Abstract

The giant resonance in $^{12}\text{C}$ occurs at 22.5 MeV. Above it at approximately 26 MeV is a strong second resonance thought to be built upon the 2+ first excited state of $^{12}\text{C}$. Both of these have been observed in photoexcitation and radiative capture studies, but the present theoretical explanations of $^{12}\text{C}$ have failed to duplicate the detailed shape of either resonance.

We have investigated the charged particle decays of $^{12}\text{C}$, $^{11}\text{B}(\text{p},p_0)^{11}\text{B}$, $^{11}\text{B}(\text{p},p_1)^{11}\text{B}(2.14)$, $^{11}\text{B}(\text{p},\alpha)^{8}\text{Be}$ to determine whether structure observed in the $(\text{p},\gamma)$ and $(\gamma,\text{p})$ studies implies activity in the particle channels. The angle and energy dependence of the proton and alpha decays for incident proton energies between 7.5 and 21.5 MeV is presented.

Particle-hole interpretations of the $^{12}\text{C}$ structure inadequately describe the elastic scattering; whereas, an optical model for low Z nuclei shows good agreement with the $p_0$ data. The inelastic processes are also best analyzed with a direct reaction theory such as the DWBA. Two alpha cluster models for $^{12}\text{C}$ are cited and a PWBA which includes the pickup of the target core is shown to reproduce the $(\text{p},\alpha)$ angular distributions.
A STUDY OF THE CHARGED PARTICLE EMISSIONS
OF $^{12}$C FOLLOWING PROTON CAPTURE BY $^{11}$B

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Abstract

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With a great deal of love I dedicate this thesis to my wife, Sylvia, to my children William and Melanie, to Gerald Bostock, and to the smiling people on this curious planet.
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Chapter I
Introduction

A. Theoretical Considerations

Once the features of a particular nuclear reaction have been delineated, an interpretation of those features requires the selection of a model to describe the results. This model must use as few parameters as possible. The true test of a model's efficacy is not merely that it reproduce the data, but that it yield a reasonable physical description of the system being studied.

Nuclear reaction models fall into two basic categories, compound and direct. A compound model assumes that when two identifiable pieces of nuclear matter interact for a time longer than a typical nuclear transit time a new compound system is formed in which the identity of the original constituents is lost. A further assumption of compound theories is that, within certain limitations, the decay of the new system is independent of its mode of formation. This compound theory has had considerable success in the study of light closed-shell or near-closed shell nuclei at low excitation energies where there are few degrees of freedom. The category of direct reaction theories includes plane-wave, distorted-wave, and optical model approximations. These theories involve a description of nuclear systems in terms of macroscopic (and often empirical) parameters such as well depth, diffuseness, and reflectivity. Clearly in nature
there will be an overlap between the domains of the compound and the direct theories.

The present experiment was designed to study the particle decay modes of excited states in $^{12}\text{C}$ at and above the giant dipole resonance and to examine their correlation with states observed in $^{11}\text{B}(p,\gamma)^{12}\text{C}$ experiments (AL64, KE69, BR70). Since compound theories such as those of Gillet and Arima (GI64, KA67, BR70) were available which predicted the gross structure for resonances found in these radiative capture experiments, it was of interest to try to explain $^{12}\text{C}$ particle decay modes within the same compound framework. It was also of interest to determine if the fine structure of the $(p,\gamma)$, $(\gamma,p)$, and $(\gamma,n)$ excitation functions (WU68, BA57) could be explained in terms of the behavior of the particle channels, since currently available particle-hole or coupled-channel calculations (MA69) have not been able to describe this fine structure.

A macroscopic analysis of the $^{11}\text{B}(p,p\gamma)$, $^{11}\text{B}(p,p\pi)$, $^{11}\text{B}(p,\alpha)$ reactions was also undertaken. This was accomplished with a recent optical model calculation (WA69) which successfully predicts elastic scattering results in light nuclei using a minimum of free parameters. From these optical model parameters one could then predict reaction and inelastic scattering yields with a distorted wave (DWBA) calculation. Prior to the development of the DWBA, pickup and stripping mechanisms were described in terms of the interaction between nuclear clusters on the target surface and the incident plane wave of the projectile. Only the reduced widths and spectroscopic factors for these clusters were needed in this method. Increased computer size and speed caused
this PWBA to be replaced by a DWBA which included the distorting
effect of the nuclear and Coulomb potentials on incident and
exiting waves. Recently the PWBA has been revived for use in the
analysis of multi-nucleon processes involving pickup or knockout
not only of surface clusters but also of the core. Projectile-
heavy particle interactions have only just begun to be introduced
into DWBA because of the difficulty of performing finite range
calculations for clusters having relative motion with non-zero
angular momentum. In our analysis we have used a PWBA including
contributions such as $^7\text{Li}$ pickup to interpret the $^{11}\text{B}(p,\alpha)^8\text{Be}$
reaction.

We present in what follows a particle-hole S-matrix derivation
of proton elastic scattering cross sections, the predictions of an
optical model and a DWBA for elastic and inelastic proton scattering
on $^{11}\text{B}$, and the results of a plane-wave calculation which includes
heavy particle interactions for the $^{11}\text{B}(p,\alpha)^8\text{Be}$ reaction. Experimental
data are compared with the various theories in an effort to determine
how best to interpret reactions induced in a $^{11}\text{B}$ target by a beam of
protons.
B. Experimental Methods

The giant dipole resonance in $^{12}$C occurs at 22.5 MeV, and a second giant resonance (thought to be a coupling of particle-hole configurations to the 2+ 4.44 MeV State in $^{12}$C) exists at $\sim$26 MeV. To supplement data accumulated below the giant resonance (RI68, BL62, BL64, and SE65) and to extend the available information beyond this second resonance, we designed our experiment to study the reactions $^{11}$B($p,p'_{0}$)$^{11}$B, $^{11}$B($p,p'_{1}$)$^{11}$B(2.14), and $^{11}$B($p,\alpha_0$)$^{8}$Be for proton energies between 7.5 and 21.5 MeV corresponding to $^{12}$C excitation energies between 22.5 and 35.5 MeV. Any large anomaly in the proton elastic scattering would effect the results of the available particle-hole theory—(BR70) for radiative capture and any correlation of proton yields with gamma yields—could be analyzed with this theory. We, therefore, were particularly interested in the ($p,p_{0}$) reaction. The $\alpha$ decay from $^{12}$C was also of interest since the previous gamma ray studies were sensitive mainly to the $T=1$ component of $^{12}$C levels, whereas alpha particles to the ground state of $^{8}$Be originate in $T=0$ natural parity levels of $^{12}$C. Thus it was hoped that by studying the ($p,p_{0}$), ($p,p_{1}$) and ($p,\alpha_0$) reactions we could learn more about the importance of isospin mixing and particle-hole configurations while at the same time investigating the role of direct reaction mechanisms at high $^{12}$C excitations.
Chapter II
Experimental Procedure

A. Detection of the charged particles

Figure 1 shows a level diagram for $^{12}$C indicating a number of particle emission thresholds. For protons incident on $^{11}$B the only open, charged-particle thresholds below 25.0 MeV are those for protons and alphas. This limitation on the emitted species allows one to separate decay products simply by a proper selection of detector thicknesses. The Q-values for the $^{11}$B(p,$^{3}$He)$^{9}$Be and $^{11}$B(p,d)$^{10}$B reactions are sufficiently negative, 10.32 and 9.23 MeV, respectively, that even after these channels open their reaction products never interfere with the analysis of the proton or alpha particle spectra. To extract elastic and inelastic proton yields we used single Si(SB) detectors of between 1000µ and 2000µ to detect protons with energies up to 17 MeV. For protons having more energy than 17 MeV two transmission mounted detectors were placed in a telescope array and their summed output was then analyzed. To displace the alpha peaks and the broad alpha background from regions of interest in the proton spectra 100µ Al foils were placed at the front of each detector. A typical particle spectrum (figure 2) contains elastic and inelastic peaks from $^{11}$B and the contaminants $^{10}$B, $^{12}$C, $^{15}$N, $^{16}$O etc. as well as the alpha peaks associated with decays to the $^{8}$Be ground state and to the $^{8}$Be first excited state which appears as a broad peak merged with the
breakup alpha particles underlying the elastic protons. In figure 3 we show a typical proton spectrum with the Al foil in place.

Alpha particle spectra were obtained with thin detectors which were thick enough (100-500\(\mu\)) to stop the most energetic alpha particles, but thin enough so that the protons deposited only a fraction of their maximum energy in them. The alpha spectrum of figure 4 shows the alpha background generated by \(^6\)Be and direct \(^{12}\)C breakup in addition to alpha particles to the ground and first excited states of \(^6\)Be.
B. Hardware and Electronics

In a typical experiment six detectors were placed on the baseplate of an ORTEC target chamber at 17°, 32°, 62°, 118°, 158° and 168°. By moving the baseplate 20° data were also collected at 37°, 52°, 82°, 98°, 138° and 148° relative to the incident protons. Several of these angles were selected because in the middle energy range their corresponding center-of-mass angle is a zero of a Legendre polynomial; for example at 12.00 MeV 82° is approximately 90° in the center-of-mass and hence a zero of all odd, polynomials. In addition 62°, 116° and 148° are zeroes for polynomials with \( L = 4 \), 2 and 5.

The target with which most of the data were taken was measured with an alpha gauge (C066) to be 310±20 \( \mu \text{g/m}^2 \) thick. Subsequent analysis indicated that of this 310± \( \mu \text{g/m}^2 \) 78% was \(^{11}\text{B} \), 15% \(^{16}\text{O} \), 3% \(^{12}\text{C} \), and 2% \(^{10}\text{B} \). The target fabrication technique and contaminant effects are discussed in Appendices 1 and 2.

The beam energy was varied between 7.5 and 21.5 MeV in 50 keV steps. Thus with two detector arrangements, one for the protons and one for the alpha particles, we obtained excitation functions at up to twelve angles for the reactions \(^{11}\text{B}(p,p_0)\) \(^{11}\text{B} \), \(^{11}\text{B}(p,p_1)\) \(^{11}\text{B}(2.14)\) and \(^{11}\text{B}(p,\alpha_0)\)\(^{8}\text{Be}\) for \(^{12}\text{C} \) excitations between 22.5 and 35 MeV.

Data were accumulated in six of eight octants of a 4096 channel analyzer. Routing pulses were generated by threshold discriminators and low voltage signals were biased out to minimize the analyzer.
dead time.

This dead time was determined by routing the digital output of
our beam current integrator into the eighth octant of the analyzer.
The ratio of routed to unrouted BCI pulses gave a measure of the
analyzer dead time. For a typical 300 nA proton beam this dead
time ranged between 4% and 16%. When a fixed charge had been
collected (100-500 μC), data were transferred from the analyzer
onto magnetic tape for future processing.
C. Data Reduction

To extract cross sections from these spectra the magnetic tape was read back into the analyzer memory and then sent through a data link to our IBM 360/44 where it was rewritten on 1600 bpi tape for editing and reduction. The reduction consisted of displaying each spectrum on a CRT, identifying the peaks of interest, defining limits of integration with a light pen, subtracting a linear background, and then determining the resultant cross section. To identify contaminant or unknown peaks, a special routine was incorporated into the data reduction program. This routine performed an energy calibration of the spectrum and was useful in kinematic studies. Two peaks from known reactions were identified with the light pen and the energy difference between these peaks established an energy/channel calibration. One could then select with the light pen or enter through the teletype other possible reactions; the expected peak position for these reactions were then indicated on the display by two arrows as is shown in figure 4.
D. Absolute Cross Sections and Uncertainties

In addition to the statistical uncertainties associated with the extraction of peak yields, absolute errors include uncertainties inherent in the measurement of target thickness and inclination, detector position and solid angle, charge collection, and analyzer dead time. An error of typically ±10% is associated with all data points. For data points without error bars the uncertainty is less than the size of the data point. Error bars for the angular uncertainty have not been shown. These should be taken as ±0.2° for detector geometries of between 0.4° and 1.4° of angular acceptance.

Additional corrections and uncertainties had to be included in the proton elastic scattering cross sections for angles forward of 82° where we did not separate the elastic yields from the 12C or 16O contaminants. Since the electronics were not setup to resolve the elastic peaks of carbon and oxygen for detector positions of less than 90°, the yields for 10B, 12C, and 16O were included in our integration of the elastic peak. The 12C and 16O yields at these angles were determined from the 12C(p, p') 12C and 16O(p, p') 16O angular distributions of LeVine (LE69) and Skwiersky (SK72). Legendre polynomial fits to their data were normalized to the present data at backward angles (where the three elastic peaks were resolved). The normalized Legendre fits were then used to predict forward angle 12C and 16O elastic yields and these were then subtracted from the unresolved elastic data.
Lowest lying charge particle thresholds for $^{12}\text{C}$ and selected energy level schemes for $^{11}\text{B}$ and $^{12}\text{C}$. The 22.6 MeV state in $^{12}\text{C}$ indicates the centroid of the first giant resonance.
12C CHARGED-PARTICLE THRESHOLDS

\[ \begin{array}{c|c}
10^7 \text{B+d} & 25.19 \\
5.02 & 3/2^- \\
4.44 & 5/2^- \\
2.12 & 1/2^- \\
\hline
11^7 \text{B+p} & 15.957
\end{array} \]

\[ \begin{array}{c|c}
9^7 \text{Be} + 3^3 \text{He} & 26.28 \\
\hline
16.11 & 2^+, 1 \\
15.11 & 1^+, 1
\end{array} \]

\[ \begin{array}{c|c}
8^7 \text{Be} + \alpha & 7.369 \\
\hline
4.439 & 2^+ \\
\hline
12^6 \text{C} & 0^+
\end{array} \]
Particle spectrum for protons on a $^{11}$B target. The detector thickness of 1500 $\mu$m was capable of stopping all emitted charged particles. Arrows show the peak positions of reaction products generated by the proton beam on the target and its contaminants.
FULL PARTICLE SPECTRUM
TAKEN WITH A THICK DETECTOR
$E_p = 12.00$ MeV, $\theta_{\text{lab}} = 118^\circ$.
Figure 3

Proton spectrum accumulated with a 2000 micron detector positioned behind a 100 micron Al foil. The foil was used to reduce the energy of the alpha particles sufficiently to remove them from the regions of interest.
DEGRADED PROTON SPECTRUM

$E_p = 15.00$ MeV, $\theta_{\text{lab}} = 118^\circ$
An alpha particle spectrum taken with a thin detector (500μ) so that protons which have greater than a 500 micron range in silicon would deposit only a small fraction of their energy in the detector. This method produced discrete alpha particle peaks well separated from the low energy proton background.
TYPICAL THIN DETECTOR ALPHA SPECTRUM

$E_p = 15.00 \text{ MeV}, \theta_{\text{lab}} = 62^\circ$
Photograph of a typical CRT display which includes a spectrum scaled according to a square root plot routine. The characters in the two right hand columns are used to establish the emitted species and the analyzer dead time, while the V's indicate the positions of identified peaks. The characters in the upper left of the display gave the center-of-mass cross section after a peak integration had been completed.
-Chapter III-
Results

A. The Energy Dependence of $^{11}\text{B} + \text{p}$ Cross Sections

Figures 6, 7, 8, 9 and 10 show the energy variation of the $^{11}\text{B}(\text{p},\text{p}_0)$, $(\text{p},\text{p}_1)$, and $(\text{p},\alpha_0)$ reaction cross sections at several laboratory angles. The $^{12}\text{C}$ excitation energy is given at the top border of each figure and the incident proton energy at the bottom. For comparison figure 11 is presented to summarize the available information on these and other reactions (TH69, TH71, BL62, BL64, BR67, BR70, KE69, AL64, SE65, RI68 and OV65). At low $^{12}\text{C}$ excitation energies (18.0 - 25.0 MeV) correlations between the particle and gamma channels are evident. The implication of these correlations to $^{12}\text{C}$ isospin mixing and to p-h theories have been discussed by other authors (e.g. WU68, BE63, and GI62). Proton emissions for higher $^{12}\text{C}$ excitations (above 25.0 MeV) show no significant resonant behavior which has not previously been observed; however, the $\alpha_0$ decays do show minor anomalies at 26.5, 28.3, 30 and 32 MeV. Structure in the $\alpha_0$ channel is also evident at 25.2 MeV in the $118^\circ$ excitation function. It is interesting to note that in the center-of-mass system the laboratory angle $118^\circ$ is a zero of the $P_2$ Legendre polynomial and that a $0^+$, $^{12}\text{C}$ quartet excitation has been predicted at approximately this energy by Arima and co-workers (AR70). Also predicted at such $^{12}\text{C}$ excitation energies is a $6^+$ member of an $\alpha$-cluster band built on the 7.65 MeV, $0^+$ state. According the Reynolds et al. (RE71) this state
should appear at 32.2 MeV. Our data indicate the possibility of a weak state at approximately this energy. Although none of the α-structure mentioned here appears in the current energy level literature (AJ68), no energy or width assignments for these levels were made because of the lack of significant detail in the excitation functions.

Above 29 MeV the excitation functions for all charged particle decays begin to show a damped structureless energy dependence. This absence of resonance behavior is indicative of direct reaction processes and should best be described by DWBA or optical model theories rather than microscopic particle-hole theories. For high excitation energies, with many modes of single particle motion available, the number and width of the microscopic p-h state becomes large, and their lifetime short, so that statistical, collective, or macroscopic descriptions of the nucleus are needed.
B. Angular Distributions

In figures 12, 13, and 14 the angular dependence for the three reactions is given for several representative energies. A smooth curve has been drawn through the data to guide the eye. Note that the $p_0$ and $p_1$ angular distributions are similar to one another and peaked at forward angles; whereas, the $\alpha_0$ angular variation is more oscillatory and peaked at backward angles.

This difference is reflected in the fitted values of the Legendre coefficients for the $p_1$ and $\alpha_0$ data. Reasonable fits to the $p_1$ data could be obtained using polynomial expansions with $L \leq 4$; the $\alpha_0$ angular distributions, however, required the inclusion of polynomials up to $L = 6$. The energy dependence of these Legendre coefficients, $P_L$, is given for the $p_1$ and $\alpha_0$ data in figures 15 and 16. These graphs represent a smoothed curve through the coefficients and discontinuities associated with angular distributions having too few data points have been neglected. Both sets of data were analyzed using a linear least square fitting routine which included all polynomials with $L$ between 0 and 6. In figure 15 the coefficients $A_5$ and $A_6$ have been omitted because, to within their uncertainties, their values were zero. Also omitted are $A_3$ and $A_4$ both of which were constant at $0.3 \pm 0.6$ and $-0.15 \pm 0.3$, respectively. The total cross sections and plotted coefficients of figures 15 and 16 have uncertainties ranging from 10% to 50%. Such large uncertainties arise when the number of data points is close to the
number of fitting parameters, $A_i$.

It was possible to obtain coefficients which gave adequate simulation of the data throughout the experimental range but which resulted in negative cross sections at zero degrees. To avoid this problem we introduced a weakly weighted spurious data point at zero degrees and thus forced a more realistic fit to the data. This technique was also used to fit the $^{12}$C and $^{16}$O proton elastic scattering results of LeVine (LE69) and Skwiersky (SK72) which we used to extract interpolated contaminant yields as described earlier.

The energy dependence of the total cross sections for the $p_1$ and $\alpha_0$ decays (figures 15 and 16) reflects the quiescence of the excitation functions and shows no distinct resonance phenomena above a $^{12}$C excitation energy of 28.0 MeV. A least square fit to the $p_0$ data indicated no discernable resonant behavior for $\sigma_{\text{total}}(p_0)$ as well. This complete absence of compound structure in all three reactions for the energy range 27.0 to 35.0 MeV was not anticipated. The data of figure 11 below 27 MeV, which are exemplary of all particle and gamma ray activity in this region, often do correlate well with many of the compound levels predicted by Gillet (GI63), Marangoni (MA69) and Arima (KA67). However, although both $T = 0$ and $T = 1$ p-h states are anticipated theoretically between 27 and 40 MeV (v. figure 5-3a of reference GI63) we observed none experimentally. The main reason that these states are predicted and not observed is that the initial lp-lh calculations neglected multiple particle-hole excitations and core deformations which cause the pure configurations to mix and disperse their strength.
Figure 6

Center-of-mass differential cross sections as functions of energy for the $^{11}\text{B}(p,p')^{11}\text{B}(\text{g.s.})$ reaction. The increased error associated with the 82°, 62°, and 52° results from a subtraction of the estimated contributions to these cross sections from target contaminants. $7.50\text{ MeV} \leq E_p \leq 21.45\text{ MeV}$
Excitation functions for the reaction $^{11}\text{B}(p, p')^{11}\text{B}(\text{g.s.})$

$I^\text{C}$ excitation energy

$\theta_{\text{lab}} = 168^\circ$

$\theta_{\text{lab}} = 158^\circ$

$\theta_{\text{lab}} = 148^\circ$

$\theta_{\text{lab}} = 138^\circ$

$\theta_{\text{lab}} = 118^\circ$

$\theta_{\text{lab}} = 98^\circ$

$\theta_{\text{lab}} = 82^\circ$

$\theta_{\text{lab}} = 62^\circ$

$\theta_{\text{lab}} = 52^\circ$

$E_{p,\text{lab}}$ (MeV)
$^{11}\text{B}(p,p')$ $^{11}\text{B}(2.14)$ center-of-mass differential cross sections for $32^\circ, 62^\circ, 148^\circ, 158^\circ$ and $168^\circ$. The laboratory proton energy appears as the abscissa and the $^{12}\text{C}$ excitation energy is indicated at the top of the figure. $12.0$ MeV $\leq E_p \leq 21.45$ MeV
EXCITATION FUNCTIONS FOR THE REACTION $^{11}\text{B}(p,p')^{11}\text{B}$ (2.14)

$^{12}\text{C}$ EXCITATION ENERGY

$\theta_{\text{lab}}=168^\circ$

$\theta_{\text{lab}}=158^\circ$

$\theta_{\text{lab}}=148^\circ$

$\theta_{\text{lab}}=62^\circ$

$\theta_{\text{lab}}=32^\circ$

$E_{p,\text{lab}}$ (MeV) vs $d\sigma/d\Omega$ (mb/sr)
Figure 8

The energy dependence of \( \frac{d\sigma}{d\Omega} \) c.m. for the \( p_1 \)'s at 52°, 82°, 98°, 118° and 138°. 7.50 MeV \( \leq E_p \leq 21.45 \) MeV.
EXCITATION FUNCTIONS FOR THE
REACTION $^{11}$B(p,p)$^{11}$B(2.14)

$^{12}$C EXCITATION ENERGY

$\theta_{\text{lab}} = 138^\circ$

$\theta_{\text{lab}} = 118^\circ$

$\theta_{\text{lab}} = 98^\circ$

$\theta_{\text{lab}} = 82^\circ$

$\theta_{\text{lab}} = 52^\circ$

$E_{p,\text{lab}}$ (MeV)
Figure 9

Excitation function showing $\frac{d\sigma}{d\Omega}$ c.m. for the $\alpha_0$'s at 17°, 32°, 62°, 148°, 158° and 168°. 11.50 MeV ≤ $E_p$ ≤ 18.85 MeV.
EXCITATION FUNCTIONS FOR THE REACTION $^{11}\text{B}(p,\alpha_0)^{8}\text{Be}\ (\text{g.s.})$

$I_2^C$ EXCITATION ENERGY.

$\theta_{\text{lab}}$ = 168°
$\theta_{\text{lab}}$ = 158°
$\theta_{\text{lab}}$ = 148°
$\theta_{\text{lab}}$ = 62°
$\theta_{\text{lab}}$ = 32°
$\theta_{\text{lab}}$ = 17°

$d\sigma/d\Omega \ (\text{mb/sr})$ vs. $E_{p,\text{lab}} \ (\text{MeV})$
Figure 10

\( \frac{d\sigma}{d\Omega} \) c.m. for \( \alpha \)'s at 52°, 82°, 98°, 118° and 138°.

7.50 MeV \( \leq E_p \leq 18.85 \) MeV.
EXCITATION FUNCTIONS FOR THE REACTION $^{11}\text{B}(p,\alpha_0)^8\text{Be}\ (\text{g.s.})$

$^{12}\text{C}$ EXCITATION ENERGY

$d\sigma/d\Omega \ (\text{mb/sr})$

$E_{p,\text{lab}} \ (\text{MeV})$

$\theta_{\text{lab}} = 138^\circ$

$\theta_{\text{lab}} = 118^\circ$

$\theta_{\text{lab}} = 98^\circ$

$\theta_{\text{lab}} = 82^\circ$

$\theta_{\text{lab}} = 52^\circ$
Excitation functions for several proton induced reactions using a $^{11}\text{B}$ target. Data depicted may be found in the following references (TH69, TH70, RI68, OV65, BL62, BL64, BR67 and AL64). Note the damping and disappearance of structure beyond a $^{12}\text{C}$ excitation energy of 28.0 MeV.
ENERGY DEPENDENCE OF SOME PROTON INDUCED REACTIONS ON A $^{11}$B TARGET

$E_x$ of $^{12}$C (MeV)

Differential Cross Section for the Reaction

$^{11}$B(p,α)$^8$Be (g.s.)
$\theta_{lab}=120^\circ$
$\theta_{lab}=118^\circ$

$^{11}$B(p,$p$)$^{11}$B (2.14)
$\theta_{lab}=120^\circ$
$\theta_{lab}=118^\circ$

$^{11}$B(p,p0)$^{11}$B (g.s.)
$\theta_{lab}=120^\circ$
$\theta_{lab}=118^\circ$

$^{11}$B(p,n)$^{11}$C
$\theta_{lab}=0^\circ$

Total Cross Section for the Reaction

$^{11}$B(p,γ)$^{12}$C (4.44)

$^{11}$B(p,γ0)$^{12}$C (g.s.)

$\sigma$ (µb)

$E_{p,lab}$ (MeV)
Figure 12—

Selected typical $p_\theta$ angular distribution for the energy range $E_p = 12.00$ to $21.50$ MeV. The solid curves are included as an aid to the eye and show the similarity of these angular distributions to one another over the entire energy range.
SELECTED ANGULAR DISTRIBUTIONS FOR THE REACTION $^7\text{B}(p,p_o)^7\text{B}$

$\frac{d\sigma}{d\Omega}$ (mb/sr) vs. $\theta_{\text{c.m.}}$ (deg) at different energies:
- 19.95 MeV
- 18.00 MeV
- 16.00 MeV
- 14.00 MeV
- 12.00 MeV
Typical angular distributions for the inelastically scattered protons. The distinct lack of energy variation in the angular dependence is common to both the $p_0$'s and $p_1$'s.
SELECTED ANGULAR DISTRIBUTIONS FOR THE REACTION $^{11}$B($p,p',^{11}$B (2.14))

$d\sigma/d\Omega$ (mb/sr)

$\theta_{c.m.}$ (deg)

20.00 MeV

18.00 MeV

16.00 MeV

14.00 MeV

12.00 MeV
Several $\alpha_e$ angular distributions which show considerably more energy and angular dependence than do the proton angular distributions.
SELECTED ANGULAR DISTRIBUTIONS FOR THE $^{11}\text{B}(p,\alpha_0)^8\text{Be}$ REACTION
Figure 15

Energy dependence of the Legendre coefficients obtained by a least squares fit to the $^{11}B(p,p')^{11}B(2.14)$ angular distributions with $L_{\text{max}} = 6$. Not shown are $A_3$, $A_4$, $A_5$ and $A_6$ which were nearly constant at $0.30 \pm 0.60$, $-0.15 \pm 0.30$, $\sim 0.0 \pm 1.0$ and $\sim 0.0 \pm 1.0$, respectively, for the entire energy range.
TOTAL CROSS SECTION AND COEFFICIENTS OF P₁ AND P₂
FOR THE REACTION ¹¹B(p, p₁)¹¹B (2.14)

¹²C EXCITATION ENERGY (MeV)

A₂

A₁

σₜ (mb)

Eₚ,lab (MeV)
Legendre coefficients for the reaction $^{11}\text{B}(p,\alpha)\ ^8\text{Be}$. A more systematic variation with energy is apparent for these $A_l$ than for the $A_l$ of either the $p_6$ or $p_1$ data. Note that these coefficients remain significant even up to $L = 6$ verifying the greater complexity of the angular distributions.
TOTAL CROSS SECTION AND COEFFICIENTS OF $P_1, P_2, P_3, P_4, P_5$ AND $P_6$ FOR THE REACTION $^{11}\text{B}(p,\alpha)^8\text{Be}$

$^{12}\text{C}$ EXCITATION ENERGY (MeV)
Chapter IV
Theoretical Analysis

A. Elastic Scattering

1. Compound S-matrix Analysis

In an effort to understand the results of his $^{11}\text{B}(p,\gamma)^{12}\text{C}$ studies, Brassard (BR70) developed a modified version of Gillet's (GI62) particle-hole theory. Brassard contended that although Gillet's calculations were adequate for predicting energy levels in $^{12}\text{C}$, the theory was not designed to describe angular distributions for radiative capture. Gillet's calculations were performed with an infinitely deep harmonic oscillator potential, and therefore the nucleon radial wave functions have zero amplitude outside of the nuclear volume. This type of bound state calculation ignores the possibility of nucleons in the continuum and the concept of a particle reduced width has no meaning.

As a more realistic model of $^{12}\text{C}$, Brassard calculated the radial solutions to the Schrodinger equation for a nucleon in a potential which closely resembled the shape of the $^{12}\text{C}$ ground state as determined from electron scattering. The resultant radial wave functions were then combined with angular momentum functions of good $\ell$ and $j$ to obtain intrinsic single-particle states for $^{12}\text{C}$. On the assumption that the intrinsic states differed from those of Gillet only at the channel surface, Gillet's amplitudes for the contribution of particle-hole states to a given $^{12}\text{C}$ level were retained.

This technique was found to be useful for predicting the angular distribution of gamma ray transition to the ground state of $^{12}\text{C}$. Gamma transitions to the 4.44 and the 9.64 MeV levels were not as well predicted, perhaps because these levels consist partially of a deformed
\[ ^{12}\text{C} \] core built of multi-particle multi-hole excitations neglected by the theory.

Contained in the mathematics of this gamma transition theory are the R and S matrices required to calculate proton elastic scattering from \[ ^{11}\text{B} \]. It was necessary merely to develop a j-j coupled expression for the \[ ^{11}\text{B}(P,P_0)^{11}\text{B} \] cross section to extract results from Brassard's S-matrix.

The next section describes a procedure for obtaining from an arbitrary model of a nuclear interior the matrix elements for R and S. The R-matrix links resonant phenomena within the nuclear volume to allowed decay channels at the surface of the nucleus. Once these surface boundary conditions are defined the form of all asymptotic wave functions outside the nucleus are determined. In Section 2 we also derive the relationship between the R and the S-matrix. The S-matrix specifies how the incident flux is distributed into all exit channels and provides a direct measure of the reaction's strength. With the nuclear information contained in the S-matrix and the appropriate angular momentum coupling procedure one obtains the angular dependence of the decay fragments as is seen in Appendix 3.

In Section 3 we derive the radial wave functions for a nucleon in a realistic spherical potential resembling that of \[ ^{12}\text{C} \]. The reduced widths, associated with these wave functions are derived and the assumptions of the particle-hole theory outlined.
2. The Nuclear R and S-Matrices

An S-matrix theory of compound nuclear systems as given by Maňaux and Wiedenmueller (MA69) or by Preston (PR63) is particularly useful near closed or quasi-closed shells. In this case $^{12}$C may be considered as a quasi-closed shell nucleus, since it contains a full complement of $P_{3/2}$ nucleons.

For the nuclear systems described by a discrete set of single particle eigenfunctions within the nuclear interior using the S-matrix theory, one may calculate the cross sections for both scattering and reaction processes. The degree to which an internal eigenstate resembles a specific exit or entrance channel, $c$, determines the probability of decay or formation through that channel. The R-matrix relates the eigenstate $\lambda$ to the wave function $\psi^c_E$ in the external region where the remnant subsystems exert no polarizing effect on one another.

If the internal region can be expressed by a Hamiltonian $H$ having the eigenfunctions $\{x_\lambda\}$, then the outgoing scattered wave function in the internal region is

$$\psi^c_E(+) = \sum_\lambda A^c_\lambda (E) X_\lambda .$$  \hspace{1cm} (IV-1)

The requirement that $\psi^c_E$ be continuous and have a continuous derivative at the channel surface $S_c$ determines the $A^c_\lambda$. For a specific channel $c$ in the external region in a $j$-$j$ coupled notation one has

$$\psi^c_E(+) = \frac{1}{r_c} U^{(+)}_{c}(r_c, k_c) \left[ \sum_\mu \left( j I \mu I |JM \right) \sum_\nu \gamma_{j\nu} |\bar{\Omega}_{JM}\right],$$  \hspace{1cm} (IV-2)
with

\[ y_{l}^{u} = \sum_{m} (\frac{l}{2} \frac{m}{2} \mu - m | j, \mu \rangle \lambda_{l}^{m} \Theta_{\frac{\mu-m}{2}} \]  \hspace{1cm} (IV-3)\

\( \Omega_{A-I} \) describes the intrinsic residual nucleus final state of A-1 nucleons and \( \Theta_{\frac{\mu-m}{2}} \) is the spin function for the nucleon in channel c.

We designate as the surface wave function, \( \varphi_{c} \), that portion of equation IV-2 within the square brackets and the reduced width amplitude \( \gamma_{\lambda c} \) as

\[ \gamma_{\lambda c} = \left( \frac{\hbar}{2M_{c}a_{c}} \right)^{1/2} \int ds \varphi_{c} X_{\lambda} = a_{c}^{2} \langle \varphi_{c} | X_{\lambda} > \]  \hspace{1cm} (IV-4)\

The curved bra indicates an integration over all but the radial coordinates while the notation \( \int ds \) implies an integration over all channel surfaces (v. MA69 p. 142). Two related expressions which will be useful are

\[ d_{\lambda c} = \gamma_{\lambda c} + \left( \frac{\hbar a_{c}}{2M_{c}} \right)^{1/2} \int ds (\nabla X_{\lambda}) \varphi_{c} \]  \hspace{1cm} (IV-5)\

and

\[ B_{c} = d_{\lambda c}/\gamma_{\lambda c} \]  \hspace{1cm} (IV-6)\

\( M_{c} \) is the reduced mass of channel c and \( a_{c} \) is the channel radius. The set \( \{ B_{c} \} \) is generally real and determines \( \{ X_{\lambda} \} \).

Suppose flux is incident on the nuclear volume through channel c
then the radial wavefunction \( U'_{c'} \) for the exit channel \( c' \) is

\[
U_{c'}^{(c)}(r_{c'}, k_{c'}) = x_{c'}^{(c)} O_{c'}(r_{c'}, k_{c'}) + y_{c'}^{(c)} I_{c'}(r_{c'}, k_{c'}) ,
\]

with

\[
y_{c'}^{(c)} = y_{c}^{(c)} \delta_{c c'} .
\]

The \( I_{c'} \) and \( O_{c'} \) are the incoming and outgoing waves typically composed of Coulomb or Bessel functions. From this definition of \( U_{c'}^{(c)} \) it follows that

\[
(U_{c'}^{(c)})' O_{c'} - U_{c'}^{(c)} = -2i k_c y_{c'}^{(c)} ,
\]

if one assumes plane wave asymptotic behavior for \( O_{c'} \) and \( I_{c'} \). We should also define two more quantities

\[
\gamma_{c'}^{(c)} = \sum_{\lambda} A_{\lambda}^{(c)} \gamma_{\lambda c'}
\]

and

\[
d_{c'}^{(c)} = a_{c'}^{(c)} \gamma_{c'}^{(c)} \left( \frac{O_{c'}'}{O_{c'}} \right) a_{c'}^{(c)} - 2i k_{c'} O_{c'}^{-1} \left( \frac{a_{c'}^2}{2M_{c'}} \right)^{1/2} y_{c'}^{(c)} .
\]
It is now convenient to introduce the expression obtained by applying the nuclear Green's Theorem with $\Psi_c$ and $X_\mu$ which reduces to

$$\left(E - E\right) A^{(c)}_{\mu} = \sum_{c'} \gamma_{\mu c'} d^{(c)}_{c'} - \gamma^{(c)}_{c'} d_{\mu c'} \quad \text{(IV-12)}$$

The shift function $S_c$ and the penetrability $P_c$ are given by

$$O_c(a_c,k_c) = (k_a a_c)^{-1/2} \Omega_{c_1 c_2}^{1/2} \quad \text{(IV-13)}$$

$$S_c + i P_c = \left( \frac{a_c O_c}{O_c} \right) a_c \quad \text{(IV-14a)}$$

$$\Omega_{c_1} = \left( \frac{I_{c_1}}{O_{c_1}} \right)^{1/2} \quad \text{(IV-14b)}$$

and

$$P_{c^-} = 0 \quad \text{(IV-15a)}$$

where $c^-$ implies a closed channel,
where \( c^+ \) implies an open channel.

When all of these expressions are substituted into IV-12 one obtains

$$\sum_{\lambda} A^{(c)}_{\lambda} \left\{ (E_{\lambda} - E) \delta_{\lambda \mu} + \sum_{\lambda'} \Delta_{\lambda \mu \lambda'} \right\} = - \frac{i \hbar}{2} v_{c}^{1/2} \Omega_{c}^{1/2} \Gamma_{c}^{1/2} \cdot \gamma^{(c)}_{c} \quad (IV-16)$$

where

$$\gamma^{(c)}_{c} = \frac{\hbar k}{c} \quad (IV-17)$$

$$\Gamma_{\lambda \mu c} = 2 P_{c} \gamma_{\lambda c} \gamma_{\mu c} \quad (IV-18)$$

$$\Gamma_{\mu c} = \Gamma_{\mu c} \quad (IV-19)$$

$$\Delta_{\lambda \mu c} = (B_{c} \gamma_{\lambda c}) \gamma_{\lambda c} \gamma_{\mu c} \quad (IV-20)$$

$$\Gamma_{\lambda \mu} = \sum_{c} \Gamma_{\lambda \mu c} \quad (IV-21)$$
\[ \Delta_{\lambda \mu} = \sum_{c} \Delta_{\lambda \mu c} \quad (IV-22) \]

When \( A \) is expressed as

\[ A_{\lambda \mu} = (\lambda)_{\lambda \mu} = (\varepsilon - \Omega + \frac{1}{2} i \Gamma)^{-1} \lambda_{\mu} \quad (IV-23) \]

with

\[ (\varepsilon)_{\lambda \mu} = \varepsilon \delta_{\lambda \mu} \quad (IV-24) \]

and

\[ (\Omega)_{\lambda \mu} = \Omega \delta_{\lambda \mu} \quad (IV-25) \]

then from equations IV-1 and IV-23 it follows that

\[ \psi^{(c)} = i \hbar^{1/2} \frac{1}{v} \Omega^{1/2} \frac{1}{v} \psi^{(c)} = \sum_{\lambda} A_{\lambda \mu} \Gamma^{1/2} \lambda \quad (IV-26) \]

The collision or S-matrix is defined in the external region as the ratio of the outgoing flux to that of the incoming flux which explicitly is

\[ S_{c | c'} = \left( \frac{v_{c'}}{v} \right)^{1/2} \left( \frac{x^{(c)}}{c'} \right) \quad (IV-27) \]
If one evaluates the overlap between the outgoing wave $\psi_c^0$, $\varphi_c^0$, and the expression for $\psi$ in both the internal and external region, one then obtains

$$S_{cc'} = \Omega_c \Omega_{c'} \delta_{cc'} + i \Omega_c \Omega_{c'} \sum \lambda \mu A_{c} \lambda \mu \Gamma_{c}^{1/2} \Gamma_{c'}^{1/2}.$$  \hspace{1cm} (IV-28)

This expression for $S$ may be substituted directly into any formula which relates the differential scattering or reaction cross section to the collision matrix (v. BL52). Thus if one has a set of model parameters to describe the nuclear interior, he can predict the results of any scattering experiments with this $S$-matrix.

The usefulness of the $S$-matrix in explaining how an incoming wave is dispersed by the nuclear region into the exit channels is most easily exploited by a decomposition of the $S$-matrix into a product of the $R$-matrix with several diagonal matrices. If in equation IV-12 the simplification is made that

$$\gamma_c^0 = \frac{d_c^{(c)}}{E_c} - B_{c'} \gamma_{c'}^0,$$  \hspace{1cm} (IV-23)

then

$$A_{c}^{(c)} = \sum_{c'} \frac{\gamma_{c'}^0 d_c^{(c)}}{E_c - E_{c'}}.$$  \hspace{1cm} (IV-30)

substituting this into equation IV-1 gives
\[ \Psi = \sum_{c} \sum_{\lambda} \frac{X_{\lambda} \gamma_{\lambda c}^c}{E_{\lambda} - E} d_{c'}^{(o)} \]  

By operating on this with \( \int_{S} \phi^{*} dS \) where \( S \) is the channel surface one has

\[ \gamma_{c}^{(c)} = \sum_{c} R_{cc'} d_{c'}^{(o)} \]  

with

\[ R_{cc'} = \sum_{\lambda} \frac{\gamma_{\lambda c} \gamma_{\lambda c'}}{E_{\lambda} - E} \]  

The reality of this \( R \)-matrix is assured if the proper phases are chosen for \( U_{c} \) and \( \phi_{c} \).

To obtain a relationship between the \( R \) and \( S \)-matrix one must evaluate the amplitudes \( X_{c} \) and \( Y_{c} \) in terms of the reduced widths \( \gamma_{\lambda c} \).

From equation IV-7 it follows that

\[ \Psi = \sum_{c} (x_{c}^{(c)} O_{c'} + y_{c}^{(c)} I_{c'}) \phi_{c} \]  

\[ \gamma_{c'}^{(c)} = \left( \frac{\hbar}{2M_{c'} c'} \right)^{1/2} [x_{c}^{(c)} O_{c'} + y_{c}^{(c)} I_{c'}] \]
Combining these two relations with equation IV-32 yields

\[ a^{-1/2} \left[ \gamma \gamma O + \gamma \gamma I \right] = B \left[ a^{1/2} \left( \gamma \gamma O' + \gamma \gamma I' \right) \right] - B \left[ a^{-1/2} \left( \gamma \gamma O + \gamma \gamma I \right) \right] . \]  

(IV-37)

All matrices except \( \mathbf{R} \) in this equation are diagonal; for example,

\( (a^{-1})_{cc} = \left( \frac{1}{a} \right)^\frac{1}{2} \delta_{cc} \).  

A further simplification of IV-36 results if one makes the substitutions

\[ a \gamma O' = a \gamma O - \mathbf{R} \gamma O = \gamma \gamma I' \]  

(IV-38)

and

\[ a \gamma I' = a \gamma I - \mathbf{R} \gamma I = \gamma \gamma L^{0*} \]  

(IV-39)

so that

\[ x = - y \gamma O^{-1} \left( I - \mathbf{R} \gamma L^{0} \right)^{-1} \left( I - \mathbf{R} \gamma L^{0*} \right) I \gamma \]  

(IV-40)

which gives as the matrix equivalent of equation (IV-27)

\[ \mathbf{S} = y \gamma O^{-1} \left( I - \mathbf{R} \gamma L^{0} \right)^{-1} \left( I - \mathbf{R} \gamma L^{0*} \right) I y^{-1} \]  

(IV-41)
This equation [c.f. eqn. V-93 of (CB70)] constitutes a definition of the nuclear S-matrix in terms of the known asymptotic parameters and the reduced widths contained in \( R \). It allows the direct determination of elastic scattering angular distributions as seen in Appendix 3.

3. Determination of the Reduced Widths

In order to utilize the R-matrix in IV-41 it is necessary to have the eigenvalues \( \lambda \) for the compound model being used. What follows is an outline of the procedure used by Gillet and by Brassard for obtaining these quantities.

The full Hamiltonian \( H \) for \( A \) interacting nucleons may be written as

\[
H = \sum_i H_i + \sum_{i<j} V_{ij} = H_0 + V
\]

Rather than performing a Hartree-Folk calculation to establish the eigenvalues of \( H_0 \), Gillet used a phenomenological technique in which the energies from pure single-particle or single-hole states were substituted. For the case of \(^{12}\text{C}\), the \( 3/2^- \) state in \(^{11}\text{C}\) at 18.72 MeV was taken to represent the \( 1p_{3/2} \) neutron-hole state while the \( 1/2^+ \) state at 35 MeV was assumed to be a pure neutron-hole state in the \( 1s_\nu \) shell. The proton energies were obtained from \(^{13}\text{N}\) and \(^{11}\text{B}\). For a specific single-particle or hole state one has \( E_{A+1} = E_A + \epsilon_A + m_n c^2 \) or \( E_{A-1} = E_A + \epsilon_A - m_n c^2 \) where \( E_A \) is the ground state energy of the \( A \) nucleon system and \( m_n \) the nucleon rest mass. The eigenenergy of an unperturbed particle-hole level in \(^{12}\text{C}\) is then
\[ \epsilon_{Aa} = \epsilon_A + \epsilon_a \]  

(IV-43)

The full Hamiltonian containing the residual interaction is now diagonalized in the space of the pure particle-hole wavefunctions to derive \( ^{12}\text{C} \) energy levels and eigenfunctions. The procedure is referred to by Gillet (GI64) as Approximation I and only \( lp-lh \) excitation of one or two \( \hbar \omega \) are allowed. Note that Approximation II which is known variously as the quasi-boson approximation or the RPA, permits ground state correlations and \( 2p-2h \) excitations but it provides only slightly better agreement with experiment. Due to the increase in complexity of Approximation II, Brassard chose the \( lp-lh \) approach for his work.

Gillet used as his effective two body interaction the potential

\[ V(r_{ij}) = V_0 \, f(r_{ij} / \mu) \left( W + BP_\sigma - HP_\tau + M P_0 P_\tau \right). \]  

(IV-44)

Here \( f(r_{ij} / \mu) \) has a Gaussian form of range \( \mu \). The diagonalization of \( H \) in \( lp-lh \) space determined the \( E_\lambda \). The parameters \( \mu, H, W, B \) and \( M \) were evaluated by least squares fitting a few known states in \( ^{12}\text{C} \).

The \( Y_\lambda \) are obtained by evaluating at the channel surface the overlap of the wave function for the internal region with that for the external region. The explicit form for the state vector \( \lambda \) is given by
where the $|\beta>\$ are the particle-hole eigenvectors of $H_0$ defined by Gillet to be

\[
|n_{(l^S_A)}\rangle = \sum_{m_{l^S_A}} (-1)^m_{l^S_A} (j_{l^S_A} m_{l^S_A} |JM)(-1)^{m_{l^S_A}} (t_{l^S_A} - m_{l^S_A}) \]
\]

The kets $|(l^S s) jm>\$ are defined as

\[
|(l^S s) jm> = \sum_{m_S m_l} (l^S m_l m_S | jm) |m_l> |m_S>.
\]
An improved notation suggested by Brassared creates a $1p-1h$ state for a proton ($p$) or a neutron ($n$)- particle ($a$) or hole ($b$) from the vacuum state ($^{12}\text{C }\text{g.s.}$) as follows

$$\begin{align*}
|\beta\rangle &= \sum_{m, j, j'} (-1)^{M} \frac{J}{\sqrt{2}} \left\{ a_{j}^{p} b_{j}^{b} + a_{j}^{n} b_{j}^{n} \right\} |\ell\ell'j'JM\rangle
\end{align*}$$

These states are related to Gillet's by

$$\begin{align*}
(-1)^{J} \frac{\pi}{\Gamma} |\lambda J - \text{MTO}\rangle &= \sum_{n, j, j'} (-1)^{J} \frac{\lambda^{J} \times n, j, j'}{\ell, j, j'} |\ell\ell'j'JM\rangle
\end{align*}$$

Reference (GI62) tabulates the X's need to determine $\gamma$.

The new notation was developed so that the state vectors would be time reversible and would have the proper normalization. Implied in it is that

$$\begin{align*}
|n\elljm\rangle &= \sum_{m, \ell, m} (-1)^{\ell - s + m} \frac{\ell s j}{m m - m} \left\{ \gamma m_{\ell} \right\} |n\ell m_{\ell} - s m_{s} - m\rangle
\end{align*}$$

and

$$\begin{align*}
|n\ell m_{\ell}\rangle &= i^{\ell} \gamma m_{\ell} |\theta, \phi\rangle U_{n, \ell} (r)
\end{align*}$$
The only quantities as yet undetermined are the \( u \) which were obtained by solving the radial Schrödinger equation for a single nucleon in a realistic \( ^{12}\text{C} \) potential \( V(r) \). \( V(r) \) was selected so that it described the charge distribution measured by Fregeau (FR55, FR56) through electron scattering. The analytic form of \( V \) is

\[
V_n(r) = V_o \left(1 + \frac{4r^2}{3a^2}\right) e^{-r^2/a_0^2}
\]

(IV-53)

with \( a_0 = 1.635 \text{fm} \) and \( V_o = 65 \text{ MeV} \). The proton potential \( V_p \) is the sum of \( V_n \) and \( V_{\text{Coul}} \) where \( V_{\text{Coul}} \) has the form

\[
V_{\text{coul}}(r) = 4\pi \int_0^\infty \frac{e^2}{r} \frac{dr'}{r'} \int_0^{r'} r''^2 \rho_p(r'') dr''
\]

(IV-54)

and

\[
\rho_p(r) = k V_n
\]

(IV-55)

To establish \( k \) one has assumed that the proton and neutron density distribution are given by \( V_n \) and requires that

\[
e 4\pi \int_0^{\infty} \rho_p(r'') r''^2 dr'' = Z_{\text{H,B}} e^2 = 7.190 \text{ MeV}_F
\]

(IV-56)

or equivalently that
Because the spin-orbit strength has been neglected \( y_\lambda (r) = r u_\lambda (r) \) may be simply extracted from the Schrödinger equation

\[
y''_\lambda (r) = \left\{ \frac{\ell (\ell + 1)}{r^2} + \frac{2m}{\hbar^2} (V(r) - E) \right\} y_\lambda (r),
\]

with \( V(r) = V_n (r) \) or \( V_P (r) \). \( y_\lambda \) can be solved for numerically with the boundary condition that the \( u_\lambda \) are finite at \( r = 0 \) and \( y_\lambda \rightarrow 0 \) for \( r \rightarrow 0 \) for bound states. In the case of unbound states one has that

\[
R \frac{u_\lambda' (r_c)}{u_\lambda (r_c)} = b_c = -1 \text{ at the channel radius } R_c \text{ of } 4.5f. \]

This value of \( b_c \) was chosen to minimize the level shift \( \Delta \) described earlier and brings the resonance energy close to that of the eigenvalue \( E_\lambda \).

It is now possible to evaluate the reduced widths \( \gamma_{\lambda c} \) and hence \( R \). Recall that \( \gamma_{\lambda c} \) represents the overlap of the surface function \( \phi_c \) or \( |c\rangle \) with the state vector \( |\lambda\rangle \) which implies that

\[
\gamma_{\lambda c} = \left( \frac{\hbar^2}{2M R_c} \right)^{1/2} \sum_{\beta} X_{\beta} \langle c | \beta \rangle \quad (IV-59)
\]

\[
\gamma_{\lambda c} = \left( \frac{\hbar^2}{2M R_c} \right)^{1/2} \sum_{\beta} X_{\beta} \gamma_{\beta c}, \quad (IV-60)
\]
for

\[ \Psi_\lambda = \sum_\beta X_\beta |\beta> \]  

(IV-61)

\(|c\) is composed of the intrinsic residual nucleus wavefunction and the angular and spin wavefunctions for the exiting nucleon

\[ |c\rangle = |I> |\ell jm \rangle \]

(IV-62)

If one assumes that the pure 1p-1h configuration \(|\beta\rangle\) may be decomposed into two components \(|\beta_1\rangle\) and \(|\beta_{11}\rangle\) resembling the direct product of a nucleon state \(|\beta_1\rangle\) coupled with an eleven nucleon system to total angular momentum \(J\) and that

\[ |\beta_1\rangle = U_{\beta_1} (r) |\ell jm \rangle , \]

(IV-63)

then

\[ \gamma_{\beta c} \sim (\ell jm |\beta_1\rangle < I |\beta_{11}\rangle \]

\[ \sim U_{\beta_1} (r) |R_c < I |\beta_{11}\rangle \]

(IV-64)
Antisymmetrizing $\beta$ and including the isospin coupling yields

$$\gamma_{\beta c} = \pm \left( \frac{\hbar^2}{2M_c R_c c^2} \right)^{1/2} <1|\beta_{11} > R_c U_{\beta_1} (R_c), \quad (IV-65)$$

where the minus sign is used only for an antisymmetric neutron channel. The quantity $R_c U_{\beta_1} (R_c)$ can be extracted directly from IV-58.

The evaluation of $<1|\beta_{11} >$, however, is not straightforward. $<1|\beta_{11} >$ vanishes if $J_I$ differs from $J_{\beta_{11}}$ or if the hole configurations of $|\beta_{11} >$ and $|I >$ are not the same. For the channels in which $|I >$ is either the $^{11}C$ or $^{11}B$ ground state the value of $<1|\beta_{11} >$ is a measure of the degree to which $|I >$ resembles a hole in the $1p_{3/2}$ shell. Brassard chose $<^{11}C(g.s.)|\beta_{11} >$ and $<^{11}B(g.s.)|\beta_{11} >$ to be unity for a $\beta_{11}$ which represents a $p_{1/2}^{-1}$ configuration. In the case where $|\beta_{11} >$ represents an $s_{1/2}^{-1}$ configuration it was contended that if sufficient energy, $\sim 35 $ MeV, is available to create the $s_{1/2}$ hole from the $p_{3/2}^{-1}$ initial state of $^{11}B$ then any of the eight $p_{3/2}$ particles could be promoted to the $p_{3/2}$ orbital or ejected into the continuum. The possibility of any of the $p_{3/2}$'s particles contributing to $|\beta_I >$ causes an enhancement of $\gamma_{\beta c}$ by an amount $\omega$ for the case of $|\beta_{11} >$ being an $s_{1/2}$ hole. One then has

$$\gamma_{\beta c} = \pm \left( \frac{\hbar^2}{2M_c R_c c^2} \right)^{1/2} R_c U_{\beta_1} (R_c) \cdot \omega. \quad (IV-66)$$
4. Predictions of the particle-hole theory

With an explicit relation for the reduced widths it is now possible to evaluate the R matrix for the channels of interest. It is of course necessary to limit the number of states and channels included in the determination of the R matrix. For the details of this truncation and for the explicit form of $Q$, $I$ and $V$ the reader is referred to reference (BR70). It should be mentioned here that the allowed set of reaction channels includes only those involving nucleons in the continuum since other channels will have zero overlap with the lp-1h internal wave functions. As a consequence this theory cannot predict the cross sections for multiple nucleon emissions. Also not included in $\{C\}$ are those target or residual excitations which are known not to be pure hole configurations. This is a severe limitation of the theory and prevents the calculation of proton inelastic scattering cross sections for the first $\frac{1}{2}^-$ state in $^{11}$B at 2.14 MeV which probably is a $p_{\frac{3}{2}}$ hole configuration coupled to the $2^+$ collective state in $^{12}$C.

The $S$ matrix of equation IV-41 which describes scattering from one channel in $\{C\}$ to another, can be substituted directly into the last expression of Appendix 3. In figures 17 and 18 the predicted elastic scattering cross sections are given for the energies indicated. The data and optical model predictions are also given. Note that the magnitude of the particle-hole cross sections tend toward agreement with the data at forward angles and diverge rapidly thereafter. At lower energies there is slightly better agreement in the shapes of the two
predictions (p-h and O.M.). At all energies, however, there is a great discrepancy between the magnitudes of the two theories.

As mentioned before, there is no provision in the p-h theory for decays into any channel which doesn't resemble a core plus a nucleon in the continuum. This constraint is analogous to an optical model without an imaginary channel, in that there is no way to account for a loss of flux due to decays into channels not included in the calculation. Thus at backward angles where the influence of nuclear effects is large the discrepancy is worse. At forward angles the elastic scattering is governed mostly by Coulomb effects and agreement is better. If the theory contained an energy dependent background term within the R matrix, the results would have been improved.

5. Elastic scattering according to the optical model

Recently Watson and collaborators (WA69) analyzed a wide range of elastic scattering data in lp-shell nuclei. Using a potential form

\[ V_{\text{opt}} = -V_{R} f(r) + iW_{v} f(r) + 4iaW_{s} f'(r) + \sigma \cdot \frac{1}{V_{so}} \left( \frac{\hbar r}{\sqrt{m_{c}}} \right)^{2} \frac{1}{\frac{1}{3} A \left( r_{so} A \right)^{1/3}} f'(r) + V_{\text{Coul}}(r) \]  

(IV-67)

where

\[ f(r) = \left[ 1 + \exp \left[ \left( r - \frac{A^{1/3}}{r_{o}} \right) / a_{o} \right] \right]^{-1} \]  

(IV-68)
they determined a reliable set of parameters which satisfactorily reproduced proton elastic scattering data from $^6$Li to $^{16}$O for a variety of proton energies. The solid curves of figures 17 and 18 are the angular distributions determined by the optical model code JIB3 (SI70) with Watson's parameters as input. Although the data at 8.5, 9.5 MeV and 10.5 MeV are sparse, the quality of the optical model fits is best at higher energies where less resonance structure is exhibited in the excitation functions. The overall agreement of the optical model results with our data indicates that even for such low Z nuclei as $^{11}$B direct reaction mechanisms can successfully describe proton elastic scattering.

Although we present only a few representative optical model angular distributions, we have made comparisons of the Watson JIB3 predictions with our data at 1.0 MeV increments from $E_{lab} = 8.0$ to 21.0 MeV. Because Watson's optical potentials contain an energy dependence which helps to parameterize non-local phenomena in a local calculation, the theory and our data follow one another very well with energy. The slow energy variation of the angular distributions can be seen in our figures 17 and 18, although it is more graphically depicted for $^{10}$B in figure 3 of (WA69). Note in the figure of Watson et al the similarity
of the $^{10}$B elastic scattering data to that of the $^{11}$B data. This similarity is an additional confirmation that for proton energies above 10 MeV nucleon elastic scattering has no real resonance behavior. Graphs of the energy dependent optical parameters (figures 4, 5 and 7 of WA69) obtained by fitting much p shell data show nearly a linear energy variation. From this we conclude that the smooth variation of our data is typical of that in 1p shell nuclei and that above approximately 10 MeV protons interact with targets of low Z as if these targets were composed of homogeneous nuclear matter.
B. Analysis of the Inelastic Scattering Data

At the bottom of figure 19 a typical angular distribution for the $^{11}\text{B}(p, p')^{11}\text{B}(2.14)$ reaction is shown. As is implied by the structureless excitation function for the $p_1$ Legendre coefficients, this angular distribution is typical of those found throughout the entire energy region. It is similar to those for the elastic scattering and lacks the oscillations predicted for the $p_0$ reaction by the particle-hole theory. The success of the optical model in the $p_0$ case encouraged us to attempt a DWBA analysis of the $p_1$ data using the distorted wave code DWUCK (KU71) and the optical model parameters of Watson et al. (WA69). The solid curve through the $^{11}\text{B}(p, p')^{11}\text{B}$ data represents the unadjusted predictions of this calculations for a pure $\ell = 2$ momentum transfer. Since $\ell = 0$ is the only other momentum transfer allowed and it failed to fit either the shape or magnitude of the data, we did not attempt to calculate any angular distributions for the case of mixed $\ell$ transfer.

The success of this DWBA for an $\ell$ transfer of two may be due to the similarity of the $\frac{1}{2}^-$ (2.14 MeV) in $^{11}\text{B}$ state to the $2^+(4.44\text{MeV})$ state in $^{12}\text{C}$. If the $\frac{1}{2}^-$ state resembles a $p_{\frac{3}{2}}$ proton hole coupled to the $^{12}\text{C} 2^+$ state, then proton inelastic scattering to the $2^+$ level should show a similarity to our $p_1$ data.

To corroborate this, detailed angular distributions were taken at 12.0 and 14.0 MeV for the reaction $^{12}\text{C}(p, p')^{12}\text{C}(4.44)$. The $^{12}\text{C}$ data and DWUCK predictions for $\ell = 2$ momentum transfer are given at the top of figure 19 and tend to confirm our hypothesis for the $^{11}\text{B} \frac{1}{2}^-$ level.
In both the $^{12}$C and $^{11}$B DWBA estimates shown in figure 19 there has been no adjustment of the spectroscopic factors to normalize the theory to the data. We again selected Watson's optical model parameters as input to DWUCK. The agreement of the DWUCK and JIB3 elastic scattering predictions with the data of LeVine and Parker (LE69) and the success of the DWBA code in reproducing both sets of inelastic data even though they differed from one another by a factor of $\sim 10$ gave us confidence in this parameter set.
C. Alpha Decay of $^{12}$C and the $(p,\alpha)$ Reaction

1. Possible Quartet and Alpha Cluster Structure in $^{12}$C

The excitation functions (figures 9 and 10) for the $(p,\alpha)$ reaction have more detail than do the $(p,p_1)$ or $(p,p_1)$ excitation functions. When distinct structure rises above a diffuse background, it may indicate that a state of simplified structure having a selectively long life time is formed at this particular energy. Since the $(p,\alpha)$ reaction has more structure than the two proton channels, it is possible that $^{12}$C has a significant alpha particle consituency at these high excitation energies.

A recent theory proposed by Arima et al. (AR70) discusses the energies associated with quartet-hole excitations in several $N = Z$ nuclei. At $\sim 25.2$ MeV these authors predict the existence of a $0^+$ state corresponding to the excitation of two quartets into the $1f-2p$ shell. Although the energy associated with this state could be significantly different from 25.2 MeV, it is interesting to observe that at $118^\circ$ (where $P_2$, the Legendre polynomial for $l = 2$ has a zero) there appears a sharp anomaly at exactly 25.2 MeV. Whether or not this can be associated with such a quartet excitation is uncertain; it is clear, however, that there is significant resonance behavior in the $\alpha_0$ decays near this energy.

An alternative to this theory which also might explain the enhanced structure in the $\alpha_0$ excitation functions was developed by Sheline and
Wildermuth (SH60). They visualized $^{12}\text{C}$ as an alpha cluster orbiting in the effective central anharmonic nuclear potential of $^{8}\text{Be}$. In this model the $^{12}\text{C}$ ground state consists of $^{8}\text{Be}$ in its ground state and an alpha cluster in a 3s orbital. A second $0^+$ state would result from an excitation of the alpha cluster to the 4s orbital. Both the ground and first excited $0^+$ states would have rotational levels built upon them. Reynolds et al. (RE71) believe to have found the $2^+$ and $4^+$ members of the second $0^+$ rotational band at 11.16 and 19.39 MeV and predict the $6^+$ level to be at approximately 32 MeV. This or a similar $\alpha$-cluster level built upon the $^{8}\text{Be}$ first excited state (SH60) may be responsible for the increased detail visible in figures 9, 10 and 16 at excitation energies beyond 25 MeV.

Although these theories can forecast the positions of a few isolated resonances, they were not developed to show detailed energy or angular dependence for the differential cross sections. In the next section a procedure for obtaining angular distributions from a direct reaction mechanism is proposed.
2. A Plane Wave Theory for the $^{11}\text{Be}(p,\alpha)^8\text{Be}$ Reaction

In an earlier section the results of linear least squares fit to the 
$(p,\alpha)$ angular distributions using Legendre polynomials showed that 
both even and odd coefficients for $\ell \geq 4$ were non-negligible and that 
the reaction cannot be well described by simple compound mechanisms. 
Because of the success of both the optical model and the distorted 
wave theory at describing our $(p,p_0)$ and $(p,p_1)$ data we felt that 
perhaps a direct reaction theory should also be applied to the $(p,\alpha)$ 
data. Such a theory has been proposed by Cavaignac et al. (CA71) who 
suggest that a PWBA calculation which includes heavy particle pickup is 
perferrable to a DWBA approach which omits it.

Our PWBA calculation includes contributions from three of four 
possible projectile-target initial state interactions. These inter­
actions and their associated matrix elements are diagrammed in figures 19 
and 20. The exact expression for the differential cross section con­
tributed by these matrix elements would have the form

$$\frac{d\sigma}{d\Omega} = \frac{\mu_i \mu_f}{(2\pi\hbar^2)^2} \frac{k_f}{k_i} \frac{1}{S^2 S^2} \frac{1}{P^2 T^2} \sum_{i=1}^{4} A_i (\nu_p, \nu_T, \nu_\alpha, \nu_R) \left| \sum_{i=1}^{4} A_i (\nu_p, \nu_T, \nu_\alpha, \nu_R) \right|^2$$

(IV-70)  

However, Cavaignac and collaborators had determined from preliminary 
 studies that the contribution to this cross section from the heavy 
 particle knockout was almost isotropic. Although calculations by 
 Gambarini et al. (GA69) tend to disagree slightly with this conclusion,
we have followed the suggestion of Honda and co-workers and have set
the heavy particle knockout term to zero. A further simplification
was made in which only the alpha knockout contribution, \( A_{\text{akko}} \), or the
triton pickup contribution, \( A_{\text{tpu}} \), is added coherently to the heavy
particle pickup. Since \( A_{\text{akko}} \) and \( A_{\text{tpu}} \) are very similar analytically,
and the number of free parameters is thus significantly reduced, such
a simplification seems justifiable.

What follows is a detailed calculation of a typical matrix element,
the heavy particle pickup term, \( A_{\text{hppu}} \), and a presentation of the
explicit relationships for \( A_{\text{akko}} \) and \( A_{\text{tpu}} \). The amplitude for the heavy
particle pickup contribution can be written as

\[
A_{\text{hppu}} = \langle \psi_{\text{f}} | V_{\text{pc}} | \psi_{\text{i}} \rangle = \iiint d\vec{r}_1 d\vec{r}_2 d\vec{r}_3 d\vec{r}_4 d\vec{r}_5 \psi_{\text{f}}(\vec{r}_1) V_{\text{pc}}(\vec{r}_2, \vec{r}_3, \vec{r}_4, \vec{r}_5) \psi_{\text{i}}(\vec{r}_1), \quad (IV-71)
\]

with

\[
\psi_{\text{i}} = \varphi_{\text{p}}(\vec{k}_1, \vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \sum_{\ell m \alpha \alpha} \langle \ell mL m | L_{\text{pc}} L_{\text{pc}} \rangle \psi_{\text{i}}(\vec{r}_1)
\]

\[
\cdot X_{\alpha}^i(\vec{r}_c) \cdot Y_{\ell}^{m\alpha}(\vec{r}_{\alpha c}) \cdot Q_{\alpha}(\vec{r}_{\alpha c}), \quad (IV-72)
\]
The \(\phi\)'s are plane waves for the entrance and exit channels, the \(X\)'s are intrinsic spin functions, and \(\gamma_{\ell, m}^n\) gives the angular and radial dependence of a cluster relative to its core. \(\psi_\alpha\) describes the probability of finding a cluster of four nucleons resembling an alpha particle in the target nucleus and from it one can define the quantity

\[
O_\alpha = \int \psi_\alpha^* (r_\alpha) X^f (r_\alpha) \, dr_\alpha .
\]

The radial dependence of cluster relative motion can be further expanded as

\[
\mathcal{R}_f = \left( \frac{3}{R_1} \right)^{1/2} \Theta_{\ell i} S_{\ell i} \left\{ \begin{array}{c} h_{\ell i} (ik_r) \\ h_{\ell i} (ik_{R_1}) \end{array} \right\} \text{ for } r > R_1 .
\]

\(Q_{\ell i}\) has an asymptotic dependence governed by \(h_{\ell i} (ik_{R_1})\), a Hankel function of the first kind, and is normalized to unity at \(r = R_1\) provided one assumes a radial reduced width \(\theta_{\ell i}\) and a spectroscopic factor \(S_{\ell i}\).

With this notation \(A_{ppp}\) reduces to
\[ A_{\text{hppu}} = \frac{-\hbar^2}{2\mu pc} \left( \frac{9}{R_3 R'_3 R_3} \right)^{1/2} \left( 4\pi \right)^2 \sum_{\ell_p \ell'_p} \frac{S_{\ell_p} S_{\ell'_p} \Theta_{\alpha} \Theta_{\alpha}}{p_{\alpha} \alpha} \]

\[ \frac{i (\ell_p - \ell'_p \alpha)}{K_p^2 + \kappa_{\alpha}^2} \cdot J_{\ell_p} (K R, \kappa R) J_{\ell'_p} (K R, \kappa R) \]

\[ = \sum_{\mu_p \mu_p'} \frac{S_{\mu_p} S_{\mu_p'} \left| j_{\mu_p} (K R) \right| \left| j_{\mu_p'} (K R) \right|}{(2 \ell_p + 1 + \kappa R)} \]

\[ \cdot Y_{\ell_p}^{\mu_p} (K_R) Y_{\ell'_p}^{\mu_p'} (K_R) \cdot (IV-76) \]

The J's are Wronskians defined by

\[ J_{\ell} = KR j_{\ell+1} (KR) + C_{\ell} (KR) j_{\ell} (KR) \cdot (IV-77) \]

\[ C_{\ell+1} = (\kappa R)^2 / (2\ell + 1 + \kappa R) \cdot (IV-78) \]

\[ \kappa = \left( 2 \mu E_{b.e.} \right) / \hbar, \quad E_{b.e.} = \text{cluster binding energy} \cdot (IV-79) \]

\[ K_p = k_p + \frac{mp_{\alpha}}{mR} \cdot (IV-80) \]

\[ K_{\alpha} = k_{\alpha} + \frac{m_{\alpha} \kappa}{mT} \cdot (IV-81) \]
Similarly $\Lambda_{\text{tpu}}$ and $\Lambda_{\text{ko}}$ are

$$\Lambda_{\text{tpu}} = -\frac{n^{2/3}}{2\mu_{pt}} (8\pi)^{1/2} (\kappa_{pt} R_{t})^{1/2}$$

$$\sum_{\ell_i \ell_t} S_{\ell_i \ell_t} O_{\ell_i \ell_t \alpha} \frac{J_{\ell_t} (K_{R_{t}} R_{t})}{(K_{R_{t}})^2 + (\kappa_{R_{t}} R_{t})^2}$$

$$\sum_{m_t \mu_t} (S_{\nu} S_{\nu} |00\rangle (S_{\nu} S_{\nu} \ell_t m_t |j_{\mu_t}) (j_{\mu_t} L_{R_{t}} R_{t} T_{R_{t} T_{t}})$$

$$Y_{\ell_t}^m (K_{R_{t}}) , \quad \text{(IV-82)}$$

for

$$K_{R_{t}} = k_{\alpha} - \frac{mR}{mT} k_{p} \quad \text{(IV-83)}$$

and

$$\Lambda_{\text{ko}} = - (4\pi)^{3/2} V_{p}\alpha R_{t}^3 \sum_{\ell_p \ell_t} S_{\ell_p \ell_t} \Theta_{\ell_t \alpha} \alpha_{\ell_t} \Theta_{0 \alpha} \alpha_{-L}$$

$$\sum_{\ell_p \ell_t} J_{\ell_t} (K_{R_{t}} R_{t}) (K_{R_{t}} R_{t}) \frac{J_{\ell_t} (K_{R_{t}} R_{t})}{(K_{R_{t}} R_{t})^2 + (\kappa_{R_{t}} R_{t})^2}$$

$$\sum_{\mu_p \mu_p \mu_p} (S_{\nu} \ell_{p} m_{p} |j_{\mu_p}) (I_{m_{p}} \ell_{p} m_{p} |I_{m_{p}})$$

$$G_{\ell_p \ell_t} \text{ m} \text{ m} |I_{R_{t}} R_{t} \rangle \rangle (I_{m_{p}} \ell_{p} m_{p} \text{ M} |I_{R_{t}} R_{t} \rangle \rangle (-1)^{m_t} Y_{L}^{M} (K_{R_{t}}) \quad \text{(IV-84)}$$

for

$$K_{R_{t}} = \frac{Mc}{MR_{t}} k_{R_{t}} \quad \text{(IV-85)}$$
In equation IV-84 the relations

\[ V(r) = -V_p \frac{4\pi}{3} R_k^3 \delta \left( |r_p| - |r_\alpha| \right) \]

and

\[ h_{q_1}^{(iK,R)} h_{q_2}^{(iK_2,R)} \approx h_L^{(iKR)} \left( \frac{K}{K_1 K_2 R} \right) \]

for \( L = l_1 + l_2 \) and \( K = K_1 + K_2 \).

have been assumed.

The PWBA estimates of the angular distributions for the alphas were calculated for two cases. In the first approximation the coherent sum of \( A_{\alpha \text{ko}} \) and \( A_{\text{hppu}} \) as in figure 20 was squared and summed over allowed channel spins as indicated in equation IV-70. For the second case \( A_{\alpha \text{ko}} \) was replaced by \( A_{\text{tpu}} \) and the processes repeated. In figure 22 the relative importance of the triton and heavy particle pickup terms is illustrated. The triton pickup and \( \alpha \) matrix elements give a predominantly forward contributions while that for the heavy particle pickup gives a backward peaking.

At the seven energies where we have taken detailed angular distributions a least squares PWBA fits to the data were performed. The cut-off radii for the triton, proton, and alpha clusters, as well as the triton and alpha reduced widths were the only parameters allowed to vary in the fitting routine. The spectroscopic factors used are tabulated in Cavaignac's article and were derived from a p-shell intermediate coupling model such as Kurath's (KU56). The non-linear least square fitting routine developed by Bevington (BE69) was permitted to search until the change in \( \chi^2 \) was less than 1%. Results for the two cases are shown in figures 23 and 24.

In an attempt to further improve the PWBA results we modified the
plane wave code to include first order Coulomb distorting effects as suggested by Austern (AU64). The wave vectors \( \mathbf{k}_i \) were replaced by their localized WKB counterparts in which a Coulomb potential had been subtracted from the kinetic energy. In the entrance channel, for example

\[
\left| \mathbf{k}_t \right| \sim \sqrt{2\mu pt} \left( \frac{E_{pt} - V_{coul}}{p_t} \right) / h. \tag{IV-86}
\]

This modification in general produced a slight deterioration of the agreement and caused the fitting routine to increase the value of the optimum cut-off radius to compensate for the decrease due to the Coulomb correction. The Coulomb corrected angular distributions are shown in figures 25 and 26.

The shift introduced by the Coulomb correction is particularly apparent when one plots the best fit cut-off radii as a function of energy (figure 27). The solid and double dashed lines represent those radii predicted by fits with \( A_{hppu} \) plus \( A_{tpu} \) or \( A_{hppu} \) plus \( A_{ako} \) respectively. The dotted lines represent their Coulomb corrected counterparts which in every case are merely displaced from the non-corrected radii by a few tenths of a fermi.

Figure 28 gives the energy dependence of the reduced widths, \( \theta_t \), \( \theta_\alpha \), and of \( V_{p\alpha} \). These quantities serve mainly to normalize each matrix element. At the top of this figure is plotted the energy variation of \( \chi^2 \) for each case and for its Coulomb corrected version. It is seen that the best results were obtained when the triton and heavy particle
pickup terms were combined and there was no adjustment of the wave vectors.

The success of this rather simplified model of the \((p,\alpha)\) reaction is encouraging. Although finite ranged DWBA codes including heavy particle pickup are now available, this PWBA calculation with projectile-core interactions may still be the simplest means of examining the \((p,\alpha)\) reaction for large incident proton energies. The importance of the heavy particle pick up contribution to the differential cross section is evident from figure 22. In order to reproduce the angular distributions any model of the \((p,\alpha)\) reaction should contain some projectile-heavy particle interaction. The success of the PWBA is, in part, due to the inclusion of such a term, i.e. the heavy-particle pickup.
Optical model (solid curve) and particle-hole (dashed curve) predictions for the proton elastic scattering angular distribution. The calculations were performed at the energies indicated by arrows on the excitation function.
Optical model (solid curve) and particle-hole (dashed curve) predictions for the proton elastic scattering angular distributions at energies indicated by arrows in the excitation function.
DWBA estimates for the inelastic scattering of protons by a $^{11}$B and a $^{12}$C target. The DWBA results which use the optical parameters of Watson (WA69) and $\ell$ transfer of 2 have not been normalized to the data.
ANGULAR DEPENDENCE OF PROTONS INELASTICALLY SCATTERED FROM $^{12}$C AND $^{11}$B

$^{12}$C($p$,p$^\prime$)$^{12}$C ($4.44$)
$E_{p,\text{lab}} = 14.00$ MeV

$^{12}$C($p$,p$^\prime$)$^{12}$C ($4.44$)
$E_{p,\text{lab}} = 12.00$ MeV

$^{11}$B($p$,p$^\prime$)$^{11}$B ($2.14$)
$E_{p,\text{lab}} = 15.50$ MeV

$\sigma/d\Omega$ (mb/sr)
$\theta_{\text{c.r.}}$ (deg)
Feynman diagrams, Hamiltonian and matrix elements for the heavy-particle pickup and alpha knockout terms included in the PWBA calculation.
\[ H = T_{pT} + T_{αC} + V_{pα} + V_{pC} + V_{αC} \]

\[ H_0 = T_{pT} + T_{αC} + V_{αC} \]

\[ H - H_0 = V_{pC} + V_{pα} \]

\[ \langle hppu \rangle = \langle \Psi_f | V_{pC} | \Psi_i \rangle \]

\[ \langle αko \rangle = \langle \Psi_f | V_{pα} | \Psi_i \rangle \]
Figure 21

Feynman diagrams, Hamiltonian and matrix elements for the triton pickup and the heavy-particle knockout terms needed for the PWBA calculation.
PWBA MATRIX ELEMENTS

TRITON PICKUP

HEAVY PARTICLE KNOCKOUT

\[ H = T_{pT} + T_{tC} + V_{pt} + V_{pC} + V_{fC} \]

\[ H_0 = T_{pT} + T_{tC} + V_{fC} \]

\[ H - H_0 = V_{pC} + V_{pt} \]

\[ \langle tpu \rangle = \langle \Psi_f | V_{pt} | \Psi_i \rangle \quad \langle hko \rangle = \langle \Psi_f | V_{pC} | \Psi_i \rangle \]
The forward angle (|tpu|^2) and backward angle (|hppu|^2) contributions to the (p,α₀) angular distributions and their coherent sum (dσ/dΩ) for Ep = 12.0 and 18.0 MeV.
CONTRIBUTIONS TO THE PWBA DIFFERENTIAL CROSS SECTIONS FROM THE TRITON PICKUP AND HEAVY PARTICLE PICKUP MATRIX ELEMENTS FOR REACTION $^{11}\text{B}(p,\alpha)^{8}\text{Be}$

$E_{p,lab}=12.0$ MeV

$E_{p,lab}=18.0$ MeV
Angular distributions predicted by a PWBA which included triton pickup and heavy-particle pickup terms plotted with the data. No Coulomb correction has been introduced.
$^{11}\text{B}(p,\alpha)^8\text{Be}$ CROSS SECTIONS FOR TRITON
PLUS HEAVY PARTICLE PICKUP
WITHOUT COULOMB CORRECTIONS
Angular distributions predicted by a FWBA including alpha knockout and heavy-particle pickup contributions without Coulomb correction.
12B(p,α)8Be CROSS SECTIONS FOR ALPHA KNOCKOUT PLUS HEAVY PARTICLE PICKUP WITHOUT COULOMB CORRECTIONS
Figure 25

The Coulomb corrected results of a PWBA containing both triton and heavy-particle pickup matrix elements.
$^{11}\text{B}(p,\alpha)^8\text{Be}$ CROSS SECTIONS FOR TRITON PLUS HEAVY PARTICLE PICKUP WITH COULOMB CORRECTIONS

\[ \frac{d\sigma}{d\Omega}_{\text{c.m.}} \text{ (mb/sr)} \]

\[ \theta_{\text{c.m.}} \text{ (deg)} \]

\[ E_p \text{ (MeV)} \]

\[ 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.01 \]
Figure 26

The Coulomb corrected results of a PWBA containing both alpha knockout and heavy particle pickup matrix elements.
$^{11}\text{B}(p,\alpha)^8\text{Be}$ CROSS SECTIONS FOR ALPHA KNOCKOUT PLUS HEAVY PARTICLE PICKUP WITH COULOMB CORRECTIONS
The energy dependence of the optimum cutoff radii as determined by a non-linear least squares fit of the PWBA results to the data. Note that the cutoff radii obtained by a Coulomb adjustment (CC) of the wave vectors are merely shifted a few tenths of a Fermi from their uncorrected (NC) values.
ENERGY DEPENDENCE OF CUTOFF RADII PREDICTED BY LEAST SQUARE FITTING $^{11}\text{B}(p,\alpha)^8\text{Be}$ ANGULAR DISTRIBUTIONS WITH A PWBA

\begin{align*}
\text{\textbackslash \textasciitilde tpu\textgreater \textasciitilde hpu\textgreater }|_{\text{NC}}^2 \\
\text{\textbackslash \textasciitilde tpu\textgreater \textasciitilde hpu\textgreater }|_{\text{CC}}^2
\end{align*}

\begin{align*}
\text{\textbackslash \textasciitilde ako\textgreater \textasciitilde hpu\textgreater }|_{\text{NC}}^2 \\
\text{\textbackslash \textasciitilde ako\textgreater \textasciitilde hpu\textgreater }|_{\text{CC}}^2
\end{align*}
The energy dependence of $\chi^2$ and the cluster reduced widths which result from a least squares fit to the $(p,\alpha)$ data using several forms of the PWBA.
DEPENDENCE OF $\chi^2$ AND THE CLUSTER REDUCED WIDTHS ON ENERGY

$|<\text{tpu}> + <\text{hppu}>|^2_{\text{NC}}$  
$|<\text{ako}> + <\text{hppu}>|^2_{\text{NC}}$  
$|<\text{tpu}> + <\text{hppu}>|^2_{\text{CC}}$  
$|<\text{ako}> + <\text{hppu}>|^2_{\text{CC}}$

$\chi^2$ vs $E_{p,\text{lab.}}$ (MeV)

$\theta^\perp$ vs $E_{p,\text{lab.}}$ (MeV)

$\theta^\alpha$ vs $E_{p,\text{lab.}}$ (MeV)

$V_{p\alpha}$ vs $E_{p,\text{lab.}}$ (MeV)
multi-nucleon emission threshold, where the compound nucleus has a large number of decay modes available to it, and the lifetime of a compound state ($\tau / \langle \Gamma \rangle$) approximates the projectile transit time ($\approx 10^{-22}$ sec). The Bohr assumptions of the compound nucleus are no longer valid.

In figure 11 one can see that for $^{12}$C the transition from a nucleus with resonant features such as at 20 MeV to one in which $^{11}$B interacts as homogeneous matter occurs at $\approx 27$ MeV. The twelve or so levels below the proton threshold and those upto nearly 27 MeV excitation have various descriptions, as analogue resonances (AJ57), particle-hole configurations (GI64), shell model states of intermediate coupling (KU56), or alpha cluster states (AR70). Also in this region are states approximately described by these models but which have additional rotational or vibrational characteristics. There are a finite number of states below 27 MeV and they all can be parameterized by microscopic quantities. Above 27 MeV levels begin to mix and overlap, and there is a diffusion of the microscopic information. Given sufficiently elaborate compound models and much computer time perhaps one could filter out this information. Its import would be obscured in the complexity of the model and one must be satisfied with the alternative approach of direct reaction theories.

Before abandoning all compound analysis in favor of these theories, the investigator should not overlook the possibility of the collection of nucleons clustering into a state of high energy but low entropy. For such a state (as is observed in nuclear molecules) the excitation
energy is great but the degrees of freedom are few and resonance phenomena will appear. Such an occurrence in $^{12}$C may be responsible for the resonance features observed at approximately 29, 30 and 32 MeV excitation energy by Shay in his $^3$Be($^3$He,$\gamma$)$^{12}$C studies (SH72).
B. Implications of the (p,p) and (p,α) Studies

1. The Elastic Scattering

Although considerable improvement may be possible in the particle-hole S-matrix theory, the discrepancies between it and the data indicate that for the energy range of our experiment effort is best spent in an optical model analysis. The reason is that an improved multi-particle multi-hole theory, such as proposed by Dreschel (DR67) or Goswami and Pal (GO63), having ground state and collective correlations has an associated increase in ad hoc parameters and its usefulness at high excitation energies is still questionable. The optical model, however, can reproduce the elastic scattering cross sections while indicating some of the properties of $^{11}$B. The parameters of the optical model, the real, imaginary, volume, surface and spin-orbit potentials; the nuclear radii; and the diffuseness characterize the target in generic rather than intrinsic quantities. Although these quantities do predict certain of the $^{11}$B's nuclear properties one would like to know if they are unique or a function of the experiment performed. Generally optical parameters are not unique and the problem of finding an absolute minimum in their parameter space is difficult. They may also be sensitive to the type of experiment being performed, for example, the best fit nuclear radius is a function of what projectile, nucleon, electron, alpha particle etc. is probing the target. So in this transition from a compound to a direct analysis of the elastic
scattering there is a loss of information about the internal nuclear
properties, but a gain of information about the specific bulk properties.
This new information depends on the type and energy of the probe as
well as the assumptions made concerning the model. The optical model
results should therefore be interpreted carefully.

2. The Inelastic Scattering

Because the $\frac{1}{2}^-$ first excited state of $^{11}$B is not a simple particle-
hole configuration, the theory of Gillet and Brassard could not directly
be applied. It is unlikely that an improved calculation performed
with Gillet's Approximation II, essentially on RPA, would have enhanced
the results, since both Approximations I and II gave poor fits to the
first excited $2^+\, ^{12}$C state which is felt to be similar to the $\frac{1}{2}^-$ state
in $^{11}$B. The results of the particle-hole work for the elastic scattering
indicated that a direct rather than a compound analysis should be used.
Presumably this would hold true for the inelastic case as well.

The similarity of the angular distributions for the $^{12}$C(p,p')$^{12}$C(4.44)
and the $^{11}$B(p,p')$^{11}$B(2.14) reactions is evident in figure 19 and implies
that quadrupole excitation of both $^{11}$B and $^{12}$C has an identical effect
on the proton flux. Such a similarity is not necessarily a feature of
nucleon inelastic scattering on neighboring nuclei; for example angular
distributions for 14 MeV neutrons inelastically scattered by $^7$Li, $^9$Be,
and $^{12}$C have markedly different patterns, c.f. figure 19-3 of Preston
(PR63). Although the $^{11}$B and $^{12}$C cross sections differ by almost a
factor of 10, the optical parameters of Watson et al. (WA69) allow the DWBA predictions, which are unnormalized and not adjusted to fit the data, to give both the correct shape and magnitude in each case. The success of the DWBA for light nuclei warrants a further investigation of nucleon inelastic scattering in this mass region.

3. The (p,α) Reactions

For the proton induced alpha emission there also was no compound theory which could adequately describe the angular distributions. Consequently, we adopted an alternative plane wave model of the interaction. This was not an unreasonable decision in view of the direct character of the (p,p₀) and (p,p₁) reactions. The success of this simple parameterization is demonstrated in our reproduction of the rather complicated angular distributions throughout the entire energy range. The monotonic trend of all the parameters beyond 15 MeV suggest that the compound manifestations have now become sufficiently damped out. The particular feature of this PWBA which makes it appealing is the inclusion of the projectile heavy particle interaction. Since this technique is not limited to the type of cluster species, it has universal applicability and could be utilized, at least for surveillance studies, in other reactions and mass regions. It will also be of interest to analyze such reactions using a true finite range DWBA code which included these projectile-heavy particle exchanges.
Thin boron foils with thicknesses between 50 and 500 \( \mu \text{gm/cm}^2 \) are difficult to fabricate because stresses created during the evaporation induce flaking and curling in the films when they are released from the substrate. The effect of these stresses can be minimized by evaporating slowly onto a heated substrate. The processes outlined below describes a technique to produce stable self-supporting boron targets.

Boron sublimes at 2600°C (at atmospheric pressure); consequently, electron bombardment and not ohmic heating methods must be used to vaporize it. By molding amorphous boron powder enriched in \(^{11}\text{B}\) and slurried in methanol into a small conical pellet we could obtain intense local heating of the boron. The pellet was placed in the center of a cooled copper crucible at the focus of a 200 milliamper - 1 keV electron beam. To reduce the contaminant level evaporations were performed with a vaccum of \(10^{-6}\) torr or less. The optimum pellet temperature for even deposition was \(\sim 1900°C\), as estimated with an optical pyrometer. Above 1900°C the boron tended to fuse and the increased currents associated with this fusing produced showers of boron fragments. Targets exposed to such a shower showed pitting and non-uniformities.

The boron vapor was condensed onto a glass slide positioned about five inches above the pellet. To inhibit stressing film we warmed the substrate to \(\sim 200°C\) before the evaporation began and maintained this
temperature for at least six hours after the evaporation was completed. In addition deposition rates were kept below 1 μg/m² per min. A thin layer of NaCl evaporated onto the glass before the boron served as a release agent. After cooling, a slide would be gradually immersed in water dissolving the salt and allowing the foils to float freely on the surface. A typical 2" x 3" slide could be sectioned into six target squares each large enough to span a 3/8" hole in the target holders.

Target thicknesses were roughly measured with an alpha thickness gauge (CO66). Areal densities derived from this technique were accurate to 5% for pure targets.

Gauged and mounted targets were stored in evacuated dessicators. Undessicated targets seemed more fragile and frequently broke during storage or when subsequently placed in vacuüm.

A more detailed description of the vapor deposition equipment and the effects of substrate heating can be found in references (MR65) and (AR66).
Appendix 2
Uncertainties in the Differential Cross Section

The differential cross section \( \frac{d\sigma}{d\Omega} \) can be expressed in mb/sr as,

\[
\frac{d\sigma}{d\Omega} = \frac{Y \cdot A \cdot q_p \cdot \cos \phi \cdot 10^{27}}{(1-D_t) \cdot L \cdot Q \cdot T \cdot P_c \cdot d\Omega_{\text{lab}} / d\Omega_{\text{cm}}} \cdot \left| \frac{d\Omega_{\text{lab}}}{d\Omega_{\text{cm}}} \right| \quad (\text{Ap. 2-1})
\]

where

- \( Y \) = measured yield in the spectrum,
- \( A \) = target atomic weight (gm/mole),
- \( q_p \) = projectile charge (Coul/projectile),
- \( \phi \) = angle target incline to the beam (degrees),
- \( L \) = Avagadro's number (atoms/mole),
- \( Q \) = total charge collected (Coul),
- \( T \) = target thickness (gm/cm²),
- \( d\Omega \) = detector solid angle (steradim),
- \( \left| \frac{d\Omega_{\text{lab}}}{d\Omega_{\text{cm}}} \right| \) = Jacobian transforming from lab to center-of-mass,

\( D_t \) = analyzer dead time

\( P_c \) = per cent of the foil composed of target material

\( d\Omega = \pi(d/2)^2/r^2 \)

\( d \) = aperture diameter (cm),

\( r \) = distance from aperture to target (cm)
The uncertainty \( \Delta f/\phi \) associated with a function \( f \) dependent on the variables \( q_j \) is given by

\[
\frac{\Delta f}{\phi} = \left( \sum_{j} \left( \frac{\delta f}{\delta q_j} \right)^2 \right)^{1/2} \tag{Ap. 2-2}
\]

Tabulated below are typical values of the \( q_j \) entering into Equation (Ap2-1).

<table>
<thead>
<tr>
<th>( q_j )</th>
<th>( \Delta q/q )</th>
</tr>
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<tbody>
<tr>
<td>( \phi = 45 \pm 2^\circ )</td>
<td>( \Delta \phi/\phi = 0.035 )</td>
</tr>
<tr>
<td>( Q = 100 - 300 \pm 5 \mu C )</td>
<td>( \Delta Q/Q = 0.01 )</td>
</tr>
<tr>
<td>( T = 310 \pm 20. \mu g/m/cm^2 )</td>
<td>( \Delta T/T = 0.06 )</td>
</tr>
<tr>
<td>( P_c = 78 \pm 1.0% )</td>
<td>( \Delta P/P_c = 0.02 )</td>
</tr>
<tr>
<td>( d = 0.2 - 0.5 \pm 0.0/cm )</td>
<td>( \Delta d/d = 0.05 )</td>
</tr>
<tr>
<td>( x = 20 - 25 \pm 0.2 \text{ cm} )</td>
<td>( \Delta x/x = 0.02 )</td>
</tr>
</tbody>
</table>

If a linear background is subtracted from the total counts within a peak, then the yield \( Y \) for \( X_i \) counts in the \( i \)th channel is

\[
Y = \sum_{i=1}^{N} X_i - \frac{1}{2} \left( \frac{X_1 + X_N}{N} \right) N \tag{Ap. 2-3}
\]

where \( \Delta X_i \) is generally taken as \( 1/X_i \). For a yield of \( Y \) at least 1000 counts the statistical uncertainty is 3% and the total uncertainty 8.5%. From equation Ap. 2-2 one sees that the lower bound on the uncertainty is 6.5% determined by the quality of the physical measurements and that the upper bound is a function of the counting statistics. We adjusted the amount of charge collected to maintain yields at 100 counts.
or more; thus the maximum total uncertainty could not exceed 13%.

It is possible given the cross section for $^{12}$C and $^{16}$O to
determine exactly how much increase in thickness the presence of the
contaminants introduces into the estimated areal density $T$ of the
target. Equation Ap.2-1 can be simplified and rewritten for a pure
target $k$ with $N_k$ scattering centers /cm$^3$($N_k = L / A_k$) as

\[
\frac{d\sigma}{d\Omega} = \frac{\nu_k q_p \cos \psi}{Q \, d\Omega \, N_k}.
\]  
(Ap. 2-4)

Since in Equation Ap.2-4 only $N_k$ is unknown, one may simply solve for
$T_k$, the areal density of the contaminant $k$. By subtracting all $T_k$'s
from the estimated thickness $T_i$ the percent by mass, $P_c$, of the foil
composed of the isotope of interest can be obtained. This process
requires a few iterations, however, because the specific energy loss
$S(E)$ entering into the equation for an alpha gauged target,

\[
T_{\text{target}} = \rho_{\text{air}} \, d_{\text{air}} \, (S(E_{\text{a}})/S(E_{\text{t}})) \quad \text{(Ap.2-5)}
\]

varies with the element and isotope involved. For our $^{11}$B target
the ratio $P_c$ of the actual areal density of $^{11}$B to the areal density
predicted by assuming the range shift, $d_{\text{air}}$, due to a pure $^{11}$B target
was $0.78 \pm 0.01$. 
When our cross section for the $p_0$, $p_1$, and $\alpha_0$ reactions were compared with those of other authors (SE65, RI68, SE63) and the above corrections had been introduced, agreement was usually within 5%.

No correction was made for contributions to the areal density due to $^{10}$B or $^{15}$N. Their low yields in the proton spectra and the good agreement of our data with other sources indicate that their effect was minimal.
-Appendix 3-
Differential Reaction Cross Section in a $j$-$j$ Coupling Notation

In order to utilize the $S$-matrix elements of Brassard for estimates of proton elastic scattering cross sections, it is necessary to derive an expression for $d\sigma/d\Omega$ in a $j$-$j$ coupling notation. A procedure similar to that of Blatt and Biedenharn (BL52) will be followed, except that Wigner 3- and 6- symbols will be used in place of Clebsch-Gordon and Racah coefficients.

An unperturbed flux of plane-waves is assumed such that

$$\psi_{\alpha} = \frac{1}{\tilde{I}\tilde{S}\sqrt{\nu_\alpha}} e^{ikz} |l\nu> |s\sigma> . \quad (\text{Ap.3-1})$$

The factor $\tilde{I}\tilde{S} \equiv \left[ (2l+1)(2s+1) \right]^{1/2}$ was introduced in order that the total beam be normalized to unity and not merely each spin state. The quantity $e^{ikz}$ can also be expressed as

$$e^{ikz} = (4\pi)^{1/2} \sum_l \tilde{j}_l(kr) i^l Y^0_l(\Omega) . \quad (\text{Ap.3-2})$$

In a $j$-$j$ coupling scheme the system consisting of a particle of spin $s$ and one of spin $I$ in an eigenstate $\alpha$ (where $\alpha$ may define the energy, the isospin, or several other quantum numbers) is represented explicitly by

$$|\alpha(ls)jLM>_{\text{in}} = \sum_{\mu\nu\sigma m} (-1)^{(l-s+\mu)} (-1)^{(j-I+M)} \tilde{j} \tilde{\alpha}$$

$$\binom{l,s}{m\sigma - \mu} \binom{j,l,l_j}{\mu\nu - M} \frac{k_{\alpha}}{\sqrt{\nu_\alpha}} i^l Y^m_l(\Omega_\alpha) \frac{f_l(k_r)}{k_{\alpha}} |l\nu> |s\sigma> \quad (\text{Ap.3-3})$$
Employing the orthogonality relations for 3-j symbols (ME62) one has

\[
\frac{k_x}{\sqrt{\alpha}} \sum_{\ell} \frac{\mathbf{j}_\ell}{\ell} \frac{\mathbf{y}_m}{\ell} \frac{\mathbf{f}_\ell(k_r)}{\mathbf{f}_{\ell+1}(k_r)} |I\nu| |s\sigma> = \\
\sum_{j\mu J M} (-1)^{(\ell-s+\mu)}(-1)^{(J-I+M)} \mathbf{j}_\ell \left( \begin{array}{ccc} \ell & s & j \ 
\mu & \nu & j \end{array} \right) \left( \begin{array}{ccc} j & M \\
0 & 0 & M \end{array} \right) |\alpha(\ell s)jIM\rangle_{\text{in}}
\]

(Ap. 3-4)

The \( j_\ell(kr) \) in equation Ap. 3-4 have the asymptotic form

\[
j_\ell(kr) \sim \frac{1}{2kr} \left[ \{ e^{i(kr-(\ell+1)\pi/2)} \} + \{ e^{-i(kr-(\ell+1)\pi/2)} \} \right]
\]

(Ap. 3-5)

Let \( f_\ell(kr) \) contain the radial dependence of the wave function in the external region, then, asymptotically \( f_\ell = e^{-i(kr-(\ell+1)\pi/2)} \) and

\[
e^{ikz} \sim (4\pi)^{1/2} \sum \mathbf{j}_\ell \left( \frac{1}{2kr} \right) \left( f_\ell^{*} + f_\ell \right) \mathbf{y}_\ell(\Omega).
\]

(Ap. 3-6)

Combining equations 3-1, 3-2, and 3-4 one gets

\[
\psi_{\alpha}^{\nu \sigma} \sim \frac{(4\pi)^{1/2}}{\ell S} \sqrt{\alpha} \sum_{j\mu J M} \mathbf{j}_\ell \left( \frac{r \cdot J M}{2k_r} \right) (-1)^{(\ell-s+\mu)}
\]

\[
\cdot (-1)^{(J-I+M)} \left( \begin{array}{ccc} \ell & s & j \ 
0 & \sigma & j \end{array} \right) \left( \begin{array}{ccc} I & J \\
0 & 0 & M \end{array} \right) \left\{ |\alpha(\ell s)jIM\rangle_{\text{out}}^\dagger + |\alpha(\ell s)jIM\rangle_{\text{in}} \right\}.
\]

(Ap. 3-7)

The asymptotic value of the unperturbed wave function \( \psi_{\alpha}^{\nu \sigma} \) for two particles of spin \( S \) and \( I \) may also be expressed as

\[
\psi_{\alpha}^{\nu \sigma} \sim \sum_{J M, \nu \sigma} A_{JM, \nu \sigma} \left\{ |\alpha(\ell s)jIM\rangle_{\text{out}}^\dagger + |\alpha(\ell s)jIM\rangle_{\text{in}} \right\}.
\]

(Ap. 3-8)

By matching the coefficients of the spherical wave \( |\alpha(\ell s)jIM\rangle \) at infinity we see that

\[
A_{JM, \nu \sigma} = \frac{\mathbf{j}_\ell}{\ell S} \frac{\sqrt{\pi}}{k_r} (-1)^{(\ell+jS+I-\nu)} \left( \begin{array}{ccc} \ell & s & j \\
0 & 0 & j \end{array} \right) \left( \begin{array}{ccc} I & J \\
0 & 0 & M \end{array} \right)
\]

(Ap. 3-9)
After an interaction occurs the asymptotic expression describing the system in channel \( \alpha \) with spins \( S \) and \( I \) and projections \( \sigma \) and \( \nu \) becomes

\[
\psi_{\alpha}^{\nu\sigma}(\text{total}) = \sum_{fJJM} \{A_{\alpha fJJM}^{JM;\nu\sigma} |\alpha(fs)jIJM\rangle_{\text{in}} + B_{\alpha fJJM}^{JM;\nu\sigma} |\alpha(fs)jIJM\rangle_{\text{out}} \}. \quad (Ap.3-10)
\]

The incident flux we had defined to be

\[
\psi_{\alpha}^{\nu\sigma}(\text{incident}) = \sum_{fJJM} \{A_{\alpha fJJM}^{JM;\nu\sigma} |\alpha(fs)jIJM\rangle_{\text{in}} + A_{\alpha fJJM}^{JM;\nu\sigma} |\alpha(fs)jIJM\rangle_{\text{out}} \}. \quad (Ap.3-11)
\]

The flux in channel \( \alpha' \) due to an interaction is

\[
\psi_{\alpha'}^{\nu'\sigma'}(\text{reaction}) = \psi_{\alpha'}^{\nu'\sigma'}(\text{total}) - \psi_{\alpha'}^{\nu'\sigma'}(\text{incident}). \quad (Ap.3-12)
\]

In what follows the primed notation will refer to final state wave functions and unprimed notation to initial state wave functions. The relationship between the incident flux of channel \( \alpha \) and that of channel \( \alpha' \) is defined by the transformation

\[
B_{\alpha'fj}^{JM;\nu'\sigma'} = \sum_{\alpha f'j'} S_{\alpha'f'j'\nu'\sigma';\alpha fj\nu\sigma} A_{\alpha fJJM}^{JM;\nu\sigma}. \quad (Ap.3-13)
\]
Because there is no flux incident in channels other than $\alpha$, the sum over $\alpha$ is superfluous. For the reactions under consideration here, $S$ is independent of $\nu$, $\sigma$, $\nu'$, and $\sigma'$. $S$ also does not depend on $M$ (c.f. BL52). These conclusions result in the following simplification of (Ap. 3-13)

\[ B^{JM;\nu'\sigma'}_{\alpha'\ell'j'} = B^{JM}_{\alpha'\ell'j'} = \sum_{\ell j} S^{\ell j}_{\alpha'\ell'j';\alpha\ell j} \sum_{\nu\sigma} A^{JM;\nu\sigma}_{\alpha\ell j} \]  

(Ap. 3-14)

Combining equations Ap. 3-4, 10, 11, and 12 gives

\[ \psi^{\nu'\sigma'}_{\alpha'} (\text{reaction}) = \sum_{\ell'j'JM} \{-A^{JM;\nu'\sigma'}_{\alpha'\ell'j'}\} |\alpha'...>_{\text{in}} + B^{JM;\nu'\sigma'}_{\alpha'\ell'j'} |\alpha'...>_{\text{out}} \] 

\[ + \sum_{\ell'j'JM} A^{JM;\nu'\sigma'}_{\alpha'\ell'j'} \{ |\alpha'...>_{\text{in}} |\alpha'...>_{\text{out}} \}. \] 

(Ap. 3-15)

Where $|\alpha'...>_{\text{in}}$ implies

\[ |\alpha'...>_{\text{in}} = |\alpha'(...j')j'JM>_{\text{in}} \] 

(out) 

\[ |\alpha'...>_{\text{out}} = |\alpha'(...j')j'JM>_{\text{out}} \] 

(out)

This gives

\[ -\psi^{\nu'\sigma'}_{\alpha'} (\text{reaction}) = \sum_{\ell'j'JM} \{-A^{JM;\nu'\sigma'}_{\alpha'\ell'j'} + B^{JM;\nu'\sigma'}_{\alpha'\ell'j'} \} |\alpha'...>_{\text{out}}. \] 

(Ap. 3-17)
\( \psi_{\alpha'}^{\nu'} \sigma' \) (reaction) = \( \sum_{\ell' j' l j} (-1) [A_{\alpha \ell j}^{JM} \sigma \{ \delta_{\ell' j' j} \delta_{\ell j} \delta_{\alpha' \alpha} - S_{\ell' j' j}^{JM} \alpha' \ell j \alpha \ell j \}] |\alpha' \ldots >_{\text{out}}. \) (Ap. 3-18)

Replacing \( |\alpha' \ldots >_{\text{out}} \) with its actual representation yields

\[ \psi_{\alpha'}^{\nu'} \sigma' \] (reaction) = (-1) \[ \sum_{\mu' \ell' j'} \{ \delta_{\ell' j' j} \delta_{\ell j} \delta_{\alpha' \alpha} - S_{\ell' j' j}^{JM} \alpha' \ell j \alpha \ell j \} \cdot A_{\alpha \ell j}^{JM} \nu \sigma \]

\[ \left[ \frac{1}{\sqrt{\alpha'}} \right] \left( \begin{array}{c} j' \ell' J \\ \mu' \nu' - M \end{array} \right) \cdot \frac{k_{\alpha'}}{\sqrt{\nu}} \frac{1}{Y_{\ell' j'}^{M \ell \ell'}(\Omega_{\alpha'})} \frac{f_{\ell' j'}(k_{\alpha'} r_{\alpha'})^*}{k_{\alpha'}^{\nu'} \alpha'} \left| I' \nu' > | s' \sigma' > \right. \] (Ap. 3-19)

Asymptotically this reduces to

\[ \psi_{\alpha'}^{\nu'} \sigma' \] (reaction) = \[ \frac{\sqrt{\pi} i e^{i k_{\alpha'} r_{\alpha'}}}{k_{\alpha'}^{\nu' \alpha'} \sqrt{\nu}} \left| I' \nu' > | s' \sigma' > \right. \]

\[ \times \sum_{\ell' j' l j} \{ \delta_{\alpha' \alpha} \delta_{\ell' j' j} - S_{\ell' j' j}^{JM} \alpha' \ell j \alpha \ell j \} \left[ \frac{\hat{\delta} \hat{\delta}}{1} \right] (-1)^{(l' + s' + j') \mu' - s' + j' + l' + s' + j'}

(-1)^{(l + j + s + I - \nu)} \left( \begin{array}{c} j j \ell j \\ \sigma - \sigma \end{array} \right) \left( \begin{array}{c} j I J \\ \sigma \nu - M \end{array} \right) (-1)^{(l' - s' + j') \mu'}

(-1)^{(l' - \nu' + M)} \left( \begin{array}{c} j' \ell' j' \\ \mu' \nu' - M \end{array} \right) \frac{\sqrt{\pi} Y_{l' j'}^{M \ell \ell'}(\Omega_{\alpha'})} \] (Ap. 3-20)
The quantity inside the large set of parentheses represents the reaction amplitude \( q_{\alpha'\nu'\sigma';\alpha\nu\sigma} \) and according to Blatt et al. (BL52) is related to the scattering cross section by

\[
d\sigma_{\alpha'\nu'\sigma';\alpha\nu\sigma} = \chi_{\alpha}^2 |q_{\alpha'\nu'\sigma';\alpha\nu\sigma}|^2 \frac{d\Omega}{\Omega}. \tag{Ap. 3-21}
\]

Where \( \chi_{\alpha} = k_{\alpha}^{-1} \). If the reaction is insensitive to spin polarizations then

\[
d\sigma_{\alpha'\nu'\sigma';\alpha\nu\sigma} = \sum_{\nu'\sigma'} d\sigma_{\alpha'\nu'\sigma';\alpha\nu\sigma}. \tag{Ap. 3-22}
\]

Substituting,

\[
q_{\alpha'\nu'\sigma';\alpha\nu\sigma} = \sum_{J,M,J'} \left\{ \delta_{\alpha'\alpha} \delta_{J,J'} \delta_{M,M'} - S_{J,J'}^{J'} \right\}
\]

\[
\cdot \left[ \frac{\hat{\gamma} \hat{\gamma}}{\hat{\gamma}^2} \right] \cdot (-1)^{\left(\ell + j + s + I - \nu\right)} \left(\frac{\ell}{\ell' \nu} \frac{j}{j'} \frac{\sigma}{\sigma'} \frac{J}{J'} \right) \cdot \left(\frac{\ell}{\ell' \nu} \frac{j}{j'} \frac{\sigma}{\sigma'} \frac{J}{J'} \right)
\]

\[
\cdot (-1)^{\left(\ell - \nu + \mu\right)} \left(\frac{\ell'}{\ell' \nu} \frac{j'}{j'} \frac{\sigma'}{\sigma'} \frac{J'}{J'} \right) \cdot \left(\frac{\ell' \nu'}{\ell' \nu'} \frac{j'}{j'} \frac{\sigma'}{\sigma'} \frac{J'}{J'} \right)
\]

\[
\cdot \left(\frac{\ell \nu \sigma}{\ell \nu \sigma} \right)^{\left(\ell \nu \sigma \mu\right)} \sqrt{\pi} Y_{J'}^{m'}(\Omega_{\alpha'})\right]. \tag{Ap. 3-23}
\]

Into (Ap. 3-22) and collapsing \( Y_{J'}^{m'} \)'s and 3-j's wherever possible gives,
\[ \frac{d\sigma}{d\Omega} (\alpha \rightarrow \alpha') = \pi \lambda^2 \sum_{\ell_1, \ell_2, j_1, j_2} \{ [\delta^\alpha_{\ell_1} \delta^\beta_{\ell_2} \delta^j_{j_1} \delta^j_{j_2} - S^j_{\ell_1 j_1 \ell_2 j_2}]^* \} \cdot \frac{\hat{j}_1 \hat{j}_2 \hat{L}_1 \hat{L}_2}{(\hat{\Omega})^2 (\hat{s})^2 \sqrt{4\pi}} \]

\[ \cdot \left( \hat{j}_1 \right)^2 \left( \hat{j}_2 \right)^2 (-1)^{(L'+s)} (-1)^{(L'+I')} (-1)^{(L'+I')} (-1)^{(L_2-L_1)} \]

\[ \cdot (-1)^{(L_2-L_1)} (-1)^{(L_1+I)} (-1)^{(L_2-1)} \left( \begin{array}{c} \ell_1 \ell_2 \ell_1 \ell_2 \\ j_1 j_2 \end{array} \right) \left( \begin{array}{cc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{c} \ell_1 \ell_2 \ell_1 \ell_2 \\ j_1 j_2 \end{array} \right) \]

\[ \left( \begin{array}{c} j_1 L_1^I \\ j_2 L_2^I \end{array} \right) \left( \begin{array}{c} j_1 L_1^I \\ j_2 L_2^I \end{array} \right) \left( \begin{array}{c} j_1 L_1^I \\ j_2 L_2^I \end{array} \right) \left( \begin{array}{c} j_1 L_1^I \\ j_2 L_2^I \end{array} \right) \]

\[ \cdot Y_{L_1}^{\ast}(\Omega). \quad \text{(Ap. 3-24)} \]

It is this expression which was used to determine from the S-matrix of Chapter IV the particle-hole elastic scattering cross sections.
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