COLOR GLASS CONDENSATE
Nucleon at rest:
- Complicated non-perturbative object
- Contains fluctuations at all scales smaller than its own size
- Only the fluctuations that are longer lived than the external probe participate in the interaction process
- The only role of short lived fluctuations is to renormalize the masses and couplings
- Interactions are very complicated if the constituents of the nucleon have a non trivial dynamics over time-scales comparable to those of the probe
Nucleon at high energy:

- Dilation of all internal time-scales of the nucleon
- Interactions among constituents now take place over time-scales longer than the characteristic time-scale of the probe
  - The constituents behave as if they were free
- Many fluctuations live long enough to be seen by the probe. Nucleon appears denser at high energy (contains more gluons)
- Pre-existing fluctuations are totally frozen over the time-scale of the probe, and act as static sources of new partons
PARTON SATURATION

At low energy, only valence quarks in the hadron wave function
PARTON SATURATION

• When energy increases, new partons are emitted

• The emission probability is $\alpha_s \int \frac{dx}{x} \sim \alpha_s \ln(1/x)$ with $x$ the longitudinal momentum fraction of the gluon

• At small $x$ (i.e. high energy), these logs need to be resummed
As long as the density of constituents remains small, the evolution is linear:
The number of partons produced at a given step is proportional to the number of partons at the previous step (BFKL)

Sov. Phys. JETP 44 (1976) 443; 45 (1977) 199;
Eventually, the partons start overlapping in phase-space → parton recombination occurs

Then the evolution becomes non-linear:
The number of partons created at a given step depends non-linearly on the number of partons present previously

Iancu, Leonidov, McLerran (2001)
SATURATION CRITERION

L.V. Gribov, E.M. Levin and M.G. Ryskin, Physics Reports 100, Nos. 1 & 2 (1983) 1—150

• Number of gluons per area:

\[ \rho \sim \frac{xG(x, Q^2)}{\pi R^2} \]

• Recombination cross section:

\[ \sigma_{gg \rightarrow g} \sim \frac{\alpha_s}{Q^2} \]

• Recombination important when \( \rho \sigma_{gg \rightarrow g} \gtrsim 1 \), i.e. \( Q^2 \lesssim Q_s^2 \)

\[ Q_s^2 \sim \frac{\alpha_s xG(x, Q_s^2)}{\pi R^2} \sim A^{1/3} x^{-0.3} \]

• At saturation the phase-space density is:

\[ \frac{dN_g}{d^2x_\perp d^2p_\perp} \sim \frac{\rho}{Q_s^2} \sim \frac{1}{\alpha_s} \]
DEGREES OF FREEDOM


• Small $x$ modes have a large occupation number → they can be described by a classical color field $A^\mu$
• Large $x$ modes, slowed down by time dilation, are described as static color sources $\rho$
• The classical field obeys Yang-Mills equations:

$$D_\nu F^{\nu\mu} = J^\mu = \delta^{\mu+} \delta(x^-) \rho(\vec{x}_\perp)$$

The color sources $\rho$ are random, and described by a statistical distribution $W_{x_0}[\rho]$, where $x_0$ is the separation between “small $x$” and “large $x$”
• An evolution equation (JIMWLK) controls the changes of $W_{x_0}[\rho]$ with $x_0$ (generalizes BFKL to the saturated regime)
SEMANTICS

McLerran (mid 2000)

• Color: Gluons carry color charge

• Glass: The system has degrees of freedom whose time-scale is much larger than the typical time-scales for interaction processes. Moreover, these degrees of freedom are stochastic variables, like in “spin glasses” for instance

• Condensate: The soft degrees of freedom are as densely packed as they can (the density remains finite, of order $\alpha_s^{-1}$, due to repulsive interactions between gluons)
CORRELATION LENGTH

• In a nucleon at low energy, the typical correlation length among color charges is of the order of the nucleon size, i.e. \( \Lambda_{QCD}^{-1} \sim 1 \text{ fm} \). Indeed, at low energy, color screening is due to confinement, controlled by the non-perturbative scale.

• At high energy (small \( x \)), partons are much more densely packed, and it can be shown that color neutralization occurs in fact over distances of the order of \( Q_s^{-1} \ll \Lambda_{QCD}^{-1} \).

• This implies that all hadrons and nuclei behave in the same way at high energy. In this sense, the small \( x \) regime described by the CGC is universal.
COLOR GLASS CONDENSATE

- **BFKL equation**: evolution with $x$ of parton distributions, in the linear regime
- **McLerran-Venugopalan model**: a model in which the degrees of freedom are separated in fields (small-$x$ partons) and color sources (large-$x$ partons). This model assumes a fixed, gaussian, distribution of the color sources
- **CGC and non-linear evolution**: from renormalization group arguments, one derives the non-linear evolution equation for the distribution of color sources (JIMWLK). A mean field approximation of this is the Balitsky-Kovchegov equation
IPSAT MODEL


• I will not discuss in detail the derivation and form of the JIMWLK or BK evolution equations

• Instead, we will study a simple saturation model, that parametrizes the $x$-dependence and includes the spatial geometry of the target

• This model will be used to provide input to the IP-Glasma model for heavy ion collisions
IPSAT MODEL

- Deeply inelastic scattering off a proton
- Factorized processes: electron emits $\gamma^*$, $\gamma^*$ splits into quark-antiquark pair of spatial size $r_\perp$, this dipole interacts elastically with the target (eikonal process)
- The splitting is determined by the photon light-cone wave function computed in light-cone perturbation theory

- The elastic scattering of the dipole with transverse momentum transfer $\Delta^2=-t$ described by the scattering amplitude $A_{q\bar{q}}^{\text{el}}(x, r, \Delta)$
- As discussed before, the total cross section is given by
  $$\sigma_{q\bar{q}}(x, r) = \text{Im}iA_{q\bar{q}}^{\text{el}}(x, r, 0) = 2 \int [1 - \text{Re}S(b)] d^2b$$
  note that relates to the previously used $f$ by $f = \frac{iAk}{4\pi}$
IPSAT MODEL

• \( \sigma_{q\bar{q}}(x, r) = \text{Im} i A_{cl}^{q\bar{q}}(x, r, 0) = 2 \int [1 - \text{Re} S(b)] d^2 b \)

• Here \( S(b) \) is the S-matrix at distance \( b \) from the center
• The total cross section for a small dipole to pass through a dilute gluon cloud is proportional to the dipole area, the strong coupling constant, and the number of gluons in the cloud


\[
\sigma_{q\bar{q}} = \frac{\pi^2}{N_c} r^2 \alpha_s(\mu^2) xg(x, \mu^2)
\]

where \( xg(x, \mu^2) \) is the gluon density at some scale \( \mu^2 \)

• If the target is dense, the probability that the dipole does not scatter inelastically at impact parameter \( b \) is

\[
P(b) = 1 - \frac{\pi^2}{N_c} r^2 \alpha_s(\mu^2) xg(x, \mu^2) \rho(b, z) dz
\]
IPSAT MODEL

total prob. for no inel. interaction


\[
P(-L < z \leq L) = \lim_{n \to \infty} \prod_{i=0}^{n-1} P(z_i < z \leq z_{i+1})
\]

\[
= \lim_{n \to \infty} \prod_{i=0}^{n-1} \left( 1 - \sigma_{q \bar{q}} \rho(b, z_i < z \leq z_{i+1}) dz \right)
\]

\[
= \lim_{n \to \infty} \prod_{i=0}^{n-1} \exp(-\sigma_{q \bar{q}} \rho(b, z_i < z \leq z_{i+1}) dz)
\]

\[
= \exp(- \lim_{n \to \infty} \sum_{i=0}^{n-1} \sigma_{q \bar{q}} \rho(b, z_i < z \leq z_{i+1}) dz)
\]

\[
= \exp \left( - \int_{-L}^{L} \sigma_{q \bar{q}} \rho(b, z) dz \right)
\]

\[
= \exp(-\sigma_{q \bar{q}} T(b)) = P_{tot}(b) \quad \text{letting } L \to \infty
\]
IPSAT MODEL

• So the probability for the dipole not to interact inelastically passing through the entire target is:

\[ |S(b)|^2 = P_{\text{tot}}(b) = \exp \left( -\frac{\pi^2}{N_c} r^2 \alpha_s(\mu^2) xg(x, \mu^2) T(b) \right) \]

• Assuming the S-matrix element is predominantly real, we have

\[
\frac{d\sigma_{q\bar{q}}}{d^2 b} = 2(1 - \text{Re}S(b)) = 2 \left[ 1 - \exp \left( -\frac{\pi^2}{2N_c} r^2 \alpha_s(\mu^2) xg(x, \mu^2) T(b) \right) \right]
\]

• This is the Glauber-Mueller dipole cross section

• \(T(b)\) and \(xg(x, \mu^2)\) are determined from fits to HERA DIS data \((b, x, \text{and initial scale } (\mu_0)^2 \text{ dependence})\) and DGLAP evolution in \(\mu^2\)
The impact parameter dependent function $T(b)$ for a proton is assumed to be Gaussian:

$$T(b) = \frac{1}{2\pi B_G} \exp \left( \frac{-b^2}{2B_G} \right)$$

$B_G$ is assumed to be energy independent and fit yields $\sim 4$ GeV$^{-2}$

It is related to the average squared gluonic radius $\langle b^2 \rangle = 2B_G$

$b$ is smaller than the charge radius: $b = 0.56$ fm
(c.f. $R_p = 0.8751(61)$ fm)

We will later discuss how additional sub-nucleonic fluctuations of this shape affect observables
**Extracting Q_s**

\[
\frac{d\sigma_{q\bar{q}}}{d^2b} = 2(1 - \text{Re}S(b)) = 2 \left[ 1 - \exp \left( -\frac{\pi^2}{2N_c} r^2 \alpha_s(\mu^2) xg(x, \mu^2) T(b) \right) \right]
\]

---

Dipole amplitude saturates at 1!

Q_s is defined as the inverse scale where saturation effects begin

\[ N(R_s, x, b) = 1 - e^{-1/2} \]

\[ Q_s^2 = 2/R_s^2 \]
IP-GLASMA MODEL

• Incoming nuclei described within color glass condensate:
  large x d.o.f. are color sources, small x classical gluon fields

• Incoming currents need to be constructed first:
  • Sample nucleons from nuclear density distributions
    like in MC-Glauber model
  • Add the $T(b)$ at every transverse position
  • Extract $Q_s$ from the IPSat dipole amplitude
  • Obtain the color charge density: $g^4\mu^2 \sim (Q_s)^2$
    (the precise proportionality factor is of order 1, see Lappi, arXiv:0711.3039: $Q_s/(g^2\mu) = 0.75$)
• Incoming currents need to be constructed first:
  • Sample color charges $\rho^a$ from local Gaussian distributions with $\langle \rho^{a}_{A(B)}(\vec{x}_\perp) \rangle = 0$ and

\[
\langle \rho^{a}_{A(B)}(\vec{x}_\perp) \rho^{b}_{A(B)}(\vec{y}_\perp) \rangle = g^2 \mu^2_{A(B)}(x, \vec{x}) \delta^{ab} \delta^2(\vec{x} - \vec{y})
\]

• The sampled color charges comprise the eikonal color current that sources the small-\(x\) classical gluon fields

\[
J^\nu = \delta^\nu_{\pm} \rho_{A(B)}(x^\mp, \vec{x})
\]

• Gluon fields are determined via the Yang-Mills equations

\[
[D_\mu, F_{\mu\nu}] = J^\nu
\]

with $F_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu]$
In covariant gauge the Yang-Mills equations take the form of a Poisson equation:

\[ A_{A(B)}^{\pm} = -\frac{\rho_{A(B)}}{\nabla^2_{\perp}} \]

The fields before the collision are pure gauge fields and can be transformed to lightcone gauge via the Wilson line:

\[ V_{A(B)}(\vec{x}) = P \exp \left( -ig \int dx^- \frac{\rho_{A(B)}(x^-, \vec{x})}{\nabla^2_{\perp} - m^2} \right) \]

They read

\[ A_{A(B)}^i = \theta(x^{-(+)}) \frac{i}{g} V_{A(B)}(\vec{x}) \partial_i V_{A(B)}^\dagger(\vec{x}) \]

and

\[ A_{A(B)}^{- (+)} = A_{A(B)}^{+ (-)} = 0 \]
IP-GLASMA MODEL


• The fields in the forward light cone at time $\tau = 0^+$ are determined by the requirement that the equations of motion do not contain any singular terms for $\tau \to 0$


• More precisely, choose gauge $x^+ A^- + x^- A^+ = 0$, use the ansatz

$$A^i = \theta(x^-)\theta(-x^+) A^i_A + \theta(-x^-)\theta(x^+) A^i_B + \theta(x^-)\theta(x^+) \alpha^i_3$$

$$A^+ = \theta(x^-)\theta(x^+) x^+ \alpha$$

$$A^- = -\theta(x^-)\theta(x^+) x^- \alpha$$

and demand that

$$[D_\mu, F^{\mu i}] = 0 \quad \text{and} \quad [D_\mu, F^{\mu +}] = J^+$$

are not singular on the boundary as $\tau \to 0$
• We find in the forward lightcone:

\[ A^i|_{\tau=0^+} = \alpha_3 = A^i_A + A^i_B \]
\[ A^\eta|_{\tau=0^+} = \alpha = \frac{ig}{2} [A^i_A, A^i_B] \]
\[ \partial_\tau A^\eta|_{\tau=0} = \partial_\tau A^i|_{\tau=0} = 0 \]


• These are evolved in time with the source-free Yang Mills equations

• All this can be implemented on a spatial 2D lattice. Care has to be taken with parallel transporting when taking derivatives etc.

**IP-GLASMA MODEL: FIELDS**


**Action:**

\[
S = \int \mathcal{L} \sqrt{-\det(g_{\mu\nu})} d^4x = -\frac{1}{4} \int \tau F_{\mu\nu} F^{\mu\nu} d\tau dx dy d\eta
\]

because \(g_{\mu\nu} = (1, -1, -1, -\tau^2)\)

\[
S = \int \tau d\tau dx dy d\eta \left( -\frac{1}{2} F_{\tau\eta} F^{\tau\eta} - \frac{1}{2} F_{\tau i} F^{\tau i} - \frac{1}{4} F_{ij} F^{ij} - \frac{1}{2} F_{\eta i} F^{\eta i} \right)
\]

**Gauge condition:** \(A^\tau = x^+ A^- + x^- A^+ = 0\)

\[
E^i = \frac{\delta S}{\delta (\partial_\tau A_i)} = -\tau F^{\tau i} = -\tau g^{\tau \tau} g^{ij} F_{\tau j} = \tau \partial_\tau A_i
\]

is the conjugate momentum to \(A^i\)

Björn Schenke, BNL
INITIAL FIELDS

\[ E^i = \frac{\delta S}{\delta (\partial_\tau A_i)} = -\tau F^{\tau i} = -\tau g^{\tau \tau} g^{ij} F_{\tau j} = \tau \partial_\tau A_i \]

\[ E^n = \frac{\delta S}{\delta (\partial_\tau A_\eta)} = -\tau F^{\tau n} = -\tau g^{\tau \tau} g^{n\eta} F_{\tau \eta} = \frac{1}{\tau} \partial_\tau A_\eta \]

So the initial conditions are

\[ A^\eta|_{\tau=0^+} = \alpha = \frac{ig}{2}[A_A^i, A_B^i] \]

\[ E^n|_{\tau=0} = \frac{1}{\tau} \partial_\tau (-\tau^2 A^\eta|_{\tau=0}) = -2A^\eta|_{\tau=0} = -\tau \partial_\tau A^\eta|_{\tau=0} \]

\[ A_\eta|_{\tau=0} = -\tau^2 A^\eta|_{\tau=0} = 0 \quad \text{and} \quad E^i|_{\tau=0} = 0 \]
INITIAL FIELDS - SUMMARY


- The initial color electric and magnetic fields are

\[ E^i = \tau \partial_\tau A_i = 0 \]

\[ B^{x/y} = -\frac{1}{2} \epsilon^{(x/y)jk} F_{jk} = 0 \]

because \( A_\eta = 0 \) initially and gradients in \( \eta \) vanish \( j,k \in \{x,y,\eta\} \)

\[ E^n = -ig[A^i_A, A^i_B] \]

\[ B^n = F^{yx} = F_{yx} = \partial_y A_x - \partial_x A_y \]

Initially, the color electromagnetic fields only have longitudinal components!
Finally, let’s compute the stress energy tensor, defined by

\[ T_{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} + \frac{1}{4} g^{\mu\nu} g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} \]

The energy density is \( T^{T\tau} \)

\[ T^{T\tau} = \frac{1}{2} (E_{\eta})^2 + \frac{1}{2\tau^2} [(E^x)^2 + (E^y)^2] + \frac{1}{2} F_{xy} F_{xy} + \frac{1}{2\tau^2} (F_{x\eta}^2 + F_{y\eta}^2) \]

- Transverse electric field
- Longitudinal electric field
- Longitudinal magnetic field
- Transverse magnetic field
$T^{\eta \tau} = \frac{1}{2} (E^\eta)^2 + \frac{1}{2 \tau^2} [(E^x)^2 + (E^y)^2] + \frac{1}{2} F_{xy} F_{xy} + \frac{1}{2 \tau^2} (F_{x \eta}^2 + F_{y \eta}^2)$