zeta^2 \chi^2 \frac{N(N-1)(N-2)(N+5)(N+6)}{21(2N+4)(N+1)^2}. \] (4.23)

Inspection of Eqs. (4.22)-(4.23) shows that addition of the \((d^\dagger \times d)^{(2)}\) term does not change the previously allowed transitions but rather allows population of other levels.

In order to compare the results with experiment, we use \(\chi\) as a free parameter to fit the data. In Table XVIII, we present the B(E2) values for \(^{191,193}\text{Ir}\) obtained from Table XVII. The results in Table XVIII show that the agreement between the calculations and experiment is considerably improved. We also note that the \((d^\dagger \times d)^{(2)}\) term relaxes the selection rule \(\Delta \tau_1 = \pm 1, 0\) and allows E2 transitions otherwise forbidden by the Spin(6) symmetry.

The intensities for the one-nucleon transfer reaction \(^{193}\text{Ir} \rightarrow ^{194}\text{Pt}\), obtained from Eqs. (4.22)-(4.23), are listed in Table XIX. Again, presence of the \((d^\dagger \times d)^{(2)}\) term relaxes the \(\Delta \tau_1 = \pm \frac{1}{2}\) selection rule and allows population of levels otherwise forbidden. However, as can be seen from Table XIX, only part of the discrepancy in the reaction \(^{193}\text{Ir} \rightarrow ^{194}\text{Pt}\) is explained by the modified operator with reasonable values of \(\chi\) (\(\simeq 1\)). A large value of \(\chi\) (\(\simeq 4\)) can explain this discrepancy, but it would be inconsistent with the physical picture where SU(3) limit acts only as a perturbation.
V. Spinor Symmetries in Molecular Spectra

Although not directly related, we include a brief description of spinor symmetries in molecular spectra. Recently, it has been suggested that algebraic methods, in the spirit of the IBM, may be useful in describing molecular rotation-vibration spectra [Ia 81b]. In the conventional approach, the solution of the Schrödinger equation is complicated beyond the simple case of a diatomic molecule, whereas in the algebraic approach even complex molecules can be solved rather simply [RDI 82]. Since the molecular interactions are dominated by the dipole degree of freedom, in this model, the spectrum is generated through excitations of scalar bosons (J=0) to vector bosons (J=1). The single component of the scalar boson, denoted by $\sigma$, and the three components of the vector boson, denoted by $\pi^\mu$ ($\mu=0, \pm 1$), span a four dimensional space, yielding $U(4)$ as the spectrum generating algebra of the problem. This algebra is most conveniently realized in terms of the 16 bilinear products, $G_{\alpha\alpha'}=b^\dagger_\alpha b_{\alpha'}$, where $b^\dagger_\alpha (b_\alpha)$ denotes the boson creation (annihilation) operators ($b_\alpha=\sigma, \pi^\mu, \alpha=1,2,3,4$). The most general $U(4)$ Hamiltonian, containing at most one and two-body interactions, can be written as [Ia 81c]

$$
H = \epsilon_\sigma \langle \sigma^+\sigma \rangle + \epsilon_\pi \langle \pi^+\pi^- \rangle + \sum_{L=0,2} \frac{c_L}{2^{L+1}} \left[ \langle \pi^+\pi^+ \rangle \times \langle \pi^-\pi^- \rangle \right] \langle 0 \rangle \\
+ u_0 \left[ \langle \sigma^+\sigma^+ \rangle \times \langle \sigma\sigma \rangle \right] \langle 0 \rangle + u_1 \left[ \langle \pi^+\pi^+ \rangle \times \langle \pi\pi \rangle \right] \langle 0 \rangle \\
+ u_2 \left[ \langle \pi^+\pi^+ \rangle \times \langle \sigma\sigma \rangle \right] \langle 0 \rangle + \langle \sigma^+\sigma^+ \rangle \times \langle \pi^+\pi^- \rangle \langle 0 \rangle \ , \quad (5.1)
$$
where $\tilde{\pi}_\mu = (-)^{1-\mu} \pi_\mu$. In general the Hamiltonian (5.1) should be diagonalized numerically in the basis provided by the totally symmetric irreducible representations $[N]$ of $U(4)$. However, under special conditions, corresponding to dynamical symmetries of $H$, the eigenvalue problem can be solved analytically. For the group $U(4)$ there are two possible such symmetries represented by the group chains

\begin{align}
U(4) &\supset O(4) \supset O(3) \supset O(2) \quad I, \\
U(4) &\supset U(3) \supset O(3) \supset O(2) \quad II.
\end{align}

(5.2)

The group chain I, called $O(4)$ symmetry, has found practical applications in rotation-vibration spectra of diatomic molecules [Ia 81b], and nuclear quasi-molecular resonances in the $^{12}\text{C}-^{12}\text{C}$ system [Ia 81c, EB 81].

In this chapter, we will extend the $O(4)$ boson symmetry to the Spin(3) Bose-Fermi symmetry by coupling an extra fermion with $j=1/2$ to the boson core. For fermions occupying a $j=1/2$ single particle level, the group chain is

\begin{equation}
U^{F}(2) \supset SU^{F}(2) \supset SO^{F}(2) .
\end{equation}

(5.3)

This can be combined with the group chain I, Eq. (5.2), into

\begin{equation}
U^{B}(4) \otimes U^{F}(2) \supset O^{B}(4) \otimes SU^{F}(2) \supset O^{B}(3) \otimes SU^{F}(2) \supset Spin(3) \supset Spin(2)
\end{equation}

(5.4)
where we have introduced Spin(3), the covering group of O(3), which is isomorphic to SU(2). Possible applications of the Spin(3) symmetry include diatomic molecules with a valence fermion in the j=1/2 orbital, and the quasi-molecular resonances in the $^{12}\text{C}-^{13}\text{C}$ system where the extra neutron occupies the lp_{1/2} single particle level.

In the following section, consequences of the Spin(3) symmetry will be worked out, in analogy to the Spin(6) symmetry of Chapter III. In particular, we will derive closed expressions for energies and electromagnetic (E1) transition rates.
A. Spin(3) Symmetry

We start with the labeling problem of the Spin(3) states. In order to uniquely characterize the states according to the group chain (5.4), the following quantum numbers are needed;

(a) The number of boson quanta, \([N]\), which labels the totally symmetric ir. reps. of \(U^B(4)\).

(b) The number of fermions, \([M]\), labeling the totally antisymmetric ir. reps. of \(U^F(2)\), \(M=0,1,2\).

(c) A label, \(\omega\), to characterize the ir. reps. of \(O^B(4)\) contained in \(U^B(4)\), which takes the values

\[
\omega = N, N-2, \ldots, 1 \text{ or } 0 \quad (N \text{ odd or even}) \tag{5.5}
\]

(d) The boson angular momentum, \(L\), characterizing the ir. reps. of \(O^B(3)\). The values of \(L\) contained in a representation \(\omega\) of \(O^B(4)\) are given by

\[
L = \omega, \omega-1, \ldots, 1, 0 \tag{5.6}
\]

(e) The angular momentum, \(J\), and its projection, \(M_J\), which characterize the ir. reps. of Spin(3) and Spin(2) respectively. \(J\) takes the values

\[
J = L, \quad (M=0) ; \quad J = L + \frac{1}{2}, \quad (J = \frac{1}{2} \text{ for } L=0), \quad (M=1) \tag{5.7}
\]
Thus the set of labels, \( |N\rangle, \{M\}, \omega, L, J, M_J \rangle \), uniquely classify the states. The most general Spin(3) Hamiltonian, containing one and two-body terms, can be written as

\[
H = -A \hat{C}_4 + B' C_3' + B C_3 \quad .
\]

(5.8)

where \( \hat{C}_4 \), \( C_3' \) and \( C_3 \) are the quadratic Casimir operators of \( O^B(4) \), \( O^B(3) \) and Spin(3) respectively. Physically, \( \hat{C}_4 \) corresponds to the attractive pairing interaction and \( C_3 \) represents the rotational energy. In terms of the group generators

\[
\begin{align*}
D^{(1)}_{\mu} &= \left[ \pi^+ \times \sigma + \sigma^+ \times \tilde{\pi} \right]^{(1)}_{\mu} , \\
G^{(1)}_{\mu} &= \left[ \pi^+ \times \tilde{\pi} \right]^{(1)}_{\mu} , \\
\hat{J} &= \sqrt{2} \left[ \pi^+ \times \pi \right]^{(1)} - \frac{1}{\sqrt{2}} \left[ a^+ \times \varepsilon \right]^{(1)} ,
\end{align*}
\]

(5.9)

of \( O^B(4) \) and Spin(3), they can be written as

\[
\begin{align*}
C_4 &= \frac{1}{2} \left[ G^{(1)} \cdot G^{(1)} + D^{(1)} \cdot D^{(1)} \right] , \\
C_3' &= 2 G^{(1)} \cdot G^{(1)} , \\
C_3 &= \hat{J} \cdot \hat{J} .
\end{align*}
\]

(5.10)

Here, we have introduced the creation (annihilation) operator, \( a^+ (a) \).
for \( j = 1/2 \) fermions, and 
\[
a_{j\mu} = (-)^{j-\mu} a_{j,-\mu}
\]
as before. Eigenvalues of \( C_4, C_3 \) and \( C_3 \) in the representations \( \omega, L, J \) are given by
\[
C_4 = \frac{1}{4} \omega(\omega + 2),
\]
\[
C_3 = L(L + 1),
\]
\[
C_3 = J(J + 1).
\]

Thus the Hamiltonian (5.8) has the eigenvalues
\[
E([N], [M], \omega, L, J) = -\frac{A}{4} \omega(\omega + 2) + B' L(L + 1) + B J(J + 1). \tag{5.12}
\]

For \( M = 0 \), Eq. (5.12) is the same as Eq. (4) of Ref. [Ia 81c], apart from a constant term \((A/4)N(N+2)\). The structure of the spectrum for \( M = 0 \) and \( M = 1 \) are shown in Figs. 16 and 17.

For properties other than the energies, one needs the wave functions explicitly. Spin(3) wave functions can be expanded into wave functions of the product group \( O^B(3) \otimes SU^F(2) \). Since here the groups meet at the level of Spin(3), the expansion coefficients are simply given by the Clebsch-Gordan coefficients. Because we are only interested in the reduced matrix elements of the operators, we will drop any dependence on magnetic quantum numbers, and write the Spin(3) states as
\[
|\omega, L, J> = |\omega, L> \times |\frac{1}{2}>
\]

where the states, in order, represent the Spin(3), \( O^B(4) \) and \( SU^F(2) \) states.
Electromagnetic transitions in molecular spectra are governed by the electric dipole (E1) operator. A one-body E1 operator can be written as

\[ T^{(E1)} = q_1 D^{(1)} \]  

(5.14)

where \( D^{(1)} \) is given in Eq. (5.9). Since \( D^{(1)} \) is a generator of \( O(8,4) \) and transforms like an axial vector under rotations, its matrix elements have the selection rules \( \Delta \omega = 0, \Delta L = \pm 1 \). \( \Delta L = 0 \) is excluded because axial vector operators have negative parity. The non-vanishing matrix elements of \( D^{(1)} \) are given by

\[
<\omega, L+1, J \parallel D^{(1)} \parallel \omega, L, J' > = (-)^{J'+L+1} \frac{1}{2} \left[ (2J+1)(2J'+1) \right]^{1/2} 
\]

\[
\begin{pmatrix}
L+1 & J & 1/2 \\
J' & L & 1
\end{pmatrix} <\omega, L+1 \parallel D^{(1)} \parallel \omega, L > . 
\]  

(5.15)

Matrix elements of the E1 operator in the O(4) basis have already been calculated in Eq. (21.32) of Ref. [Wy 74] as

\[
<\omega, L+1 \parallel D^{(1)} \parallel \omega, L > = [(\omega+L+2)(\omega-L)(L+1)]^{1/2} . 
\]  

(5.16)

Substituting Eq. (5.16) and the 6-j symbols in Eq. (5.15) we obtain the following E1 transition rates

\[
B(E1; L+1, J+1 \rightarrow L, J) = q_1^2 \frac{J}{2J+3} (\omega-L)(\omega+L+2) 
\]  

(5.17)

\[
B(E1; L+1, J \rightarrow L, J) = q_1^2 \frac{J}{2J+1} \frac{(\omega-L)(\omega+L+2)}{(2J+1)(2J+2)} . 
\]
One of the early successes of the interacting boson model has been the prediction of a SO(6) symmetry in transitional nuclei. The nuclei in this region have been very difficult to understand in terms of the geometrical models, whereas the IBM has provided a simple yet comprehensive description of these nuclei in terms of the SO(6) symmetry and small departures therefrom.

In a parallel development, the IBM has been extended to the interacting boson-fermion model which provides a unified description for collective states in odd-A nuclei. In this work, we have studied one of the spinor symmetries of the IBFM. This symmetry, Spin(6), arises when the bosons have SO(6) symmetry and the dimensionality of the fermion space is m=4 (j=3/2), m=20 (j= 1/2, 3/2, 5/2, 7/2), etc. As discussed in Chapter II, the Spin(6) symmetry is then a consequence of the isomorphism SO(6)\cong SU(4).

In Chapter III, using group theoretical techniques we have derived closed expressions for energies, electromagnetic transition rates and nucleon transfer intensities in the framework of the Spin(6) symmetry. An inspection of the single particle levels shows that a Spin(6) symmetry could occur for odd-A nuclei in the transitional regions Os-Pt, Xe-Ba and Kr-Sr. We have analyzed the experimental information in the first of these regions and concluded that qualitative features of $^{191,193}$Ir and $^{191,193}$Au can be described by the Spin(6) symmetry. However, discrepancies remain in a detailed comparison between experiment and calculations, especially for M1 transitions and one-
nucleon transfer intensities. These deviations can be investigated by introducing symmetry breaking effects such as mixing of the other single particle orbitals and of other limiting boson symmetries. In Chapter IV, we have discussed two of these effects in perturbation theory; mixing of the $3s_{1/2}$ single particle orbital and of the boson $SU(3)$ limit. The perturbation calculations indicated that the Spin(6) results can be improved considerably if the symmetry breaking effects are taken into account properly. The discrepancies in some M1 transition rates and one-nucleon transfer intensity in the reaction $^{193}$Ir $\rightarrow ^{194}$Pt persisted even after the perturbation calculations. Before reaching to a definite conclusion, however, one should carry out a full scale numerical computation which includes all single particle levels in the major valence shell and distinguishes between proton and neutron-bosons. Although we have attempted such a calculation using the computer program ODDA, we were not able to produce these effects as a perturbation of the Spin(6) symmetry. The difficulty seems to be lying in the simplified form of the boson-fermion interaction used in the program.

Finally, we mention that the Spin(6) symmetry discussed in this work can be further generalized to supersymmetries by imbedding the product group $SO^B(6) \otimes SU^F(4)$ into the graded Lie group $U(6/4)$ [BBI 81]. Since the spinor groups Spin(6)$\supset$Spin(5)$\supset$Spin(3) are contained as subgroups of $U(6/4)$, most of the results derived here can also be used for the study of the supersymmetries.
Table I. Angular momentum content of the lowest Spin(5) representations

<table>
<thead>
<tr>
<th>$(\tau_1, \tau_2)$</th>
<th>$\nu_\Delta = 0$</th>
<th>$\nu_\Delta = \frac{1}{2}$</th>
<th>$\nu_\Delta = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>$J = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1, 0)</td>
<td>$J = 2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2, 0)</td>
<td>$J = 4, 2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3, 0)</td>
<td>$J = 6, 4, 3$ ;</td>
<td>$J = 0$</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$M = 0$

<table>
<thead>
<tr>
<th>$(\frac{1}{2}, \frac{1}{2})$</th>
<th>$J = \frac{3}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\frac{3}{2}, \frac{1}{2})$</td>
<td>$J = \frac{7}{2}, \frac{5}{2} ;$</td>
</tr>
<tr>
<td>$(\frac{5}{2}, \frac{1}{2})$</td>
<td>$J = \frac{11}{2}, \frac{9}{2}, \frac{7}{2} ;$</td>
</tr>
<tr>
<td>$(\frac{7}{2}, \frac{1}{2})$</td>
<td>$J = \frac{15}{2}, \frac{13}{2}, \frac{11}{2}, \frac{9}{2} ;$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

$M = 1$
Table II. A selected number of isoscalar factors for the group chain $SU(4) \supset Sp(4) \supset SU(2)$ which are used in construction of the Spin(6) wave functions for N bosons and one fermion $(N, M=1)$.

\[
\begin{align*}
N, \tau_1 - \frac{1}{2}, 2\tau_1 - 1 \\
\xi \quad &\quad N + \frac{1}{2}, \tau_1, 2\tau_1 + \frac{1}{2} \\
&\quad = - \left[ \frac{(N + \tau_1 + \frac{1}{2})}{(2N + 4)} \right]^{1/2}
\end{align*}
\]

\[
\begin{align*}
N, \tau_1 + \frac{1}{2}, 2\tau_1 + 1 \\
\xi \quad &\quad N + \frac{1}{2}, \tau_1, 2\tau_1 + \frac{1}{2} \\
&\quad = \left[ \frac{(N-\tau_1 + \frac{1}{2}) (2\tau_1 + 2) (4\tau_1 + 3)}{(2N + 4) (2\tau_1 + 4) (4\tau_1 + 1)} \right]^{1/2}
\end{align*}
\]

\[
\begin{align*}
N, \tau_1 + \frac{1}{2}, 2\tau_1 - 1 \\
\xi \quad &\quad N + \frac{1}{2}, \tau_1, 2\tau_1 + \frac{1}{2} \\
&\quad = -\left[ \frac{(N-\tau_1 + \frac{1}{2}) (4\tau_1 - 2) (4\tau_1 + 1)}{(2N+4)(2\tau_1+4)(4\tau_1+1)} \right]^{1/2}
\end{align*}
\]

\[
\begin{align*}
N, \tau_1 - \frac{1}{2}, 2\tau_1 - 1 \\
\xi \quad &\quad N + \frac{1}{2}, \tau_1, 2\tau_1 - \frac{1}{2} \\
&\quad = - \left[ \frac{(N + \tau_1 + \frac{1}{2})}{(2N+4)} \right]^{1/2}
\end{align*}
\]

\[
\begin{align*}
N, \tau_1 + \frac{1}{2}, 2\tau_1 + 1 \\
\xi \quad &\quad N + \frac{1}{2}, \tau_1, 2\tau_1 - \frac{1}{2} \\
&\quad = \left[ \frac{(N-\tau_1 + \frac{1}{2}) (4\tau_1 - 2) (\tau_1 + \frac{1}{2}) (4\tau_1 + 3)}{(2N+4) (2\tau_1 + 4) (2\tau_1 + 1) (4\tau_1 + 1) (\tau_1 - \frac{1}{2})} \right]^{1/2}
\end{align*}
\]

\[
\begin{align*}
N, \tau_1 + \frac{1}{2}, 2\tau_1 - 1 \\
\xi \quad &\quad N + \frac{1}{2}, \tau_1, 2\tau_1 - \frac{1}{2} \\
&\quad = \left[ \frac{(N-\tau_1 + \frac{1}{2}) (2\tau_1 + 2)^2}{(2N+4) (2\tau_1 + 4) (4\tau_1 + 1) (\tau_1 - \frac{1}{2})} \right]^{1/2}
\end{align*}
\]

\[
\begin{align*}
N, \tau_1 + \frac{1}{2}, 2\tau_1 - 2 \\
\xi \quad &\quad N + \frac{1}{2}, \tau_1, 2\tau_1 - \frac{1}{2} \\
&\quad = -\left[ \frac{(N-\tau_1 + \frac{1}{2}) (\tau_1 - \frac{3}{2}) (4\tau_1 + 1)}{(2N+4)(2\tau_1 + 4)(\tau_1 - \frac{1}{2}) (2\tau_1)} \right]^{1/2}
\end{align*}
\]
\[
\begin{align*}
N, t_1 - \frac{1}{2}, 2t_1 - 1 & = - \left[ \frac{N + t_1 + \frac{7}{2}}{2N + 4} \right]^{1/2} \\
N, t_1 - \frac{1}{2}, 2t_1 - 3 & = - \left[ \frac{N + t_1 + \frac{7}{2}}{2N + 4} \right]^{1/2} \\
N, t_2 + \frac{1}{2}, 2t_1 - 1 & = - \left[ \frac{3(N - t_1 + \frac{1}{2})}{2N + 4} \right]^{1/2} \\
N, t_1 + \frac{1}{2}, 2t_1 - 2 & = - \left[ \frac{8(N - t_1 + \frac{1}{2})}{2N + 4} \right]^{1/2}
\end{align*}
\]
Table III. Some isoscalar factors used in construction of two-quasiparticle states \((N, M=2)\) in the Spin(6) limit.

\[\begin{align*}
N, \tau+1, 2\tau+2 \quad & = \quad \left[ \frac{(N-\tau)(N-\tau+1)(\tau+1)(4\tau+5)}{(N+1)(2N+4)(2\tau+5)(4\tau+1)} \right]^{1/2} \\
N+1, \tau, 2\tau \quad & = \quad -\left[ \frac{(N-\tau)(N-\tau+1)(4\tau+2)}{(N+1)(2N+4)(2\tau+5)(2\tau-1)} \right]^{1/2} \\
N, \tau+1, 2\tau-1 \quad & = \quad \left[ \frac{(N-\tau)(N-\tau+1)(2\tau+2)}{(N+1)(2N+4)(2\tau+3)(2\tau-1)} \right]^{1/2} \\
N+1, \tau, 2\tau \quad & = \quad -4 \left[ \frac{(N-\tau)(N-\tau+1)(2\tau(\tau-1)(\tau-2)}{(N+1)(2N+4)(2\tau+5)(2\tau-1)(4\tau-1)(4\tau+1)} \right]^{1/2} \\
N, \tau, 2\tau \quad & = \quad -\left[ \frac{(N+1)(N+\tau+4)}{(N+1)(2N+4)} \right]^{1/2} \\
N+1, \tau, 2\tau \quad & = \quad -\left[ \frac{(N+1)(N+\tau+4)}{(N+1)(2N+4)(2\tau+3)} \right]^{1/2}
\end{align*}\]
Table IV. Some $B(E2)$ values and quadrupole moments in the Spin(6) limit.

\[
B(E2; N+ \frac{1}{2}, \tau_1 + \frac{1}{2}, 2 \tau_1 + \frac{3}{2} + N+ \frac{1}{2}, \tau_1, 2 \tau_1 + \frac{1}{2}) = (q_2)^2 (N+\tau_1 + \frac{1}{2})^2 \frac{(\tau_1 + \frac{1}{2})}{(2\tau_1 + 4)}
\]

\[
B(E2; N+ \frac{1}{2}, \tau_1 + \frac{1}{2}, 2 \tau_1 + \frac{3}{2} + N+ \frac{1}{2}, \tau_1, 2 \tau_1 - \frac{1}{2}) = (q_2)^2 (N-\tau_1 + \frac{1}{2})^2 \frac{(\tau_1 - \frac{1}{2})}{(2\tau_1 + 4)} (4\tau_1 + 5)
\]

\[
B(E2; N+ \frac{1}{2}, \tau_1 + \frac{1}{2}, 2 \tau_1 + \frac{1}{2} + N+ \frac{1}{2}, \tau_1, 2 \tau_1 + \frac{1}{2}) = (q_2)^2 (N-\tau_1 + \frac{1}{2})^2 \frac{16(\tau_1 - \frac{1}{2})}{(2\tau_1 + 4)} (4\tau_1 + 3)
\]

\[
B(E2; N+ \frac{1}{2}, \tau_1 + \frac{1}{2}, 2 \tau_1 + \frac{1}{2} + N+ \frac{1}{2}, \tau_1, 2 \tau_1 - \frac{3}{2}) = (q_2)^2 (N-\tau_1 + \frac{1}{2})^2 \frac{(\tau_1 - \frac{3}{2})}{(2\tau_1 + 4)} (4\tau_1 - 1)
\]

\[
Q(N+ \frac{1}{2}, \tau_1, 2 \tau_1 + \frac{1}{2}) = + (q_2) \left( \frac{2N+5}{2\tau_1 + 4} \right) \left[ \frac{(\tau_1 + \frac{1}{2})(2\tau_1 + 2)(4\tau_1 + 3)}{(2\tau_1 + 1)(4\tau_1 + 1)} \right]^{1/2}
\]

\[
Q(N+ \frac{1}{2}, \tau_1, 2 \tau_1 - \frac{1}{2}) = - (q_2) \left( \frac{2N+5}{2\tau_1 + 4} \right) \left[ \frac{(\tau_1 - \frac{3}{2})(2\tau_1 + 2)(4\tau_1 - 3)}{(2\tau_1 + 1)(4\tau_1 - 1)} \right]^{1/2}
\]
Table V. Some B(M1) values and magnetic moments in the Spin(6) limit.

\[
B(M1; N^+ \frac{1}{2}, \tau_1 + 1, 2\tau_1 + \frac{3}{2} + N^+ \frac{1}{2}, \tau_1 + 2\tau_1 + \frac{1}{2}) = (q_1 - \tau_1)^2 \frac{(N+\tau_1 + \frac{1}{2}) (N+\tau_1 + 3)}{2(N+4)^2} \frac{(N+\tau_1 + 9)}{10(2\tau_1 + 4)} \frac{(2\tau_1 + 5)}{10(2\tau_1 + 4)}
\]

\[
B(M1; N^+ \frac{1}{2}, \tau_1 + 1, 2\tau_1 + \frac{1}{2} + N^+ \frac{1}{2}, \tau_1 + 2\tau_1 + \frac{1}{2}) = (q_1 - \tau_1)^2 \frac{(N+\tau_1 + \frac{1}{2}) (N+\tau_1 + 9)}{2(N+4)^2} \frac{8(\tau_1 + \frac{1}{2}) (2\tau_1 + 2)}{5(2\tau_1 + 4)^2 (4\tau_1 + 1)}
\]

\[
B(M1; N^+ \frac{1}{2}, \tau_1 + 1, 2\tau_1 + \frac{1}{2} + N^+ \frac{1}{2}, \tau_1 + 2\tau_1 - \frac{1}{2}) = (q_1 - \tau_1)^2 \frac{(N+\tau_1 + \frac{1}{2}) (N+\tau_1 + 9)}{2(N+4)^2} \frac{3(2\tau_1 + 2) (4\tau_1 + 3)}{5(2\tau_1 + 4)^2 (4\tau_1 + 1)}
\]

\[
B(M1; N^+ \frac{1}{2}, \tau_1, 2\tau_1 + \frac{1}{2} + N^+ \frac{1}{2}, \tau_1, 2\tau_1 - \frac{1}{2}) = (q_1 - \tau_1)^2 \frac{(N+\tau_1 + \frac{1}{2}) (N+\tau_1 + 9)}{2(N+4)^2} \frac{12(\tau_1 + \frac{1}{2}) (\tau_1 + \frac{3}{2})}{5(2\tau_1 + 4)^2 (4\tau_1 + 1)}
\]

\[
B(M1; N^+ \frac{1}{2}, \tau_1, 2\tau_1 - \frac{1}{2} + N^+ \frac{1}{2}, \tau_1, 2\tau_1 - \frac{3}{2}) = (q_1 - \tau_1)^2 \frac{(N+\tau_1 + \frac{1}{2}) (N+\tau_1 + 9)}{2(N+4)^2} \frac{8(\tau_1 + \frac{1}{2}) (\tau_1 + \frac{3}{2})}{5(2\tau_1 + 4)^2 (4\tau_1 + 1)}
\]

\[
\langle N^+ \frac{1}{2}, \tau_1 + 1, 2\tau_1 + \frac{3}{2} \rangle \ll \Delta(E2/M1) \ll N + \frac{1}{2}, \tau_1 + 2\tau_1 + \frac{1}{2} \rangle = \bar{\Delta}(2N+4) \left[ \frac{30}{(4\tau_1 + 1)(4\tau_1 + 5)} \right]^{1/2}
\]

\[
\langle N^+ \frac{1}{2}, \tau_1 + 1, 2\tau_1 + \frac{3}{2} \rangle \ll \Delta(E2/M1) \ll N + \frac{1}{2}, \tau_1 + 2\tau_1 + \frac{1}{2} \rangle = \bar{\Delta}(2N+4) \left[ \frac{10(\tau_1 + \frac{3}{2})}{(2\tau_1 + 2)(2\tau_1 + 4)} \right]^{1/2}
\]

\[
\langle N^+ \frac{1}{2}, \tau_1 + 1, 2\tau_1 + \frac{3}{2} \rangle \ll \Delta(E2/M1) \ll N^+ \frac{1}{2}, \tau_1 + 2\tau_1 + \frac{1}{2} \rangle = \bar{\Delta}(2N+4) \left[ \frac{250}{3(4\tau_1 + 1)(4\tau_1 + 3)} \right]^{1/2}
\]

\[
\mu(N^+ \frac{1}{2}, \tau_1, 2\tau_1 + \frac{1}{2}) = \left[ (4\tau_1 - 2) q_1 + 3 \tau_1 + 4 \frac{(N-\tau_1 + \frac{1}{2})(2\tau_1 + 2)}{(2N+4)(2\tau_1 + 4)} (q_1 - \tau_1) \right] \left[ \frac{(\tau_1 + \frac{3}{2})(4\tau_1 + 3)}{10(4\tau_1 + 1)} \right]^{1/2}
\]

\[
\mu(N^+ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}) = \frac{1}{5} \left( \frac{3}{7} \right)^{1/2} \left[ 11q_1 + \frac{13}{2} \tau_1 \right] + \frac{48}{35} \left( \frac{3}{7} \right)^{1/2} \frac{(N-1)}{(2N+4)} [q_1 - \tau_1]
\]

\[
\mu(N^+ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}) = \left( \frac{3}{5} \right)^{1/2} [q_1 - \frac{1}{2} \tau_1]
\]
Table VI. Spectroscopic factors for low-lying states in $^{193}$Ir. Data are from Refs. [CB 81, IA 81]. (Spectroscopic factor for the ground state transition is normalized to unity.)

<table>
<thead>
<tr>
<th>Orbital</th>
<th>$E_x$ (KeV)</th>
<th>$S(t,a)$</th>
<th>$S(d,^3$He)</th>
<th>$S($Spin$)$</th>
<th>$(\sigma_1, \tau_1, \nu_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2d_{3/2}$</td>
<td>0</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>$(15/2, 1/2, 0)$</td>
</tr>
<tr>
<td></td>
<td>180</td>
<td>0.07</td>
<td>0.08</td>
<td>0</td>
<td>$(15/2, 5/2, 0)$</td>
</tr>
<tr>
<td></td>
<td>460</td>
<td>0.69</td>
<td>0.74</td>
<td>0.64</td>
<td>$(13/2, 1/2, 0)$</td>
</tr>
<tr>
<td>$3g_{1/2}$</td>
<td>73</td>
<td>0.31</td>
<td>0.37</td>
<td>0</td>
<td>$(15/2, 3/2, 1/2)$</td>
</tr>
<tr>
<td></td>
<td>964</td>
<td>0.35</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2d_{5/2}$</td>
<td>139</td>
<td>0.08</td>
<td>0.08</td>
<td>0</td>
<td>$(15/2, 3/2, 0)$</td>
</tr>
<tr>
<td></td>
<td>362</td>
<td>0.17</td>
<td>0.21</td>
<td>0</td>
<td>$(15/2, 5/2, 1/2)$</td>
</tr>
<tr>
<td></td>
<td>559</td>
<td>1.12</td>
<td>0.98</td>
<td>0</td>
<td>$d_{5/2}$</td>
</tr>
<tr>
<td>$1g_{7/2}$</td>
<td>358</td>
<td>0.14</td>
<td>0</td>
<td>0</td>
<td>$(15/2, 3/2, 0)$</td>
</tr>
<tr>
<td></td>
<td>621</td>
<td>0.28</td>
<td>0.21</td>
<td>0</td>
<td>$(15/2, 5/2, 0)$</td>
</tr>
</tbody>
</table>
Table VII. Comparison between calculated and experimental energy levels in $^{191,193}$Ir and $^{191,193}$Au. The calculation has been done using Eq. (3.15) with $(B/6)=40$ KeV, $C=10$ KeV for Ir and $(B/6)=40$ KeV, $C=10$ KeV for Au isotopes. Since all states belong to the same Spin(6) representation $\sigma_1=N+\frac{1}{2}$, $\sigma_2=\sigma_3=\frac{1}{2}$, only the last two terms in Eq. (3.15) contribute to the excitation energies. The experimental energies are taken from Refs. [LK 79, VL 79, Br 80, Sh 81].

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$\sigma_1$</th>
<th>$I_1$</th>
<th>$\gamma_\Delta$</th>
<th>$J^\pi$</th>
<th>$E^{th}(\text{keV})$</th>
<th>$E^{exp}(\text{keV})$</th>
<th>$\Delta(\text{keV})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$17/2$</td>
<td>$1/2$</td>
<td>$0$</td>
<td>$3/2^+$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$3/2$</td>
<td>$0$</td>
<td>$7/2^+$</td>
<td>320</td>
<td>343</td>
<td>-23</td>
<td></td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$0$</td>
<td>$5/2^+$</td>
<td>250</td>
<td>129</td>
<td>+121</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$1/2$</td>
<td>$1/2^+$</td>
<td>170</td>
<td>82</td>
<td>+88</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$5/2$</td>
<td>$0$</td>
<td>$11/2^+$</td>
<td>800</td>
<td>832</td>
<td>-32</td>
<td></td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$0$</td>
<td>$9/2^+$</td>
<td>690</td>
<td>503</td>
<td>+187</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$0$</td>
<td>$7/2^+$</td>
<td>600</td>
<td>686</td>
<td>-86</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$1/2$</td>
<td>$5/2^+$</td>
<td>530</td>
<td>351</td>
<td>+179</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$1/2$</td>
<td>$3/2^+$</td>
<td>480</td>
<td>179</td>
<td>+301</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$7/2$</td>
<td>$0$</td>
<td>$15/2^+$</td>
<td>1420</td>
<td>1418</td>
<td>+2</td>
<td></td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$0$</td>
<td>$13/2^+$</td>
<td>1270</td>
<td>1004</td>
<td>+266</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$0$</td>
<td>$11/2^+$</td>
<td>1140</td>
<td>1207</td>
<td>-67</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$0$</td>
<td>$9/2^+$</td>
<td>1030</td>
<td>945</td>
<td>+85</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$1/2$</td>
<td>$9/2^+$</td>
<td>1030</td>
<td>812</td>
<td>+218</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$1/2$</td>
<td>$7/2^+$</td>
<td>940</td>
<td>504</td>
<td>+436</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{191}$Ir$_{114}$</td>
<td>$1/2$</td>
<td>$5/2^+$</td>
<td>870</td>
<td>748</td>
<td>-122</td>
<td></td>
<td></td>
</tr>
<tr>
<td>193(<em>{77})Ir(</em>{116})</td>
<td>15/2</td>
<td>1/2</td>
<td>0</td>
<td>3/2(^+)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>3/2</td>
<td>0</td>
<td>7/2(^+)</td>
<td>320</td>
<td>358</td>
<td>-38</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>5/2(^+)</td>
<td>250</td>
<td>139</td>
<td>+111</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>1/2(^+)</td>
<td>170</td>
<td>73</td>
<td>+97</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5/2</td>
<td>0</td>
<td>11/2(^+)</td>
<td>800</td>
<td>857</td>
<td>-57</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>9/2(^+)</td>
<td>690</td>
<td>522</td>
<td>+168</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>7/2(^+)</td>
<td>600</td>
<td>621</td>
<td>-21</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>5/2(^+)</td>
<td>530</td>
<td>362</td>
<td>+168</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>3/2(^+)</td>
<td>480</td>
<td>180</td>
<td>+300</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>191(<em>{79})Au(</em>{112})</th>
<th>17/2</th>
<th>1/2</th>
<th>0</th>
<th>3/2(^+)</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>3/2</td>
<td>0</td>
<td>7/2(^+)</td>
<td>440</td>
<td>521</td>
<td>-81</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>5/2(^+)</td>
<td>300</td>
<td>252</td>
<td>+48</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>1/2(^+)</td>
<td>140</td>
<td>11</td>
<td>+129</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5/2</td>
<td>0</td>
<td>11/2(^+)</td>
<td>1140</td>
<td>1132</td>
<td>+8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>9/2(^+)</td>
<td>900</td>
<td>789</td>
<td>+111</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>7/2(^+)</td>
<td>720</td>
<td>662</td>
<td>+58</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>5/2(^+)</td>
<td>580</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>3/2(^+)</td>
<td>480</td>
<td>207</td>
<td>+273</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>193(<em>{79})Au(</em>{114})</th>
<th>15/2</th>
<th>1/2</th>
<th>0</th>
<th>3/2(^+)</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>3/2</td>
<td>0</td>
<td>7/2(^+)</td>
<td>440</td>
<td>540</td>
<td>-100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>5/2(^+)</td>
<td>300</td>
<td>258</td>
<td>+42</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>1/2(^+)</td>
<td>140</td>
<td>38</td>
<td>+102</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5/2</td>
<td>0</td>
<td>11/2(^+)</td>
<td>1140</td>
<td>1154</td>
<td>-14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>9/2(^+)</td>
<td>900</td>
<td>809</td>
<td>+91</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>7/2(^+)</td>
<td>720</td>
<td>687</td>
<td>+33</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>5/2(^+)</td>
<td>580</td>
<td>780</td>
<td>-200</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>3/2(^+)</td>
<td>480</td>
<td>225</td>
<td>+255</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table VIII. Comparison between calculated and experimental B(E2) values in $^{191,193}$Ir. The calculation has been done using Fig. 6 with $q_2=0.139$ $\text{eb}$ for both nuclei. The experimental values are taken from the compilation in Ref. [VL 79].

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$\sigma_{1I}$</th>
<th>$\tau_{1I}$</th>
<th>$J_1^m$ + $\sigma_{1f}$</th>
<th>$\tau_{1f}$</th>
<th>$J_f^m$</th>
<th>B(E2)$_{\text{exp}}(e^2b^2)$</th>
<th>B(E2)$_{\text{th}}(e^2b^2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{191}<em>77\text{Ir}</em>{114}$</td>
<td>17/2</td>
<td>3/2</td>
<td>7/2 + 17/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.250±0.020</td>
<td>0.400</td>
</tr>
<tr>
<td>$^{191}<em>77\text{Ir}</em>{114}$</td>
<td>17/2</td>
<td>3/2</td>
<td>5/2 + 17/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.593±0.040</td>
<td>0.400</td>
</tr>
<tr>
<td>$^{191}<em>77\text{Ir}</em>{114}$</td>
<td>17/2</td>
<td>3/2</td>
<td>1/2 + 17/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.166±0.018</td>
<td>0.400</td>
</tr>
<tr>
<td>$^{191}<em>77\text{Ir}</em>{114}$</td>
<td>17/2</td>
<td>3/2</td>
<td>5/2 + 17/2</td>
<td>3/2</td>
<td>1/2</td>
<td>0.052±0.023</td>
<td>0.060</td>
</tr>
<tr>
<td>$^{191}<em>77\text{Ir}</em>{114}$</td>
<td>17/2</td>
<td>5/2</td>
<td>3/2 + 17/2</td>
<td>3/2</td>
<td>1/2</td>
<td>0.349±0.174</td>
<td>0.188</td>
</tr>
<tr>
<td>$^{191}<em>77\text{Ir}</em>{114}$</td>
<td>17/2</td>
<td>5/2</td>
<td>3/2 + 17/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.110±0.010</td>
<td>0</td>
</tr>
<tr>
<td>$^{191}<em>77\text{Ir}</em>{114}$</td>
<td>17/2</td>
<td>5/2</td>
<td>5/2 + 17/2</td>
<td>3/2</td>
<td>1/2</td>
<td>0.259±0.076</td>
<td>0.323</td>
</tr>
<tr>
<td>$^{191}<em>77\text{Ir}</em>{114}$</td>
<td>17/2</td>
<td>5/2</td>
<td>5/2 + 17/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.018±0.003</td>
<td>0</td>
</tr>
<tr>
<td>$^{193}<em>77\text{Ir}</em>{116}$</td>
<td>15/2</td>
<td>3/2</td>
<td>7/2 + 15/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.235±0.025</td>
<td>0.323</td>
</tr>
<tr>
<td>$^{193}<em>77\text{Ir}</em>{116}$</td>
<td>15/2</td>
<td>3/2</td>
<td>5/2 + 15/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.423±0.038</td>
<td>0.323</td>
</tr>
<tr>
<td>$^{193}<em>77\text{Ir}</em>{116}$</td>
<td>15/2</td>
<td>3/2</td>
<td>1/2 + 15/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.222±0.024</td>
<td>0.323</td>
</tr>
<tr>
<td>$^{193}<em>77\text{Ir}</em>{116}$</td>
<td>15/2</td>
<td>5/2</td>
<td>3/2 + 15/2</td>
<td>3/2</td>
<td>1/2</td>
<td>0.341±0.311</td>
<td>0.150</td>
</tr>
<tr>
<td>$^{193}<em>77\text{Ir}</em>{116}$</td>
<td>15/2</td>
<td>5/2</td>
<td>3/2 + 15/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.085±0.010</td>
<td>0</td>
</tr>
<tr>
<td>$^{193}<em>77\text{Ir}</em>{116}$</td>
<td>15/2</td>
<td>5/2</td>
<td>5/2 + 15/2</td>
<td>5/2</td>
<td>3/2</td>
<td>0.220±0.260</td>
<td>0.009</td>
</tr>
<tr>
<td>$^{193}<em>77\text{Ir}</em>{116}$</td>
<td>15/2</td>
<td>5/2</td>
<td>5/2 + 15/2</td>
<td>3/2</td>
<td>1/2</td>
<td>0.107±0.034</td>
<td>0.258</td>
</tr>
<tr>
<td>$^{193}<em>77\text{Ir}</em>{116}$</td>
<td>15/2</td>
<td>5/2</td>
<td>5/2 + 15/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.009±0.001</td>
<td>0</td>
</tr>
</tbody>
</table>
Table IX. Comparison between calculated and experimental $B(M1)$ values in $^{191,193}$Ir. The calculation has been done using Fig. 7 with $(q_1 - t_1) = 1 \mu_N$. The experimental values are taken from the compilation in Ref. [VL 79].

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$j_{1i}$</th>
<th>$\tau_{1i}$</th>
<th>$J_{1f}^n + \sigma_{1f}$</th>
<th>$\tau_{1f}$</th>
<th>$J_{f}^n$</th>
<th>$B(M1)^{exp} (\mu_N)$</th>
<th>$B(M1)^{th} (\mu_N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{191}<em>{77}$Ir$</em>{114}$</td>
<td>17/2</td>
<td>3/2</td>
<td>1/2 $\rightarrow$ 17/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.0011$\pm$0.0001</td>
<td>0.078</td>
</tr>
<tr>
<td></td>
<td>17/2</td>
<td>3/2</td>
<td>1/2 $\rightarrow$ 17/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.042$\pm$0.010</td>
<td>0.036</td>
</tr>
<tr>
<td></td>
<td>17/2</td>
<td>5/2</td>
<td>3/2 $\rightarrow$ 17/2</td>
<td>3/2</td>
<td>1/2</td>
<td>0.133$\pm$0.043</td>
<td>0.037</td>
</tr>
<tr>
<td></td>
<td>17/2</td>
<td>5/2</td>
<td>3/2 $\rightarrow$ 17/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.0042$\pm$0.0013</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>17/2</td>
<td>3/2</td>
<td>7/2 $\rightarrow$ 17/2</td>
<td>5/2</td>
<td>5/2</td>
<td>0.058$\pm$0.039</td>
<td>0.069</td>
</tr>
<tr>
<td></td>
<td>17/2</td>
<td>5/2</td>
<td>5/2 $\rightarrow$ 17/2</td>
<td>5/2</td>
<td>3/2</td>
<td>0.106$\pm$0.016</td>
<td>0.060</td>
</tr>
<tr>
<td></td>
<td>17/2</td>
<td>5/2</td>
<td>5/2 $\rightarrow$ 17/2</td>
<td>3/2</td>
<td>5/2</td>
<td>0.0016$\pm$0.0003</td>
<td>0.034</td>
</tr>
<tr>
<td></td>
<td>17/2</td>
<td>5/2</td>
<td>5/2 $\rightarrow$ 17/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.0104$\pm$0.0016</td>
<td>0</td>
</tr>
<tr>
<td>$^{193}<em>{77}$Ir$</em>{116}$</td>
<td>15/2</td>
<td>3/2</td>
<td>1/2 $\rightarrow$ 15/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.0021$\pm$0.0003</td>
<td>0.078</td>
</tr>
<tr>
<td></td>
<td>15/2</td>
<td>3/2</td>
<td>5/2 $\rightarrow$ 15/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.058$\pm$0.011</td>
<td>0.036</td>
</tr>
<tr>
<td></td>
<td>15/2</td>
<td>5/2</td>
<td>3/2 $\rightarrow$ 15/2</td>
<td>3/2</td>
<td>1/2</td>
<td>0.158$\pm$0.107</td>
<td>0.036</td>
</tr>
<tr>
<td></td>
<td>15/2</td>
<td>5/2</td>
<td>3/2 $\rightarrow$ 15/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.0077$\pm$0.0048</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>15/2</td>
<td>3/2</td>
<td>7/2 $\rightarrow$ 15/2</td>
<td>3/2</td>
<td>5/2</td>
<td>0.081$\pm$0.053</td>
<td>0.070</td>
</tr>
<tr>
<td></td>
<td>15/2</td>
<td>5/2</td>
<td>5/2 $\rightarrow$ 15/2</td>
<td>5/2</td>
<td>3/2</td>
<td>0.029$\pm$0.015</td>
<td>0.060</td>
</tr>
<tr>
<td></td>
<td>15/2</td>
<td>5/2</td>
<td>5/2 $\rightarrow$ 15/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0.0058$\pm$0.0009</td>
<td>0</td>
</tr>
</tbody>
</table>
Table X. Comparison between calculated and experimental electric quadrupole and magnetic dipole moments in $^{191,193}\text{Ir}$. Same values of $q_2$ and $q_1-t_1$ as in Tables VII-IX are used.

The experimental data are from Refs. [VL 79, Br 80].

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$J^\pi$</th>
<th>$Q_{\text{exp}}\ (\text{eb})$</th>
<th>$Q_{\text{th}}\ (\text{eb})$</th>
<th>$\mu_{\text{exp}}\ (\mu_N)$</th>
<th>$\mu_{\text{th}}\ (\mu_N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{193}\text{Ir}_{116}$</td>
<td>$3^+\ \frac{3}{2}^-$</td>
<td>$+1.0\pm0.5$</td>
<td>$+0.84$</td>
<td>$+0.159\pm0.001$</td>
<td>$+0.27$</td>
</tr>
<tr>
<td></td>
<td>$5^+\ \frac{5}{2}^-$</td>
<td>$-0.08$</td>
<td></td>
<td></td>
<td>$+0.81$</td>
</tr>
<tr>
<td></td>
<td>$1^+\ \frac{1}{2}^-$</td>
<td>$0$</td>
<td></td>
<td>$+0.470\pm0.001$</td>
<td>$+0.36$</td>
</tr>
<tr>
<td>$^{191}\text{Ir}_{114}$</td>
<td>$3^+\ \frac{3}{2}^-$</td>
<td>$+0.81\pm0.21$</td>
<td>$+0.92$</td>
<td>$+0.146\pm0.001$</td>
<td>$+0.27$</td>
</tr>
<tr>
<td></td>
<td>$5^+\ \frac{5}{2}^-$</td>
<td>$-0.09$</td>
<td></td>
<td>$+0.62\pm0.07$</td>
<td>$+0.82$</td>
</tr>
<tr>
<td></td>
<td>$1^+\ \frac{1}{2}^-\frac{1}{2}$</td>
<td>$0$</td>
<td></td>
<td>$+0.540\pm0.005$</td>
<td>$+0.36$</td>
</tr>
</tbody>
</table>
Table XI. Comparison between calculated and experimental intensities of one-nucleon transfer reactions to and from $^{193}\text{Ir}$. Intensities are normalized to unity for each $J_f^\pi$. The calculations have been done using Eqs. (3.88), (3.91), (3.95) and (3.98). The experimental values are from Refs. [BT 79, IA 81, CB 81, VR 81].

(a) $^{192}\text{Os}_{76}^{116} \rightarrow ^{193}\text{Ir}_{77}^{116}$

<table>
<thead>
<tr>
<th>$\sigma_{1i}$</th>
<th>$\tau_{1i}$</th>
<th>$J_{f}^\pi$</th>
<th>$\sigma_{1f}$</th>
<th>$\tau_{1f}$</th>
<th>$J_{f}^\pi$</th>
<th>$E_f$</th>
<th>Spin(6)</th>
<th>$(a,t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0</td>
<td>0+</td>
<td>15/2</td>
<td>1/2</td>
<td>3/2+</td>
<td>0</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>15/2</td>
<td>5/2</td>
<td>3/2+</td>
<td>180</td>
<td>0</td>
<td>&lt;0.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>13/2</td>
<td>1/2</td>
<td>3/2+</td>
<td>460</td>
<td>0</td>
<td>&lt;0.01</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) $^{193}\text{Ir}_{77}^{116} \rightarrow ^{192}\text{Os}_{76}^{116}$

<table>
<thead>
<tr>
<th>$\sigma_{1i}$</th>
<th>$\tau_{1i}$</th>
<th>$J_{f}^\pi$</th>
<th>$\sigma_{1f}$</th>
<th>$\tau_{1f}$</th>
<th>$J_{f}^\pi$</th>
<th>$E_f$</th>
<th>Spin(6)</th>
<th>$(t,a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15/2</td>
<td>1/2</td>
<td>3/2+</td>
<td>8</td>
<td>0</td>
<td>0+</td>
<td>0</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>3</td>
<td>0+</td>
<td>956</td>
<td>0</td>
<td>&lt;0.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>1</td>
<td>2+</td>
<td>205</td>
<td>1.00</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>2</td>
<td>2+</td>
<td>489</td>
<td>0</td>
<td>0.32</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \alpha_{II} )</td>
<td>( \tau_{III} )</td>
<td>( J_1^\pi \to \alpha_{If} )</td>
<td>( \tau_{If} )</td>
<td>( J_f^\pi )</td>
<td>( E_f )</td>
<td>( \text{Spin}(6) )</td>
<td>( (d, 3^+_\text{He}) )</td>
<td>( (t, a) )</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0(^+)</td>
<td>15/2</td>
<td>1/2</td>
<td>3/2(^+)</td>
<td>0</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>15/2</td>
<td>5/2</td>
<td>3/2(^+)</td>
<td>180</td>
<td>0</td>
<td>0.08</td>
<td>0.07</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15/2</td>
<td>1/2</td>
<td>3/2(^+)</td>
<td>460</td>
<td>0.64</td>
<td>0.74</td>
<td>0.70</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \alpha_{II} )</th>
<th>( \tau_{III} )</th>
<th>( J_1^\pi \to \alpha_{If} )</th>
<th>( \tau_{If} )</th>
<th>( J_f^\pi )</th>
<th>( E_f )</th>
<th>( \text{Spin}(6) )</th>
<th>( (3^+_\text{He}, d) )</th>
<th>( (a, t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15/2</td>
<td>1/2</td>
<td>3/2(^+)</td>
<td>7</td>
<td>0</td>
<td>0(^+)</td>
<td>0</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>0(^+)</td>
<td>1267</td>
<td>0</td>
<td>0.28</td>
<td>0.33</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>2(^+)</td>
<td>328</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>2(^+)</td>
<td>622</td>
<td>0</td>
<td>1.08</td>
<td>1.25</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table XII. Comparison between calculated and experimental intensities of two-nucleon transfer reactions in Pt and Ir. The theoretical intensities are obtained from Eqs. (3.80)-(3.81) and normalized to the Ir reaction. The experimental data are from Ref. [CB 81].

<table>
<thead>
<tr>
<th>Reaction</th>
<th>$I^{\exp}$</th>
<th>$I^{th}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{192}\text{Pt}(t,p)^{194}\text{Pt}$</td>
<td>0.97±0.13</td>
<td>1.01</td>
</tr>
<tr>
<td>$^{191}\text{Ir}(t,p)^{193}\text{Ir}$</td>
<td>1.00±0.10</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Table XIII. Matrix elements of the Hamiltonian $H_c$, Eq. (4.6), mixing the $j=3/2$ and $j=1/2$ orbitals. $|3/2, 1/2\rangle$ denotes the first $3/2$ state of the level structure built on the $j=1/2$ orbital.

| $|j, m\rangle$ | $\langle j, m | H_c | j, m\rangle$ |
|-----------------|-----------------|
| $|3/2, 1/2\rangle$ | $\frac{1}{2} \left( \frac{(N+4)^2}{2N+4} \right)^{1/2}$ |
| $|3/2, 1/2\rangle$ | $\frac{3}{2} \left( \frac{1}{4} \right)^{1/2} \left( \frac{(N-1)(N+5)^2}{2N+4} \right)^{1/2}$ |
| $|3/2, 1/2\rangle$ | $\left( \frac{1}{14} \right)^{1/2} \left( \frac{(N-1)^2(N+5)}{2N+4} \right)^{1/2}$ |
| $|3/2, 1/2\rangle$ | $\left( \frac{N(N+4)(N+5)}{2(2N+4)} \right)^{1/2}$ |
| $|3/2, 1/2\rangle$ | $\frac{1}{6} \left( \frac{13}{1} \right)^{1/2} \left( \frac{(N-2)(N+6)^2}{2N+4} \right)^{1/2}$ |
| $|3/2, 1/2\rangle$ | $\frac{1}{9} \left( \frac{11}{1} \right)^{1/2} \left( \frac{(N-2)^2(N+6)}{2N+4} \right)^{1/2}$ |
| $|3/2, 1/2\rangle$ | $\frac{1}{9} \left( \frac{5}{7} \right)^{1/2} \left( \frac{(N-2)^2(N+6)}{2N+4} \right)^{1/2}$ |
| $|3/2, 1/2\rangle$ | $-\frac{3}{7} \left( \frac{3}{7} \right)^{1/2} \left( \frac{(N-1)(N+5)(N+6)}{2N+4} \right)^{1/2}$ |
| $|3/2, 1/2\rangle$ | $-\frac{1}{2} \left( \frac{(N-1)(N+5)}{2N+4} \right)^{1/2}$ |
Table XIV. Energy denominators in Eq. (4.8). \( \tilde{\delta}=(B/6) \) and \( \Delta \) is the energy difference between the lowest state of the \( j=3/2 \) configuration and the lowest state of the \( j=1/2 \) configuration.

<table>
<thead>
<tr>
<th>State</th>
<th>Energy Denominator</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{3}{2} )</td>
<td>( \Delta + 4 B + 3 C )</td>
</tr>
<tr>
<td>( \frac{7}{2} )</td>
<td>( \Delta + 5 B + 3 C )</td>
</tr>
<tr>
<td>( \frac{5}{2} )</td>
<td>( \Delta - B + 3 C )</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>( \Delta - 5 B + 3 C )</td>
</tr>
<tr>
<td>( \frac{11}{2} )</td>
<td>( \Delta + 6 B + 3 C )</td>
</tr>
<tr>
<td>( \frac{9}{2} )</td>
<td>( \Delta - 2 B + 3 C )</td>
</tr>
<tr>
<td>( \frac{7}{2} )</td>
<td>( \Delta - 2 B + 3 C )</td>
</tr>
<tr>
<td>( \frac{5}{2} )</td>
<td>( \Delta - 8 B + 3 C )</td>
</tr>
<tr>
<td>( \frac{3}{2} )</td>
<td>( \Delta - 8 B + 3 C )</td>
</tr>
</tbody>
</table>
Table XV. Effects of the mixing of the $3s_{1/2}$ orbital in the
properties of the $\frac{1}{2}^1$ level in $^{193}\text{Ir}$. $S_{1/2}$ is the
spectroscopic factor in single particle units and $E^*_1/2$ is
its excitation energy. Perturbed Spin(6) results are
obtained from Eqs. (4.14), (4.15), (4.16), (4.8) and (4.10)
in order.

<table>
<thead>
<tr>
<th></th>
<th>Experiment</th>
<th>Spin(6)</th>
<th>Perturbed Spin (6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(E2; \frac{1}{2}_1 \rightarrow \frac{3}{2}_1) (e^2b^2)$</td>
<td>0.222±0.024</td>
<td>0.32</td>
<td>0.26</td>
</tr>
<tr>
<td>$B(M1; \frac{1}{2}_1 \rightarrow \frac{3}{2}_1) (\mu^2_N)$</td>
<td>0.0021±0.0003</td>
<td>0.078</td>
<td>0.068</td>
</tr>
<tr>
<td>$\mu_{\frac{1}{2}, \frac{1}{2}}$ (\mu_N)</td>
<td>0.470±0.001</td>
<td>0.36</td>
<td>0.71</td>
</tr>
<tr>
<td>$S_{1/2}$</td>
<td>(20±10)%</td>
<td>0</td>
<td>11%</td>
</tr>
<tr>
<td>$E^*_1/2$ (keV)</td>
<td>73</td>
<td>170</td>
<td>127</td>
</tr>
</tbody>
</table>
Table XVI. Change in the Spin(6) energies in $^{191}$Ir due to the mixing of the boson SU(3) limit. The calculation has been done with $(B/6)=25$ KeV, $C=15$ KeV and $\kappa=1$ KeV. The experimental energies are the same as in Table VII.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\tau$</th>
<th>$v_{\Delta}$</th>
<th>$J^{\pi}_n$</th>
<th>$E$ (KeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>17/2</td>
<td>1/2</td>
<td>0</td>
<td>3/2$^+$</td>
<td>0</td>
</tr>
<tr>
<td>3/2</td>
<td>0</td>
<td>7/2$^+$</td>
<td>305</td>
<td>315</td>
</tr>
<tr>
<td>0</td>
<td>5/2$^+$</td>
<td>200</td>
<td>188</td>
<td>129</td>
</tr>
<tr>
<td>1/2</td>
<td>1/2$^+$</td>
<td>80</td>
<td>111</td>
<td>82</td>
</tr>
<tr>
<td>5/2</td>
<td>0</td>
<td>11/2$^+$</td>
<td>780</td>
<td>797</td>
</tr>
<tr>
<td>0</td>
<td>9/2$^+$</td>
<td>615</td>
<td>611</td>
<td>503</td>
</tr>
<tr>
<td>0</td>
<td>7/2$^+$</td>
<td>480</td>
<td>491</td>
<td>686</td>
</tr>
<tr>
<td>1/2</td>
<td>5/2$^+$</td>
<td>375</td>
<td>406</td>
<td>351</td>
</tr>
<tr>
<td>1/2</td>
<td>3/2$^+$</td>
<td>300</td>
<td>316</td>
<td>179</td>
</tr>
<tr>
<td>7/2</td>
<td>0</td>
<td>11/2$^+$</td>
<td>1005</td>
<td>1029</td>
</tr>
<tr>
<td>0</td>
<td>9/2$^+$</td>
<td>840</td>
<td>873</td>
<td>945</td>
</tr>
<tr>
<td>1/2</td>
<td>9/2$^+$</td>
<td>840</td>
<td>857</td>
<td>812</td>
</tr>
<tr>
<td>1/2</td>
<td>7/2$^+$</td>
<td>705</td>
<td>724</td>
<td>504</td>
</tr>
<tr>
<td>1/2</td>
<td>5/2$^+$</td>
<td>600</td>
<td>634</td>
<td>748</td>
</tr>
</tbody>
</table>
Table XVII. Some B(E2) values obtained with the modified E2 operator for low-lying levels. \( \lambda = \frac{q_2}{q_2} \) and changes from 0 in the O(6) limit to \( \pm \sqrt{7}/2 \) in the SU(3) limit.

\[
B(E2; N^+ \frac{3}{2}, \frac{3}{2} \rightarrow N^+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2}) = (\lambda_2)^2 \frac{1}{5} N(N+5) \left[ 1 + x \frac{\sqrt{\frac{2}{7}}}{\sqrt{2 / 7}} \frac{2N+5}{2N+4} \right]^2
\]

\[
B(E2; N^+ \frac{3}{2}, \frac{3}{2} \rightarrow N^+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2}) = (\lambda_2)^2 \frac{1}{5} N(N+5) \left[ 1 - x \frac{\sqrt{\frac{2}{7}}}{\sqrt{2 / 7}} \frac{2N+5}{2N+4} \right]^2
\]

\[
B(E2; N^+ \frac{3}{2}, \frac{3}{2} \rightarrow N^+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2}) = (\lambda_2)^2 \frac{1}{5} N(N+5) \left[ 1 + x \frac{\sqrt{\frac{2}{7}}}{\sqrt{2 / 7}} \frac{2N+5}{2N+4} \right]^2
\]

\[
B(E2; N^+ \frac{5}{2}, \frac{5}{2} \rightarrow N^+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2}) = (\lambda_2)^2 x^2 \frac{12N(N-1)(N+5)(N+6)}{7^2 35 (N+2)^2}
\]

\[
B(E2; N^+ \frac{5}{2}, \frac{5}{2} \rightarrow N^+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2}) = (\lambda_2)^2 x^2 \frac{N(N-1)(N+5)(N+6)}{70 (N+2)^2}
\]

\[
B(E2; N^+ \frac{5}{2}, \frac{5}{2} \rightarrow N^+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2}) = (\lambda_2)^2 \frac{1}{140} (2N+5)^2 \left[ 1 + x^2 \frac{2}{\sqrt{\frac{2}{7}}} \frac{N^2+5N+15}{(2N+4)(2N+5)} \right]^2
\]

\[
B(E2; N^+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rightarrow N^+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2}) = (\lambda_2)^2 \frac{6}{35} (N-1)(N+6) \left[ 1 + x^2 \frac{2}{9\sqrt{14}} \frac{2N+5}{2N+4} \right]^2
\]

\[
B(E2; N^+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rightarrow N^+ \frac{1}{2}, \frac{1}{2}, \frac{3}{2}) = (\lambda_2)^2 \frac{1}{10} (N-1)(N+6) \left[ 1 - x \frac{1}{3\sqrt{14}} \frac{2N+5}{2N+4} \right]^2
\]
Table XVIII. Comparison between calculated (with the modified E2 operator) and experimental B(E2) values in $^{191,193}\text{Ir}$.

The calculation has been done using Table XVII with $\chi = 1.2$ for $^{191}\text{Ir}$ and $\chi = 0.7$ for $^{193}\text{Ir}$. The experimental values are the same as in Table VIII.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$g_{ll}$</th>
<th>$\tau_{ll}$</th>
<th>$J^*<em>1 + \sigma</em>{1f}$</th>
<th>$\tau_{lf}$</th>
<th>$J^*_f$</th>
<th>$\begin{array}{ccc} B(E2) \text{(e}^2\text{b}^2) \end{array}$</th>
<th>Spin(6)</th>
<th>Perturb.</th>
<th>Exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{191}<em>{77}\text{Ir}</em>{114}$</td>
<td>17/2</td>
<td>3/2</td>
<td>7/2 $\rightarrow$ 17/2</td>
<td>1/2</td>
<td>3/2</td>
<td>.400</td>
<td>.327</td>
<td>.250 ± .020</td>
<td></td>
</tr>
<tr>
<td>17/2</td>
<td>3/2</td>
<td>5/2 $\rightarrow$ 17/2</td>
<td>1/2</td>
<td>3/2</td>
<td>.400</td>
<td>.616</td>
<td>.593 ± .040</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17/2</td>
<td>3/2</td>
<td>1/2 $\rightarrow$ 17/2</td>
<td>3/2</td>
<td>1/2</td>
<td>.060</td>
<td>.025</td>
<td>.052 ± .023</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17/2</td>
<td>5/2</td>
<td>3/2 $\rightarrow$ 17/2</td>
<td>3/2</td>
<td>1/2</td>
<td>.188</td>
<td>.233</td>
<td>.349 ± .174</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17/2</td>
<td>5/2</td>
<td>3/2 $\rightarrow$ 17/2</td>
<td>3/2</td>
<td>1/2</td>
<td>0</td>
<td>.040</td>
<td>.110 ± .010</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17/2</td>
<td>5/2</td>
<td>5/2 $\rightarrow$ 17/2</td>
<td>3/2</td>
<td>1/2</td>
<td>.323</td>
<td>.276</td>
<td>.259 ± .076</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17/2</td>
<td>5/2</td>
<td>5/2 $\rightarrow$ 17/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0</td>
<td>.020</td>
<td>.018 ± .003</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^{193}<em>{77}\text{Ir}</em>{116}$</td>
<td>15/2</td>
<td>3/2</td>
<td>7/2 $\rightarrow$ 15/2</td>
<td>1/2</td>
<td>3/2</td>
<td>.323</td>
<td>.288</td>
<td>.235 ± .025</td>
<td></td>
</tr>
<tr>
<td>15/2</td>
<td>3/2</td>
<td>5/2 $\rightarrow$ 15/2</td>
<td>1/2</td>
<td>3/2</td>
<td>.323</td>
<td>.421</td>
<td>.423 ± .038</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15/2</td>
<td>3/2</td>
<td>1/2 $\rightarrow$ 15/2</td>
<td>1/2</td>
<td>3/2</td>
<td>.323</td>
<td>.208</td>
<td>.222 ± .024</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15/2</td>
<td>5/2</td>
<td>3/2 $\rightarrow$ 15/2</td>
<td>3/2</td>
<td>1/2</td>
<td>.150</td>
<td>.170</td>
<td>.341 ± .311</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15/2</td>
<td>5/2</td>
<td>3/2 $\rightarrow$ 15/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0</td>
<td>.011</td>
<td>.085 ± .010</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15/2</td>
<td>5/2</td>
<td>5/2 $\rightarrow$ 15/2</td>
<td>5/2</td>
<td>3/2</td>
<td>.009</td>
<td>.002</td>
<td>.220 ± .260</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15/2</td>
<td>5/2</td>
<td>5/2 $\rightarrow$ 15/2</td>
<td>3/2</td>
<td>1/2</td>
<td>.258</td>
<td>.236</td>
<td>.107 ± .034</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15/2</td>
<td>5/2</td>
<td>5/2 $\rightarrow$ 15/2</td>
<td>1/2</td>
<td>3/2</td>
<td>0</td>
<td>.005</td>
<td>.009 ± .001</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table XIX. Comparison between calculated (with the modified transition operator) and experimental intensities of one-nucleon transfer reactions $^{193}_{77}\text{Ir} \rightarrow ^{194}_{78}\text{Pt}$. Intensities are normalized to unity for each $J^\pi_f$. The calculations have been done using Eqs. (4.22) and (4.23) with $\chi = 1$. The experimental values are the same as in Table IX.

<table>
<thead>
<tr>
<th>$^{194}<em>{78}\text{Pt}</em>{116}$</th>
<th>$^{193}<em>{77}\text{Ir}</em>{116}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{1f}$</td>
<td>$\tau_{1f}$</td>
</tr>
<tr>
<td>---------------------------</td>
<td>---------------------------</td>
</tr>
<tr>
<td>$15/2$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$15/2$</td>
<td>$5/2$</td>
</tr>
<tr>
<td>$15/2$</td>
<td>$1/2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$^{193}<em>{77}\text{Ir}</em>{116}$</th>
<th>$^{194}<em>{78}\text{Pt}</em>{116}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{1f}$</td>
<td>$\tau_{1f}$</td>
</tr>
<tr>
<td>---------------------------</td>
<td>---------------------------</td>
</tr>
<tr>
<td>$15/2$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$7$</td>
<td>$3$</td>
</tr>
<tr>
<td>$7$</td>
<td>$1$</td>
</tr>
<tr>
<td>$7$</td>
<td>$2$</td>
</tr>
</tbody>
</table>
Figure 1. Pictorial representation of the building blocks of odd-even nuclei in the interacting boson-fermion model.
Figure 1
Figure 2. Proton levels in the shell 50-82. Energies are counted from the $3s_{1/2}$ level.
PROTON LEVELS

\[ \begin{array}{c|c} \text{E (MeV)} & \text{n} \ell j \\ \hline 0 & \text{3s}_{1/2} \\ -0.35 & \text{2d}_{3/2} \\ -1.34 & \text{1h}_{11/2} \\ -1.67 & \text{2d}_{5/2} \\ -3.48 & \text{1g}_{7/2} \end{array} \]

Figure 2
Figures 3-4. Typical spectra of an even-even \((N=2, M=0)\) and an odd-even \((N=2, M=1)\) nucleus with Spin(6) symmetry. The energy levels are given by Eq. (3.15) with \((A_1/4)=0, (A/4)=90\) KeV, \((B/6)=60\) KeV and \(C=10\) KeV. The numbers on the top of the figures denote the Spin(6) quantum numbers \((\sigma_1, \sigma_2, \sigma_3)\). The numbers in parenthesis next to each level denote the Spin(5) quantum numbers \((\tau_1, \tau_2)\). The ground state is taken as zero of the energy. The lines connecting the levels denote large E2 matrix elements.
Figure 3
Figure 5. Schematic illustration of the constraints imposed by Eq. (3.16) on the boson-boson, fermion-fermion and boson-fermion couplings. The double line denotes a boson (B), the single line denotes a fermion (F) and the dashed line the interaction.
Figure 6. B(E2) values in units of \((q_2)^2\) in the Spin(6) limit for \(\sigma_1=N+1/2\) representation. The numbers on the left denote the quantum number \(\tau_1\). \(\Delta \tau_1=0\) transitions should be multiplied by \((2N+5)^2\) and \(\Delta \tau_1=1\) transitions by \((N-\tau_1+\frac{1}{2})(N+\tau_1+\frac{7}{2})\).
Figure 7. \( B(M1) \) values in units of \((q_1 - t_1)^2\) in the Spin(6) limit for \( \sigma_1 = N+1/2 \) representation. The numbers on the left denote the quantum number \( \tau_1 \). \( \Delta \tau_1 = 0 \) transitions should be multiplied by \((2N+5)^2/(2N+4)^2\) and \( \Delta \tau_1 = 1 \) transitions by \((N-\tau_1 + \frac{1}{2})(N+\tau_1 + \frac{7}{2})/(2N+4)^2\).
Figure 8. Schematic illustration of the selection rules for the one-nucleon transfer operator restricted by $\zeta' = -\sqrt{5} \zeta$, $\theta' = -\sqrt{5} \theta$. 
ONE NUCLEON TRANSFER

SELECTION RULES

\[ \Delta \sigma_1 = \pm \frac{1}{2}, \Delta \tau_1 = \pm \frac{1}{2} \]

\[ \begin{array}{c}
\tau = \frac{1}{2} \rightarrow 3/2^+ \\
(N - \frac{1}{2}, \frac{1}{2}, \frac{1}{2})
\end{array} \]

\[ \begin{array}{c}
\tau = \frac{5}{2} \rightarrow 3/2^+ \\
(N + \frac{1}{2}, \frac{1}{2}, \frac{1}{2})
\end{array} \]
Figure 9. Regions where a description in terms of the $SO^{(B)}(6)$ symmetry may be appropriate (dashed areas). The even-even nuclei shown in the chart are those for which the energy of the first excited $2^+$ state is known.
Figures 10-13. Examples of spectra with Spin(6) symmetry: $^{191}\text{Ir}$ (N=8, M=1), $^{193}\text{Ir}$ (N=7, M=1), $^{191}\text{Au}$ (N=8, M=1) and $^{193}\text{Au}$ (N=7, M=1). The energy levels in the theoretical spectra are calculated using Eq. (3.15) with $(B/6)=40$ KeV, C=10 KeV for Ir and $(B/6)=40$ KeV, C=20 KeV for Au isotopes. All states shown in the figures belong to the highest Spin(6) representation $(N+\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and they are labelled by the quantum numbers $(\tau_1, \tau_2)$ and $J^\pi$. The lines connecting the levels indicate large E2 matrix elements. The experimental data are from Refs. [Br 80, Sh 81].
Figure 12
Figure 13
Figure 14. Spin(6) assignments of the known $3/2^+$ states in $^{191}$Ir and of the known $0^+$ states in $^{192}$Pt. The states marked with an asterisk are presumably three and two quasiparticle states. Note in the $\Sigma=8$ multiplet the pairs of states with the same $\sigma_1$ value and $\tau_1=1/2$, $\tau_1=5/2$. They are separated by 179 KeV ($\sigma_1=17/2$) and 225 KeV ($\sigma_1=15/2$). The experimental values are from Ref. [Br 80].
Figure 15. A typical spectrum of an odd-even nucleus (N=2, M=1) with Spin(3) symmetry. The energy levels are calculated using Eq. (4.5) with \((A_1+A)/4=90\) KeV, \((B/6)=40\) KeV and \(C=10\) KeV.

The numbers on the top of the figure denote the \(\text{SO}^{(B)}(6)\) quantum numbers \((\sigma,0,0)\). The numbers in parenthesis next to each level denote the \(\text{SO}^{(B)}(5)\) quantum numbers \((\tau,0)\).
Figure 16. A typical molecular spectrum with $O(4)$ symmetry ($N=4$, $M=0$).

The energy levels are given by Eq. (5.12) with $A=80$ KeV, $B'=40$ KeV and $B=0$. 
Figure 16
Figure 17. A typical molecular spectrum with Spin(3) symmetry ($N=4$, $M=1$). The energy levels are given by Eq. (5.12) with $A=80$ KeV, $B'=40$ KeV and $B=10$ KeV.
Figure 17

\[
\begin{array}{c}
E \\
(\text{MeV})
\end{array}
\]

- $9/2^+$
- $7/2^+$
- $7/2^-$
- $5/2^-$
- $5/2^+$
- $3/2^+$
- $3/2^-$
- $1/2^-$
- $1/2^+$

\text{SPIN (3)}

- $1/2^+$ $\omega = 0$
- $3/2^-$
- $1/2^-$
- $5/2^+$ $\omega = 2$
- $3/2^-$
- $1/2^-$
- $1/2^+$ $\omega = 4$

Figure 17
Appendix A. Matrix Elements of Fermion Operators in the SU\(^{(F)}(4)\) Basis

Matrix elements of the fermion operators, Eq. (2.13), between \(M=0,1,2\) fermion states have been used in deriving the Spin(6) formulas in Chapter III. Here we list some of them.

\[
\langle \{1\}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \parallel A^{(j)} \parallel \{1\}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rangle = - \sqrt{2j+1} ,
\]

\(\text{(A.1)}\)

\[
\langle \{2\}, 1, \tau, L \parallel A^{(2)} \parallel \{2\}, 1, \tau', L' \rangle = - \sqrt{5} \left( \delta_{\tau,1} \delta_{\tau',0} + \delta_{\tau',0} \delta_{\tau,1} \right)
\]

Matrix elements of the fermion creation (annihilation) operator, \(a^\dagger (a)\), are also needed in calculation of one-nucleon transfer intensities and they are given by

\[
\langle \{0\}, 0, 0, 0 \parallel a \parallel \{1\}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \rangle = 2
\]

\(\text{(A.2)}\)

\[
\langle \{1\}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \parallel a \parallel \{2\}, 1, \tau, L \rangle = \sqrt{2(2L+1)}
\]
Appendix B. Matrix Elements of Boson Operators in the SO\(^{(B)}\)\(^{(6)}\) Basis

In this appendix, we collect matrix elements of the boson operators between the SO\(^{(B)}\)\(^{(6)}\) states which are extensively used in the Spin(6) calculations. The results are obtained using the formulas in Refs. [AI 79, SG 80]. We start with the operator \(n_d\)

\[
<N, \tau, L | n_d | N, \tau, L> = \frac{N(N-1) + \tau(\tau+3)}{2(N+1)}
\]  \hspace{1cm} (B.1)

From this we can derive the matrix elements of \(s^\dagger \) (\(s\)) using the relation \(s^\dagger s = N - n_d\)

\[
<N+1, \tau, L | s^\dagger | N, \tau, L> = \left[ \frac{(N+1)(N+4) - \tau(\tau+3)}{2(N+2)} \right]^{1/2}.
\]  \hspace{1cm} (B.2)

Since the operator \(B^{(1)}\) is proportional to the angular momentum operator, its matrix elements are trivially given by

\[
<N, \tau, L | B^{(1)} | N, \tau, L> = \left[ \frac{L(L+1)(2L+1)}{10} \right]^{1/2}.
\]  \hspace{1cm} (B.3)

For the operator \(B^{(2)}\) we have from Sect. 5 of [AI 79]

\[
<N, \tau+1, L | B^{(2)} | N, \tau, L'> = \left[ \frac{(N-\tau)(N+\tau+4)}{2\tau+5} \right]^{1/2} a_{LL'}.
\]  \hspace{1cm} (B.4)
where $a^{\prime}_{LL}$ denotes the matrix elements of $d^\dagger$ in the SU(5) basis

$$a^{\prime}_{LL} = \langle n_d=n+1, \nu=\nu+1, L \parallel d^\dagger \parallel n_d=n, \nu=\nu, L'\rangle$$  \hspace{1cm} (B.5)

Values of $a^{\prime}_{LL}$ for the first four bands are listed below

$$a_{2\tau+2,2\tau} = \left[\frac{(\tau+1)(4\tau+5)}{4\tau-1}\right]^{1/2}$$

$$a_{2\tau,2\tau} = -\left[\frac{(4\tau+1)(4\tau+2)}{4\tau-1}\right]^{1/2}$$

$$a_{2\tau,2\tau-2} = \left[\frac{(\tau-1)(4\tau+1)(4\tau+3)}{4\tau-1}\right]^{1/2}$$

$$a_{2\tau-1,2\tau} = -\left[\frac{2(\tau-1)(4\tau+1)}{2\tau-1}\right]^{1/2}$$

$$a_{2\tau-1,2\tau-2} = -\left[\frac{3(2\tau+1)}{\tau-1}\right]^{1/2}$$

$$a_{2\tau-1,2\tau-3} = \left[\frac{\tau(\tau-2)(2\tau+1)(4\tau-1)}{(\tau-1)(2\tau-1)}\right]^{1/2}$$

$$a_{2\tau-2,2\tau} = -4\left[\frac{2\tau(\tau-1)(\tau-2)}{(2\tau-1)(2\tau+3)(4\tau-1)}\right]^{1/2}$$

$$a_{2\tau-2,2\tau-2} = -2\left[\frac{(\tau-2)(2\tau+1)(4\tau-3)(4\tau+1)}{(\tau-1)(2\tau+3)(4\tau-1)(4\tau-5)}\right]^{1/2}$$

$$a_{2\tau-2,2\tau-3} = -2\left[\frac{(4\tau-1)(4\tau+1)(4\tau-3)}{(\tau-1)(2\tau-1)(2\tau-3)(2\tau+3)}\right]^{1/2}$$

$$a_{2\tau-2,2\tau-4} = \left[\frac{(\tau-3)(2\tau-1)(2\tau+1)(4\tau-1)(4\tau+1)}{(2\tau-3)(2\tau+3)(4\tau-5)}\right]^{1/2}$$  \hspace{1cm} (B.6)
Matrix elements of $d^\dagger (d)$ can be obtained from those of $B(2)$ by separating the two terms

\begin{align}
<N, \tau+1, L \parallel (d^\dagger \times s)(2) \parallel N, \tau, L'> &= \frac{N+\tau+3}{2N+2} <N, \tau+1, L \parallel B(2) \parallel N, \tau, L'> ,
\end{align}

(B.7)

\begin{align}
<N, \tau+1, L \parallel (s^\dagger \times d)(2) \parallel N, \tau, L'> &= \frac{N-\tau-1}{2N+2} <N, \tau+1, L \parallel B(2) \parallel N, \tau, L'> .
\end{align}

and using Eq. (B.2) for matrix elements of $s^\dagger (s)$

\begin{align}
<N, N, \tau+1, L \parallel d^\dagger \parallel [N-1], N-1, \tau, L'> &= \left[ \frac{(N+\tau+3)(N+\tau+4)}{2N+2(2\tau+5)} \right]^{1/2} a_{LL'} ,
\end{align}

(B.8)

Finally, we derive matrix elements the operator $(d^\dagger x d)(2)$. Since $d^\dagger (d)$ has selection rules $\Delta \sigma = \pm 1$, $\Delta \tau = \pm 1$, matrix elements of this operator satisfy the selection rules $\Delta \sigma = 0, \pm 2$, $\Delta \tau = 0, \pm 2$. For the representation $\sigma = N$, some of them are listed below

\begin{align}
<N, N, \tau, 2\tau \parallel (d^\dagger \times d)(2) \parallel [N], N, \tau, 2\tau > &= \left[ \frac{(2\tau+1)(4\tau+1)(4\tau+3)}{7(4\tau-1)} \right]^{1/2} \frac{N^2+4N+\tau^2+3\tau+5}{(N+1)(2\tau+5)} ,
\end{align}

\begin{align}
<N, N, \tau, 2\tau \parallel (d^\dagger \times d)(2) \parallel [N], N, \tau, 2\tau-2 > &= -\left[ \frac{2\tau(\tau-1)(4\tau+1)}{7(4\tau-1)} \right]^{1/2} \frac{N(4N+\tau^2+3\tau+5)}{(N+1)(2\tau+5)} ,
\end{align}
<[N], N, \tau+2, 2\tau+2 \parallel (d^+ \times d)^{(2)} \parallel [N], N, \tau, 2\tau>

= - \frac{1}{N+1} \left[ \frac{(N-\tau)(N-\tau-1)(N+\tau+4)(N+\tau+5)(4\tau+5)(4\tau+7)}{14 (2\tau+5)(2\tau+7)} \right]^{1/2}.

<B.9>

<[N], N, \tau+2, 2\tau+1 \parallel (d^+ \times d)^{(2)} \parallel [N], N, \tau, 2\tau>

= - \frac{1}{N+1} \left[ \frac{3(N-\tau)(N-\tau-1)(N+\tau+4)(N+\tau+5)(2\tau+3)(4\tau+3)}{14 (2\tau+5)(2\tau+7)} \right]^{1/2}.
Appendix C. ODDA Parameters Needed to Produce the Spin(6) Limit

In order to produce the Spin(6) energy spectrum, Eq. (3.15), from the computer program ODDA following parameters are needed

(a) Boson Hamiltonian parameters.

\[ \text{PAIR} = \frac{A}{2} \]
\[ \text{OCT} = \frac{B}{15} \]
\[ \text{ELL} = 2C + \frac{B}{15} \]

(b) Boson-fermion interaction parameters.

\[ \text{PDD}(1,1) = 3\sqrt{2} \left( \frac{A}{2} - \frac{B}{3} - 10C \right) = 3\sqrt{2} \left( \text{PAIR} - 5\text{ELL} \right) \]
\[ \text{PDD}(3,1) = 7\sqrt{2} \left( -\frac{A}{2} + \frac{B}{3} \right) = 7\sqrt{2} \left( -\text{PAIR} + 5\text{OCT} \right) \]
\[ \text{PSD}(1,1) = -\frac{\sqrt{5}}{2} A = -\sqrt{5} \text{PAIR} \]

In addition, mixing of the 3s_{1/2} level can be studied by introducing the parameters

\[ \text{PDD}(1,3) = -6\sqrt{5} C = -3\sqrt{5} \left( \text{ELL} - \text{OCT} \right) \]
\[ \text{PSD}(1,2) = -\text{PSD}(2,1) = \sqrt{5} \beta \]

where \( \beta \) is the coupling strength defined in Eq. (4.5).
References

Bo 51 A. Bohr, Phys. Rev. 81 (1951) 134.
Br 80 E. Browne, Nuclear Data Sheets 30 (1980) 658.
CNS 75 L. Corwin, Y. Ne'eman and S. Sternberg, Rev. Mod. Phys. 47 (1975)


B. Harmatz, Nuclear Data Sheets 23 (1978) 607.


J. Lukasiak, R. Kaczarowski, J. Jastrzebski, S. Andre' and J.


MYT 64 T. Marumori, M. Yamamura and A. Tokunaga, Prog. Theor. Phys. 31 (1964) 1009, 32 (1964) 726.


Ra 50 J. Rainwater, Phys. Rev. 79 (1950) 432.


Sh 81 V. S. Shirley, Nuclear Data Sheets, 32 (1981) 593.


Ta 81 I. Talmi, in Interacting Bose-Fermi Systems in Nuclei, F. Iachello


Wo 82 J. Wood, private communication.
