SEQUENTIAL TRANSFERS AND INELASTIC TRANSITIONS
IN HEAVY-ION INDUCED TRANSFER REACTION

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A systematic theoretical study of the higher order nuclear processes expected to play significant roles in one- and two-nucleon, heavy-ion induced transfer reactions is carried through to numerical cross section predictions and to suggestions for particularly sensitive experimental tests of the proposed mechanisms and calculations. A full-recoil formalism is incorporated with the source-term method for the analysis of heavy-ion transfer reactions within the coupled-channels Born approximation. Simple systematics of the form factors and angular distributions are discussed.

Anomalous angular distributions for the $^{48}\text{Ti}(^{16}\text{O},^{15}\text{N})^{49}\text{V}$ reaction are analyzed by considering the interference effects of multistep and direct transfer processes. Within the weak-coupling model, the population of excited states in $^{49}\text{V}(3/2^-,11/2^-,9/2^-)$ via pure two-step inelastic excitation and transfer is characterized by rather flat angular distributions, whereas a direct single-particle transfer yields a bell-shaped curve similar to the ground state distribution. Though neither of these processes taken individually is adequate to account for the anomalous angular distributions associated with the $^{49}\text{V}$ excited states, a coherent addition of the
amplitudes from both mechanisms results in the necessary interference. The consequent determination of the relative strengths of the competing routes indicates that the single-particle components in the excited state wavefunctions correspond to $\% 2\%$ of the pure single-particle spectroscopic strength.

The shapes of calculated angular distributions resulting from sequential transfer mechanisms are compared to those of competing direct mechanisms, and some observational problems associated with sequential transfers are discussed.

The few-nucleon transfer reactions induced by $^{28}\text{Si}$ on $^{48}\text{Ca}$ are suggested as particularly appropriate for the experimental observation of sequential transfer processes. Specifically, there are three well matched rearrangements for which there are no conventional direct reaction mechanisms. The $^{48}\text{Ca}(^{28}\text{Si},^{27}\text{Mg})^{49}\text{Ti}$ reaction involves the stripping of two units of charge and one unit of mass, and therefore cannot proceed as the direct transfer of a known nuclear constituent. In the $^{48}\text{Ca}(^{28}\text{Si},^{28}\text{Mg})^{48}\text{Ti}$ reaction, two units of charge are stripped; this exchange cannot be a first order effect of the usual isospin dependent interaction. The $^{48}\text{Ca}(^{28}\text{Si},^{29}\text{Al})^{47}\text{Sc}$ reaction requires the stripping of one unit of charge and the pick-up of one unit of mass. This reaction also cannot be described by the direct transfer mechanism.

There are, however, several sequential transfer reactions which lead to these final rearrangements, e.g. $^{48}\text{Ca}(^{28}\text{Si},^{29}\text{Si})^{47}\text{Ca}(^{29}\text{Si},^{27}\text{Mg})^{49}\text{Ti}$. Moreover, the intermediate transfers comprising these sequential mechanisms have small
Q-values. Thus, measured angular distributions for these rearrangements are expected to evidence sequential processes and would provide a good test for the reaction theory.
SEQUENTIAL TRANSFERS AND INELASTIC TRANSITIONS
IN HEAVY-ION INDUCED TRANSFER REACTIONS

A Dissertation
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of
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Doctor of Philosophy

by
George Bradford Sherwood
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To my Parents and Sisters
Much of the material contained in this thesis is included to clarify developments in heavy-ion induced reaction theory and specify conventions which have evolved. The reader who is already familiar with the standard formalism for elastic scattering and the source-term method of Ascuitto and Glendenning (As 69) may wish to skim or skip Chapters 2 and 3, which contain no original work. Chapters 1, 6, and 7 are of particular importance to the general reader for it is these chapters which put forth the motivation for, and the analysis of, the reactions to be studied. The description of the numerical techniques employed in this work is appended, as its inclusion is of interest to a specialized group of readers.
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CHAPTER 1
INTRODUCTION

One of the outstanding characteristics of the history of nuclear physics is the large number of what frequently appear to be contradictory models that have been successfully applied to descriptions of nuclear structure and reaction mechanisms. Much of this multiplicity has reflected the varied, sometimes contrasting aspects of nuclear phenomena in different regions of the periodic table and at different probe and excitation energies. Moreover, all these models are at best caricatures of the actual situation, with their successes depending upon the level to which they highlight the dominant aspects of the processes under study. While the concepts\(^1\) associated with these individual models generally are not complicated, the fact that the diverse models must, in many cases, coexist for adequate phenomenological descriptions introduces a high degree of complexity.

It is, for example, thus far impossible to determine uniquely the model parameters for some of the simplest nuclear reactions; and while many properties of these processes are adequately described by the simple models, other properties seem to be associated with more complicated generalizations.

Under these circumstances, it is very important that any model

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\(^1\)Brief surveys of nuclear structure and spectroscopy which may be useful as introductions are (Bo 73 and Fa 71), and more recently (Lo 76).
generalizations be subject to verification by data bearing an unambiguous signature: The most useful experiments are those which have one and only one interpretation.

These considerations have provided a major impetus for the development of heavy-ion beams as fine probes of nuclear behavior. The relatively short wavelength of the heavy-ion projectile, together with the peripheral nature of its quasielastic scattering, makes it an ideal tool for investigation of the nuclear surface. There are at present a number of examples in which the analysis of heavy-ion transfer reactions has led to a unique interpretation of the reaction dynamics involved. Thus, the heavy-ion beams offer the possibility of removing some of the ambiguity inherent in the reaction and structure models and of making apparent some of the previously ignored amplitudes.

However, together with their potential simplifications, heavy-ion transfer reactions have necessitated an additional complication - recoil. The traditional zero-range and no-recoil finite range approximations are not generally adequate to explain the collision of two systems of comparable mass, or to treat interactions involving transfers of substantial mass fractions. The inclusion of the corresponding extra degrees of freedom does not introduce any additional unknown model parameters, but it does make the analysis a much more complicated computational problem.

\[2^2\text{See, for example, (Er 74).}\]

\[3^3\text{See (Au 64, De 73) and references therein.}\]
At the heart of a full-recoil transfer reaction calculation is the bipolar expansion, the expression of a function of two coordinates as a double sum of spherical harmonics. (In this thesis the bipolar expansion also implies the associated reexpression of the multipole field as a multipole expansion of two new coordinates, related to the originals by a linear transformation.) Actually the mathematics necessary for this endeavor is not new. Lord Rayleigh worked on the related problem of the expansion of a multipole field about a shifted origin almost a century ago. (Ra 92, Da 65) The fact that it has taken as long as it has to apply the recoil formalism to various kinds of nuclear reactions is an indication not of mathematical complexity but of computational difficulty.

When higher order processes are included in a full-recoil study, the resulting hard-fought and expensive battles with a computer serve to underscore the necessity of a careful choice of reactions for measurement and analysis. But in addition, such calculations indicate that in some cases a simple, or even simplistic, structural model can illuminate certain reaction processes which might be incalculable within the best of models. The following two examples of such situations are discussed in some detail in this thesis.

The reaction $^{48}\text{Ti}(^{16}_0,^{15}_N)^{49}\text{V}$ yields anomalous angular distributions when the excited states $^{49}\text{V}(3/2^-,11/2^-,9/2^-)$ are populated. These distributions can be successfully analyzed through the use of a model in which the excited
residual state wavefunction is mostly that of a $7/2^-$ proton
coupled to a quadrupole core phonon (the weak-coupling model) with
a small component of the appropriate $3/2^-$, $11/2^-$, or $9/2^-$
single-particle state. This model is rather crude by shell
model standards, but it has heuristic value in that it allows
the reaction dynamics to be studied simply. Within this model,
small, only a postulated single-particle component of the excited
state in $^{49}$V is populated via the direct proton transfer from
the $^{48}$Ti ground state, and only the particle-phonon component
is populated by the two-step routes composed
of both inelastic excitation and transfer.

The success of the analysis to be reported herein is
not so much a confirmation of the model as it is an indication
of some of the properties of the states. In an eventual, more
realistic model, these excited states would be weakly coupled
among themselves and with the ground state. They would
have a large overlap with particle-phonon model states
and a small overlap with single-particle states. Moreover,
the relative signs of these overlaps are determined by this
analysis.

The second example to be presented concerns a longstanding
problem of two-nucleon transfer. In the past few years the theo­
retical study of heavy-ion induced one-nucleon transfer reactions
has been relatively successful in reproducing both the shapes and
magnitudes of the experimental differential cross sections.
However the straightforward extension to two-nucleon transfer
reactions has been characterized by vast underestimations of
observed absolute cross sections, by an order of magnitude at least (Ba 74a). The list of possible deficiencies in current reaction models includes, among other items, the omission of the so-called indirect interaction terms, the lack of a realistic interaction between the transferred particles, and the usual neglect of the sequential transfer mode. Model generalizations incorporating remedies to one or more of these possible deficiencies are very likely to change the overall normalization of calculated angular distributions. But at the same time, all available evidence indicates that the angular distributions are not significantly and unambiguously altered by any of these changes.

The inclusion of some of the indirect interaction terms for one-nucleon transfer yields large changes in predicted absolute cross sections and insignificant angular distribution shape changes. The extreme interaction between the two transferred nucleons in a cluster approximation and the practically nonexistent interaction characteristic of most microscopic theories give rise to similar angular distributions with different absolute normalizations. Moreover the corresponding sequential transfer mechanism produces distributions much like the above simultaneous ones. Clearly it would be desirable to untangle some of these processes, but it is already apparent that "simple" two-nucleon transfer reactions do not easily lend themselves to clear-cut conclusions.

4See, for example, (Ba 74, De 74, Ka 74, Ta 75, Fe 76, and Tu 77).
However it is possible to consider reactions in which the final states cannot be conventionally populated by the simultaneous transfer of any known nuclear constituent. (Ko 74, Ud 75) In this situation there may be accessible experimental evidence for one or more sequential processes. For reasons to be discussed later, the best examples which would unambiguously signify sequential transfer are ones which involve the transfer of more than two nucleons. Of course these reactions are ultimately more complicated than two-nucleon transfer reactions, but they appear to offer a good test of the sequential reaction mechanism if the intermediate transfer steps can be empirically normalized.

This very simple model, if successful, would provide a strong confirmation of the sequential transfer model, as it is applied to two-nucleon transfer or other reactions. There is at present one published example of such an analysis (Ud 75) for the $^{48}\text{Ca}(^{16}\text{O},^{15}\text{C})^{49}\text{Ti}$ reaction. In this case the magnitudes of the theoretical cross sections are in agreement with the data. However, the evidence supplied by the experimental angular distributions may be strengthened by the study of other, better $Q$-matched reactions.

Before a discussion of possible candidates for a sequential transfer analysis, the pertinent ideas should be made more precise by a review of the theoretical foundations. The theory of heavy-ion induced transfer reactions is developed in Chapters 2 through 5. These chapters describe respectively elastic scattering, inelastic excitation, first order transfer
mechanisms, and second order transfer mechanisms. Sections 3.5, 4.8, 4.9, and 5.3 provide the general systematics necessary for a qualitative understanding of the observed and predicted phenomena. Chapter 6 contains the analysis of the $^{48}$Ti($^{16}$O,$^{15}$N)$^{49}$V reaction, and chapter 7 proposes for observation the sequential transfer processes induced by $^{28}$Si on $^{48}$Ca.
CHAPTER 2
THE ELASTIC CHANNELS

2.1 The Bound-State Wavefunctions

The description of the scattering of a complex system of \( T \) particles by another system of \( A \) particles begins with the specification of each of these systems when they are isolated. The nonrelativistic stationary-state wavefunction for the \( A \)-system is given by

\[
(H_A - \varepsilon_A) \psi_A (\vec{a}) = 0
\]  

(2-1)

The Hamiltonian \( H_A \) and the normalized antisymmetric wavefunction \( \psi_A (\vec{a}) \) depend on the nuclear model employed. The subscript \( A \) refers to any quantum numbers necessary to specify the stationary state,\(^1\) and \( \vec{a} \) represents space, spin, and possibly isospin coordinates. For convenience the origin of the space coordinates is taken to be the center of mass of the \( A \)-system.

For the \( T \)-system,

\[
(H_T - \varepsilon_T) \psi_T (\vec{t}) = 0
\]  

(2-2)

in which the origin of the space coordinates is the \( T \)-system center of mass.

\(^1\)It is sometimes convenient to write some of the quantum numbers explicitly. In such cases, \( A \) refers to all the other quantum numbers.
The Schrödinger equation for a system of C=A+T particles is

\[(H(C)-E) \psi_C (C) = 0\]  \hspace{1cm} (2-3)

The choice of the C-system center of mass as the space coordinate origin in equation (2-3) facilitates the expansion

\[H(C) = H(A) + H(T) + T(\vec{r}_T) + V_{AT}(C)\]  \hspace{1cm} (2-4)

The vector \(\vec{r}_T\) gives the displacement of the T-system center of mass from that of the A-system. \(T(\vec{r}_T)\) is the kinetic energy operator for the relative motion between the systems, and \(V_{AT}(C)\) describes their interaction. A derivation of equation (2-4) is given in Appendix A.

If the T-system is elastically scattered by the A-system, the total wavefunction may be expressed as

\[\psi_C (C) = \frac{1}{\sqrt{\nu}} \sum (-1)^{\nu} P_{\nu} \psi_A (A) \psi_T (T) \phi_{AT}(\vec{r}_T),\] \hspace{1cm} (2-5)

in which \(\phi_{AT}(\vec{r}_T)\) is the wavefunction of the relative motion, and

\[P_{\nu} \psi_A (A) \psi_T (T) \phi_{AT}(\vec{r}_T)\]

denotes the function obtained from

\[\psi_A (A) \psi_T (T) \phi_{AT}(\vec{r}_T)\]

by \(\nu\) consecutive permutations of particle pairs between the A- and T- partitions. The details leading to equation (2-5) are specified in Appendix B.
2.2 The Free Motion Wavefunction

Substitution of the wavefunction (2-5) into the $C$ particle Schrödinger equation (2-3) yields

$\left( H(C) - E \right)_T \sum_{v} (-1)^{v} P_v \psi_A (A) \psi_T (T) \phi_{AT} (r_T) = 0$

The Hamiltonian for $C$ identical particles is symmetric with respect to permutations of particles. Therefore $H(C)$ commutes with $P_v$

$\left( C \right)_T^{-\frac{1}{2}} \sum_{v} (-1)^{v} P_v (H(C) - E) \psi_A (A) \psi_T (T) \phi_{AT} (r_T) = 0$

The expansion (2-4) is now used to obtain

$\left( C \right)_T^{-\frac{1}{2}} \sum_{v} (-1)^{v} P_v (T (r_T) + V_{AT} (C) - E) \psi_A (A) \psi_T (T) \phi_{AT} (r_T) = 0, \quad E = \varepsilon_A - \varepsilon_T$

The overlap of equation (2-6) with the bound state wavefunction

$\left( C \right)_T^{-\frac{1}{2}} \sum_{v} (-1)^{v} P_v \psi_A (A) \psi_T (T)$

completes the separation of the relative motion coordinates.

It is important to note that the overlapping procedure produces $(C)$ identical terms the sum of which cancels the normalization factor.

$(T (r_T) + V_{AT} (r_T) - E) \phi_{AT} (r_T) = 0, \quad (2-7)$

where

$V_{AT} (r_T) = (C)_T^{-\frac{1}{2}} \int d \tilde{r} (\sum_{v} (-1)^{v} P_v \psi_T (T) \psi_A (A) \psi_T (T)) \sum_{v} (-1)^{v} P_v V_{AT} (C) \psi_A (A) \psi_T (T)$

$= \int d \tilde{r} (\sum_{v} (-1)^{v} P_v V_{AT} (C) \psi_A (A) \psi_T (T))$
The conventional normalization for $\phi_{AT}(r_T)$ is such that the particle flux density is equal to the relative motion velocity (Da 65a). For example, if $V_{AT}(r_T) = 0$,

$$\mathcal{J}_T = (\hbar/2\mu_T) (\phi_{AT}^* \nabla \phi_{AT} - \phi_{AT} \nabla^\ast \phi_{AT}) = \hbar k_T/\mu_T,$$ (2-8)

When

$$\phi_{AT}(r_T) = \exp(i k_T \cdot r_T).$$

Here $\mu_T$ is the reduced mass $m_A m_T/m_C$, and $k_T^2 = 2\mu_T E_T/\hbar^2$.

It is also conventional to choose the $z$-axis of the relative coordinate system to be in the direction of the relative motion.

$$\phi_{AT}(r_T) = \exp(ik_T Z_T)$$ (2-9)

In finding the solution (2-9) to equation (2-7) (for $V_{AT} = 0$), the coordinate representation of the kinetic energy operator is used.

$$T(r_T) = -(\hbar^2/2\mu_T) V^2$$

In spherical polar coordinates,

$$T(r_T) = -\frac{\hbar^2}{2\mu_T} \frac{1}{r_T^2} \frac{\partial}{\partial r_T} \left( r_T^2 \frac{\partial}{\partial r_T} \right) + \frac{1}{2\mu_T} \frac{1}{r_T^2} L^2(r_T),$$

$$L^2(r_T) = \frac{\hbar^2}{\sin^2 \Theta_T} \left[ \frac{1}{\sin \Theta_T} \frac{\partial}{\partial \Theta_T} (\sin \Theta_T \frac{\partial}{\partial \Theta_T}) + \frac{1}{\sin \Theta_T} \frac{\partial^2}{\partial \Theta_T^2} \right]$$

The general solution for $\phi_{AT}(r_T)$ in the absence of any interaction is well known in spherical coordinates.
The constants $A_{L_T m_T}$ and $B_{L_T m_T}$ are determined by the normalization and boundary conditions. This free motion solution is discussed in many texts on quantum mechanics and mathematical physics. In particular, sections 34 and 35 of (Da 65a), chapter 9 of (Bu 68), and part I of (De 63) contain elaborations of various degrees.

The constants $A_{L_T m_T}$ and $B_{L_T m_T}$ are to be chosen such that $\Phi_{AT} (\mathbf{r}_T)$ describes a beam of particles with a definite energy and direction; that is, in the region of space under consideration,

$$\Phi_{AT} (\mathbf{r}_T) = \exp (i k_{T} \cdot \mathbf{r}_T)$$

The so-called Bauer formula,\(^2\)

$$\exp (i k_{T} \cdot \mathbf{r}_T) = \sum_{L_T} \frac{1}{(2L_T + 1)} j_{L_T} (k_T r_T) P_{L_T} (\hat{k}_T \cdot \hat{r}_T),$$

together with the expansion in spherical harmonics,\(^3\)

$$P_{L_T} (\hat{k}_T \cdot \hat{r}_T) = \frac{4\pi}{(2L_T + 1)} \sum_{m_T} Y_{L_T}^{m_T} (k_T) Y_{L_T}^{m_T} (\hat{r}_T),$$

is used to expand the plane-wave solution.

\(^2\)See problems 5, 16, and 17 of chapter 9 in (Bu 68).

\(^3\)Standard references for angular momentum and its coupling are (Ro 57, Ed 60, Br 62, and De 63).
With the usual choice of the z-axis, \( \hat{z}_T = \hat{k}_T \), the relative motion solution reduces to that of equation (2-9).

\[
\phi_{AT}(r_T) = (4\pi)^k \sum_{L_T} i^{L_T} j_{L_T}(k_T r_T) Y_{L_T}^{m_T}(\hat{k}_T) Y_{L_T}^{m_T}(\hat{r}_T).
\]

(2-10)

2.3 The Partial Wave Expansion

For the situation in which \( V_{AT}(r_T) \) is a spherically symmetric potential of finite range, the linear combinations

\[
h^{(1)}_{L_T}(k_T r_T) = j_{L_T}(k_T r_T) + i n_{L_T}(k_T r_T)
\]

and

\[
h^{(2)}_{L_T}(k_T r_T) = h^{(1)*}_{L_T}(k_T r_T)
\]

become particularly useful. \( h^{(1)}_{L_T} \) and \( h^{(2)}_{L_T} \) are called spherical Hankel functions of the first and second kind.

Outside the range of the potential, (for large \( r_T \)) the relative motion solution becomes a free motion solution. However this solution is composed of two parts. One describes the incident beam flux and is given by the expression in equation (2-10). The other part describes particles scattered by the interaction \( V_{AT}(r_T) \). This scattering part of the wavefunction has the boundary condition that the radial component of its particle flux density is directed outwards.\(^4\) The spherical Hankel function of the first kind has this property, while the spherical Hankel function of the second kind describes inward flux.

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\(^4\)Chapter 10 of (Me 62) gives a more rigorous discussion in terms of wave packets.
This statement is easily verified by using the asymptotic forms \( (k_T r_T \gg L_T) \)

\[
j_{L_T}^{\pm}(k_T r_T) \sim \sin(k_T r_T - (\pi L_T/2))/(k_T r_T)
\]

and

\[
\eta_{L_T}^{\pm}(k_T r_T) \sim -\cos(k_T r_T - (\pi L_T/2))/(k_T r_T)
\]

together with the flux density definition (equation (2-8)) in spherical coordinates. The result is

\[
\mathbf{J}_{r_T} \sim i \mathbf{J}(\mu_T k_T r_T^2)
\]

for spherical Hankel functions of the first (second) kind.

The incident beam wavefunction, given by equation (2-10), becomes

\[
(4\pi)^{\frac{1}{2}} \sum_{L_T}^\infty i^L_T (2L_T+1)^{\frac{1}{2}} (h_{L_T}^{(1)}(k_T r_T) + h_{L_T}^{(2)}(k_T r_T)) / 2 \sum_{l=0}^{\infty} \chi_{L_T}^O(r_T)
\]

in terms of the spherical Hankel functions. Thus, with the scattered wave included, outside the range of the potentials the relative motion wavefunction is

\[
\phi_{AT}(r_T) = \sum_{L_T}^\infty \sum_{l=0}^{\infty} \chi_{L_T}^O(r_T) \chi_{L_T}^O(\hat{r}_T)
\]

The complex constant \( S_{L_T} \) is called a diagonal matrix element of the scattering matrix or S-matrix. The azimuthal symmetry of the scattering problem is implicit in this expansion of the wavefunction.
The asymptotic form of the relative motion solution can also be expressed as

\[ \phi_{AT}(r_T) \sim \exp(ik_T Z_T) + f_{AT}(\theta_T) \exp(i k_T r_T/\rho_T), \]

in which the scattering amplitude \( f_{AT}(\theta_T) \) is related to the S-matrix elements by

\[ f_{AT}(\theta_T) = (2ik_T)^{-1} \sum_{\ell_T} \frac{[4\pi(2\ell_T+1)]^{1/2} (S_{\ell_T} - 1)}{\ell_T^2} Y_{\ell_T}^{0}(\hat{r}_T). \]

In terms of the scattering amplitude, the radial component of the flux density of the asymptotic scattered wave is

\[ \mathcal{G}_{\text{scat.}}^{\text{rad}} = \mathcal{A} k_T |f_{AT}(\theta_T)|^2 / (\rho_T r_T^2). \]

The differential cross section is defined to be

\[ \left( \frac{d\sigma}{d\Omega_T} \right) = r_T^2 \mathcal{G}_{\text{scat.}} / \mathcal{G}_{\text{inc.}}, \]

in which \( \mathcal{G}_{\text{inc.}} \) is the incident flux density \( \mathcal{A} k_T / \rho_T \). The cross section is expressible in terms of the scattering amplitude or the S-matrix elements.

\[ \left( \frac{d\sigma}{d\Omega_T} \right) = |f_{AT}(\theta_T)|^2 \]

\[ = \left| (2ik_T)^{-1} \sum_{\ell_T} \frac{[4\pi(2\ell_T+1)]^{1/2} (S_{\ell_T} - 1)}{\ell_T^2} Y_{\ell_T}^{0}(\hat{r}_T) \right|^2. \]  

(2-12)

2.4 The Coulomb Wavefunctions

In general, for charged particles, the potential \( V_{AT}(r_T) \) has a long-range Coulomb component. For large values of \( r_T \),
(distances much larger than the dimensions of the A- or T-system), the interaction between the systems does not decrease fast enough to define an asymptotic region of free motion. However, the asymptotic solutions, the Coulomb wavefunctions, reduce to the free motion solutions if charge is absent from either of the interacting systems. It is this relationship between the Coulomb wavefunctions and the free motion solutions which specifies the boundary conditions for the Coulomb functions. In turn, the asymptotic solution composed of the appropriate linear combination of Coulomb wavefunctions determines the boundary conditions for the relative motion wavefunction corresponding to the entire potential $V_{AT}(r_T)$.

Specifically the asymptotic form of the potential is the Coulomb interaction.

$$V_{AT}(r_T) \sim \hbar^2 k_T \eta_T / (\mu_T r_T),$$

in which the Sommerfeld parameter is

$$\eta_T = z_A z_T e^2 \mu_T / (\hbar^2 k_T),$$

with $z_A$ and $z_T$ the numbers of protons in the A- and T-systems, and $e$ the proton charge. For the case in which

$$V_{AT}(r_T^+) = \hbar^2 k_T \eta_T / (\mu_T r_T)$$

the general solution to equation (2-7) can be expressed (Ab 64) as
\[ \phi_{\alpha T}(r_T) = r_T^{-1} \sum_{L_T m_T} \left[ A_{L_T m_T} F_{L_T}(k_T r_T) + B_{L_T m_T} G_{L_T}(k_T r_T) \right] Y_{L_T}^{m_T}(\theta_T, \phi_T). \]

(2-13)

\( F_{L_T}(k_T r_T) \) and \( G_{L_T}(k_T r_T) \) are the regular and irregular Coulomb wavefunctions respectively. Their asymptotic forms are

\[ F_{L_T}(k_T r_T) \sim \sin(k_T r_T - \eta_T \ln(2k_T r_T) - (\pi L_T / 2) + \sigma_{L_T}), \]

\[ G_{L_T}(k_T r_T) \sim \cos(k_T r_T - \eta_T \ln(2k_T r_T) - (\pi L_T / 2) + \sigma_{L_T}), \]

\[ \sigma_{L_T} = \arg \Gamma (L_T + 1 + i \eta_T). \]

The Coulomb wavefunctions can be combined to form incoming and outgoing waves.

\[ I_{L_T}^*(k_T r_T) = O_{L_T}(k_T r_T) = G_{L_T}(k_T r_T) + i F_{L_T}(k_T r_T). \]

If the constants in equation (2-13) are chosen such that

\[ \phi_{\alpha T}(r_T) = \sum_{L_T} c^{L_T}_{\alpha T}(r_T) Y_{L_T}(\theta_T, \phi_T), \]

with

\[ c^{L_T}_{\alpha T}(r_T) = i^{L_T+1} \left[ 4\pi(2L_T + 1) \right]^{1/2} \exp(i\sigma_{L_T})(I_{L_T}(k_T r_T) - O_{L_T}(k_T r_T))/(2k_T r_T), \]

the solution has an asymptotic form \(|r_T - z_T| \to \infty\) composed of the sum of a distorted plane wave

\[ \exp \left[ i(k_T z_T + \eta_T \ln(k_T (r_T - z_T)) \right] \]

and a scattered wave
\[ f_{AT}^C(\theta_T) \exp \left[ i (k_T r_T - n_T \ln \left[ 2k_T r_T \right]) \right] / r_T. \]
\[ f_{AT}^C(\theta_T) = -n_T / (2k_T \sin^2(\theta_T/2)) \exp \left[ -i n_T \ln \left[ \sin^2(\theta_T/2) \right] + 2i \sigma \right], \]

(2-14)

so that the cross section is given by the Rutherford formula:

\[ \frac{d \sigma}{d \Omega} = 4k_T^2 \sin^4(\theta_T/2) \]

Moreover, in the absence of the Coulomb force, this solution reduces to the plane wave solution (2-9) or (2-10). These assertions are detailed in chapter 11 of (Me 62).

If the interaction \( V_{AT}(r_T) \) is now allowed to have a short-range (e.g. nuclear) part, outside the range of this force \( \phi_{AT}(r_T) \) contains additional outgoing terms. These terms are expressed in terms of the S-matrix.

\[ \phi_{AT}(r_T) = \sum_{L_T} \phi_{ATO}^L(r_T) \Gamma^O_L(\hat{r}_T), \]

\[ \phi_{ATO}^L(r_T) = i^{L_T+1} \left[ 4\pi (2L_T+1) \right] \frac{1}{2} \exp(i\sigma_{L_T})(I_{L_T}(k_T r_T) - S_{L_T} O_{L_T}(k_T r_T)) / (2k_T r_T) \]

(2-15)

The cross section becomes

\[ \frac{d \sigma}{d \Omega} = |f_{AT}(\theta_T)|^2, \]

(2-16)

\[ f_{AT}(\theta_T) = f_{AT}^C(\theta_T) + f_{AT}^S(\theta_T), \]

in which the Coulomb scattering amplitude \( f_{AT}^C(\theta_T) \) is given by equation (2-14), and

\[ f_{AT}^S(\theta_T) = (2ik_T)^{-1} \sum_{L_T} \left[ 4\pi (2L_T+1) \right] \frac{1}{2} \exp(2i\sigma_{L_T})(S_{L_T}^{-1} \Gamma^O_L(\hat{r}_T)) \]
2.5 The Relative Motion Solutions

In general for elastic scattering, the relative motion wavefunction is given by

$$\phi_{AT}(r_T) = \frac{\gamma}{L_T} \sigma_{AT(r_T)} Y_{0}(r_T),$$

$$\sigma_{AT(r_T)} = \frac{\gamma}{L_T} \left[ \frac{4\pi(2L_T+1)}{4\pi(2L_T+1)} \right] \exp(i\sigma_{L_T}) U_T(r_T)/(2k_T r_T).$$

The radial functions $U_T(r_T)$ are solutions of the differential equation

$$\left[ \frac{\hbar^2}{2\mu_T} \left( -\frac{d^2}{dr_T^2} + \frac{L_T(L_T+1)}{r_T^2} + V_{AT}(r_T) - F_T \right) \right] U_T(r_T) = 0 \quad (2-17)$$

Equation (2-17) is solved subject to the conditions that $U_T(r_T)$ is regular at the origin and becomes the solution given by equations (2-15) at large radii.

Thus, near the origin,

$$U_T(r_T) \sim \alpha_{L_T} \cdot (k_T r_T)^{L_T+1}/(2L_T+1).$$

Here $\alpha_{L_T}$ is a complex normalization constant.

$$U_T(r_T) = \alpha_{L_T} u_T(r_T).$$

For small radii,

$$u_T(r_T) \sim k_T r_T j_{L_T}(k_T r_T).$$

Equation (2-17) is easily numerically integrated to obtain the solutions $u_T(r_T)$: $u_T(0)$ and $u_T'(0)$ are known. At some
large value of \( r_T = \bar{r}_T \), outside the range of any short range interactions,

\[
U_T(\bar{r}_T) = a_{LT} u_T(\bar{r}_T) = I_{LT}(k_T\bar{r}_T) - S_{LT} O_{LT}(k_T\bar{r}_T)
\]

This equation, together with its derivative with respect to \( r_T \), can be solved simultaneously to find the unknowns \( a_{LT} \) and \( S_{LT} \). Since

\[
\begin{bmatrix}
  u_T(\bar{r}_T) & O_{LT}(k_T\bar{r}_T) \\
  u_T'(\bar{r}_T) & K_T O_{LT}(k_T\bar{r}_T)
\end{bmatrix}
\begin{bmatrix}
  a_{LT} \\
  S_{LT}
\end{bmatrix} =
\begin{bmatrix}
  I_{LT}(k_T\bar{r}_T) \\
  K_T I_{LT}'(k_T\bar{r}_T)
\end{bmatrix},
\]

the solutions

\[
\alpha_{LT} = \frac{\det \begin{bmatrix}
  I_{LT}(k_T\bar{r}_T) \\
  k_T I_{LT}'(k_T\bar{r}_T)
\end{bmatrix}}{\det \begin{bmatrix}
  u_T(\bar{r}_T) & O_{LT}(k_T\bar{r}_T) \\
  u_T'(\bar{r}_T) & k_T O_{LT}'(k_T\bar{r}_T)
\end{bmatrix}},
\]

\[
S_{LT} = \frac{\det \begin{bmatrix}
  u_T(\bar{r}_T) & I_{LT}(k_T\bar{r}_T) \\
  u_T'(\bar{r}_T) & k_T I_{LT}'(k_T\bar{r}_T)
\end{bmatrix}}{\det \begin{bmatrix}
  u_T(\bar{r}_T) & O_{LT}(k_T\bar{r}_T) \\
  u_T'(\bar{r}_T) & k_T O_{LT}'(k_T\bar{r}_T)
\end{bmatrix}}
\]

follow by Cramer's rule.
Substitution of these S-matrix elements into equation (2-16) yields the cross section for a given potential \( V_{A\ell}(r) \).

2.6 The Effective Interaction

In the preceding sections of this chapter, the potential \( V_{A\ell}(r) \) of equation (2-7) is treated as if it were a known function of \( r \), as opposed to a complicated term in an integro-differential equation. This rather simplistic approach to the nuclear scattering problem fails for real-valued parameterizations of the potential. In fact, the elastic channels of the wavefunction are strongly coupled to inelastic excitation channels and compound nuclear (one-partition) channels. The elastic channels may also be coupled, though usually less strongly, to transfer channels (two-partition channels composed of species different from those of the elastic channels) as well as channels of three or more partitions. Thus, the total wavefunction is written

\[
\psi_C(\mathcal{C}) = \sum_{\text{CN}} \psi_C(\mathcal{C}) +
\]

\[
+ \sum_{c-1,1} (C_{-1,1}^{c}) \psi_{c-1,1}(\mathcal{C}) \psi_{c-1}(\mathcal{C}) +
\]

\[
+ \sum_{c-2,2} (C_{-2,2}^{c}) \psi_{c-2,2}(\mathcal{C}) \psi_{c-2}(\mathcal{C}) +
\]

\[
+ \ldots + \sum_{\text{AT}} (C_{\text{AT}}^{c}) \psi_{\text{AT}}(\mathcal{C}) \psi_{\text{AT}}(\mathcal{C}) +
\]

While some of these terms may be negligible for a description
of elastic scattering, in general there are so many complicated overlap terms to be collapsed into \( V_{\Lambda T}(r_T) \) that a real, energy independent approximation to the potential is simply inadequate.

There are, however, well known techniques which have been developed to parameterize some or all of these potential terms. The rationale for this parameterization, called the optical potential, was originated by Bethe (Be 40), Feshbach (Fe 58, Fe 62), and others. These ideas are also formulated in more recent texts (Jo 63, Gl 67, Au 70).

Tractable solutions to the many body scattering problem necessarily involve a truncation of the total wavefunction. The parts of the wavefunction thought to be most important to the description of the scattering are kept, in addition to those parts which, though relatively small, are of special interest in the problem. The couplings between the retained channels and the truncated ones are incorporated into an energy dependent, complex-valued nonlocal potential.

A satisfactory description of the elastic channels is usually obtained when all other channels are truncated and a local potential is used. Perey and Buck (Pe 62) demonstrated this fact for neutron scattering. Experiments involving more complex projectiles have also been successfully analyzed with local potentials.

Generally, however, the empirically determined optical potentials must be complex and energy dependent. The imaginary
part of the potential characterizes flux between the retained and truncated channels: A real potential would not allow for this lack of conservation of flux within the retained channels. The energy dependence of the optical potential is indicative of the changing density of states available for population as the bombarding energy is varied.

The form of the potential used throughout this work is that of Woods and Saxon (Wo 54).

\[ V_A(r_T) = \frac{V}{1+e^{(r_T-r_V)/a_V}} + \frac{iW}{1+e^{(r_T-r_W)/a_W}} + V_C(r_T), \]

in which \( V \) and \( W \) are the depths of the real and imaginary potential wells, \( r_V \) and \( r_W \) their radii, and \( a_V \) and \( a_W \) their diffusenesses. The Coulomb potential is taken to be that of a uniformly charged sphere of radius \( r_C \).

\[ V_C(r_T) = \begin{cases} \frac{2k_TR_T}{2\mu_T r_C} (3 - \frac{r_T^2}{r_C^2}), & r_T \leq r_C, \\ \frac{2k_TR_T}{\mu_T r_T}, & r_T > r_C. \end{cases} \]

It is conventional to parameterize these radii in terms of the number of nucleons in the target and projectile systems. Thus, in terms of a constant nuclear density,

\[ r_V = r_{V0}(A^{1/3} + T^{1/3}), \]
\[ r_W = r_{W0}(A^{1/3} + T^{1/3}), \]
\[ r_C = r_{C0}(A^{1/3} + T^{1/3}). \]
2.7 Properties of the Potential

Certain important properties of the effective interaction follow immediately from the form of the optical potential. First, the behavior of the potential for both small and large values of $r_T$ is consistent with the previous assumptions about the radial forms of the relative motion solutions in equation (2-17). It is also apparent that most of the nuclear scattering within the elastic channels takes place in the so-called surface region, where $r_T \approx r_V \approx r_W$. If $r_T$ is appreciably larger than $r_V$, the nuclear part of the potential is negligible. If $r_T$ is less than $r_W$, the relative motion solutions are strongly damped.\(^5\) This absorption of elastic flux represents the population of compound nuclear states, and other channels, at small impact parameters.

Sometimes additional terms which are explicitly surface-peaked are added to this potential (Au 70). The functional form of these terms is that of the radial derivative of the Woods-Saxon nuclear potential. They may be used to absorb additional flux from the surface region or to describe the spin-orbital angular momentum interaction (Sa 64).

A different sort of angular momentum dependence was incorporated into the optical potential by Chatwin, Robson, et al. (Ro 71).

---

\(^5\) Exceptions to this statement occur when $|V| >> |W|$. Notable examples are among the light-ion ($T \leq 6$) potential parameters.
to describe back angle (large $\theta_T$) elastic scattering. This rather ad hoc parametrization involves the cutoff of the imaginary potential term for total angular momenta above some maximum value.

$$W = W_0/(1 + e^{(I - I_0)/A})$$

Here $\hat{I} = \hat{L}(\hat{r}_T) + \hat{J}_A + \hat{J}_T$.

These variations on the Woods-Saxon potential are not used in the present analysis to describe the relative motion. Their inclusion here is intended to be prefatory to discussions regarding the transfer channels.

Given the large number of parameters which have been introduced to describe the effective potential in the elastic channels, it is, perhaps, not surprising that there remain ambiguities in the determination of these parameters. Even if so-called four-parameter potentials (subject to the constraints $r_V = r_W = r_C$ and $a_V = a_W$) are used, it is common for experimental differential cross sections to be satisfactorily reproduced with many different sets of parameters. This situation reflects the peripheral nature of elastic scattering: Although the real nuclear parts of these equivalent
potentials may have widely varying depths in the absorptive central region, as a rule, they are nearly the same in the surface region (Ig 58).

The analysis of high quality data can sometimes remove some of the ambiguities in the potential parameters. Cage et al. (Ca 73) have studied the ambiguities and systematics of optical potentials for light ions and deduced a prescription for the preferable family of potentials. Unfortunately this procedure is not easily extended to heavy-ion interactions.

The unique determination of heavy-ion optical potentials is an ongoing problem requiring both experimental and theoretical work. (See, for example, the recent reviews (Br 76, Ha 76a).) Before beginning the description of the inelastic and transfer channels in the next two chapters, it is worthwhile to note that these reactions probe a slightly different part of the effective interaction. Inelastic and transfer reactions are still peripheral in nature, but they may provide useful information about the potential at radii somewhat smaller than that of elastic reactions. However, throughout these analyses it must be remembered that different truncations of the wavefunction imply different potential parameters. The best set of parameters for the description of only the elastic channels is generally different from the best set describing the elastic channels plus other channels, especially if the "other channels" are coupled strongly to the elastic ones.
3.1 Excitation Models

During the last thirty years a number of nuclear models have been developed and successfully applied to a wide range of phenomena. The models of interest here describe the low-lying states of either the A- or T-nucleus. The population of these states from the initial ground states by the scattering process is sometimes called quasielastic scattering.

A macroscopic description is usually employed to describe the collective excitation modes of a nucleus. These modes are characterized by deformations of the nuclear surface: Small deformations give rise to the well known vibrational spectra of even-even nuclei near closed shells; larger deformations yield the finer structure of rotational motion. A brief summary of the relevant definitions is given by Stelson and Grodzins (St 65). Tamura (Ta 65) applies this model to the scattering problem, while Eisenberg and Greiner (Ei 70) provide a detailed general development.

More sophisticated microscopic descriptions are also used to explain the lowest nuclear states in terms of elementary excitations of pairs of quasiparticles. Glendenning (Gl 67) developed the formalism for inelastic scattering involving these excitations, and Ascuitto and Glendenning (As 70, As 70a) extended the model to include transfer reactions. More
recently the quasiparticle random phase approximation was applied to heavy-ion transfer reactions in the no-recoil limit (As 74), and in the cluster transfer approximation (Le 77).

In the present work attention is restricted to macroscopic descriptions of vibrational nuclei. In particular, the so called weak-coupling limit is used. This first order approximation is adequate for the weak transitions to be studied and facilitates the calculations involving the population of relatively high spin states in the residual nucleus.

3.2 The Total Angular Momentum-Parity Basis

Now that the possibility for explicit inelastic couplings is considered, the truncated wavefunction must include low energy excited states in addition to the ground states of the A- and T-nuclei. Equation (2-5) becomes

$$\psi_C(r) = \left( \begin{array}{c} C_T \end{array} \right)^{-1/2} \sum_{\nu} \psi_T(\alpha_T) \phi_{AT}(\tau_T)$$

Because of the conservation of total angular momentum and parity throughout the processes being considered, it is convenient to label the parts of the wavefunction with the conserved quantities. Thus with the relative motion expanded

1In general the symbols A and T do not include the projections $M_A$ and $M_T$ when the total angular momentum basis is employed.
\[ \phi_{AT}(r_T) = \sum_{L_T m_T} \mathcal{F}_{ATM_T}^{L_T}(r_T) \gamma_{L_T}^{m_T}(r_T), \]
equation (3-1) can be written\(^2\)

\[ \psi_C(C) = \sum_{\pi T}^{L_T \pi A \pi T} \mathcal{F}_{ATM_T}^{L_T}(r_T) \phi_{\pi T}(r_T) \gamma_{L_T}^{m_T}(r_T), \]
in which \( \pi = (-)^{L_T \pi A \pi T} \) and

\[ \phi_T^M(\hat{T}_T, \hat{T}, \hat{A}) = \begin{pmatrix} C \end{pmatrix}^{-1} \sum_{L_T} (-1)^{\pi_L} \left[ \gamma_{L_T}(\hat{T}_T) \otimes \psi_{TJ_T}(T) \right]^{K_A \otimes \psi_{A J_A}(A)} M \]

The "C" symbols refer to Clebsch-Gordan coefficients and the square brackets denote vector coupling:

\[ \left[ \gamma_{L_T}(\hat{T}_T) \otimes \psi_{TJ_T}(T) \right]^{m_A} = \sum_{L_T} \gamma_{L_T}^{m_T}(r_T) \psi_{TJ_T}(T) \]

With the additional definition

\[ \mathcal{F}_{TM}^{L_T}(r_T) = C_{M-M_A-M_T T M_T M_A M_T} \mathcal{F}_{ATM_T}^{L_T}(r_T), \]

the total wavefunction becomes

\[ \psi_C(C) = \sum_{\pi T}^{L_T \pi A \pi T} \mathcal{F}_{TM}^{L_T}(r_T) \phi_{\pi T}(r_T) \gamma_{L_T}^{m_T}(r_T), \]

Insertion of this expression into the Schrödinger equation yields a generalization of equation (2-6).

\(^2\)The notation here is simplified in that the symbol "T" now represents the quantum numbers \( A, L_T, \) and \( K_A \) in addition to \( T, \) and \( T_0 \) denotes an entrance (elastic) channel.
With $T(r_T^+)$ expressed in spherical coordinates,

$$T(r_T^+) = -\frac{\alpha^2}{2\mu_T} \frac{1}{r_T^2} \left[ \frac{\partial}{\partial r_T} (r_T^2 \frac{\partial}{\partial r_T}) + \frac{1}{2\mu_T} \frac{1}{r_T^2} \right] L^2(\hat{r}_T^+),$$

so that

$$L^2(\hat{r}_T^+) \mathcal{Y}_{LT}^m(\hat{r}_T^+) = L_T(L_T+1) \mathcal{Y}_{LT}^m(\hat{r}_T^+),$$

the radial relative motion can be isolated from equation (3-3) by overlapping with a term of the form $\phi_{T',\pi} M' I' (\hat{r}_{T',\pi}^+, \hat{r}_T^+, \hat{r})$.

The result for channel $T$ is

$$-\frac{\alpha^2}{2\mu_T} \left[ \frac{\theta}{r_T^2} \frac{d}{dr_T} (r_T^2 \frac{d}{dr_T}) + \frac{L_T(L_T+1)}{r_T^2} \right] \mathcal{J}_{TT}^I (r_T^+) = -\sum_{T',\pi} \mathcal{V}_{TT'}^\pi (r_T^+) \mathcal{F}_{T' M}^\pi (r_T^+).$$

$$\mathcal{V}_{TT'}^\pi (r_T^+) = \int d\hat{r}_T^+ d\hat{r}_T^+ \left[ \mathcal{Y}_{LT}^* (\hat{r}_T^+) \mathcal{Y}_{TT}^* (\hat{r}_T^+) \right] K_A^* A_J_A (\hat{r}) \mathcal{M} \times$$

$$\times \sum_{\pi} (-1)^\pi \mathcal{V}_{AT}^\pi (\hat{r}_T^+) \left[ \mathcal{Y}_{LT}^* (\hat{r}_T^+) \mathcal{Y}_{TT}^* (\hat{r}_T^+) \right] K_A^* A_J_A (\hat{r}) \mathcal{M}$$

The reduced form for equation (3-4) follows immediately from the definition.
The radial equation is reexpressed as

\[
\frac{\hbar^2}{2\mu_{TT}} \left( -\frac{d^2}{dr_T^2} + \frac{LT(L_T+1)}{r_T^2} + V_{TT}^{\pi I}(r_T) - \xi_T^I \right) U_T^{\pi I}(r_T) = 0
\]

(3-5)

If equation (3-5) is solved subject to the asymptotic boundary conditions

\[
U_T^{\pi I}(r_T) \sim \delta_{TT}^{\pi I} L_T^{\pi I}(k_T r_T) - (k_T / k_m)^{\frac{1}{2}} \eta_{TT}^{\pi I} O_T^{\pi I}(k_T r_T),
\]

the total wavefunction (3-2) has the asymptotic form

\[
\psi_C^{\pi I}(r) \sim \sum \delta_{LM}^{\pi I} L_L^{\pi I} J_L^{\pi I} K_A^0 \epsilon_{T M}^{\pi I} M_T^0 M_o^0 M^0 \times
\]

\[
\times i^{L_T+1} \left[ \frac{4\pi (2L_T+1)}{L_T(L_T+1)} \right]^{\frac{1}{2}} \exp(i\sigma_{L_T}^{\pi I}) \left[ \delta_{TT}^{\pi I} L_T^{\pi I}(k_T r_T) - S_{TT}^{\pi I} O_T^{\pi I}(k_T r_T) \right] \times
\]

\[
\times \phi_{TT}^{\pi I}(\xi_T^{\pi I}, \xi_T^{\pi I}) / \left[ 2(k_T / k_m)^{\frac{1}{2}} r_T \right].
\]

Now with

\[
\delta_{TT}^{\pi I} L_T^{\pi I}(k_T r_T) - S_{TT}^{\pi I} O_T^{\pi I}(k_T r_T)
\]

rewritten as

\[
\delta_{TT}^{\pi I} L_T^{\pi I}(k_T r_T) - O_T^{\pi I}(k_T r_T) - (S_{TT}^{\pi I} - \delta_{TT}^{\pi I}) O_T^{\pi I}(k_T r_T),
\]
it is straightforward to obtain

\[ \psi_C(\zeta) \sim \psi_{\text{inc.}} \ast \psi_{\text{scat.}}, \]

in which

\[ \psi_{\text{inc.}} = \{ \exp[i(k_T z_T + \eta_T \ln[k_T (r_T - z_T)])] + \]

\[ + f_{A_T}^C(\theta_T) \exp[i(k_T r_T - \eta_T \ln[2k_T r_T])] / r_T \} \times \]

\[ \times (C_T)^{-\frac{1}{2}} \sum \} (-1)^V P_V \psi_{A_T}^{(T)} \psi_A(\lambda), \]

\[ f_{A_T}^C(\theta_T) = -\eta_T / (2k_T \sin^2(\theta_T/2)) \exp[-i\eta_T \ln[\sin^2(\theta_T/2)] + 2i\sigma_0], \]

\[ \psi_{\text{scat.}} = \sum \] \[ \frac{(2i)^{-1}(K_T K_T)^{-\frac{1}{2}}[4\pi(2L_T + 1)]^\frac{1}{2} L_{T_0} - L_T e^{(\sigma_{L_T} + \sigma_{L_T})} x \]

\[ \times (K_T J_T K_{A_T} - K_{A_T} J_{A_T} I \pi I) \]

\[ \times \left[ M_{T_0} M_T A_{T_0} M_{A_T} (S_{T_0} - \delta_{T_T}) \right] \times \]

\[ \exp[i(k_T r_T - \eta_T \ln[2k_T r_T])] / r_T \phi_{\text{scat.}} \]

In terms of the scattering amplitude,

\[ \psi_{\text{scat.}} = \sum f_{A_T}^S(\hat{\phi}_T) \exp[i(k_T r_T - \eta_T \ln[2k_T r_T])] / r_T \times \]

\[ \times (C_T)^{-\frac{1}{2}} \sum \} (-1)^V P_V \psi_{A_T}^{(T)} \psi_A(\lambda). \]

Therefore,

\[ f_{A_T}^S(\hat{\phi}_T) = (2i)^{-1}(K_T K_T)^{-\frac{1}{2}} \sum \] \[ \frac{L_{T_0}}{L_T} K_{A_T} L T_A I [4\pi(2L_T + 1)]^\frac{1}{2} \times \]

\[ L_{T_0} - L_T e^{(\sigma_{L_T} + \sigma_{L_T})} C_{M_{T_0} M_T A_{T_0} M_{A_T} M_{T_0} + M_{A_T}} \]
Finally

\[
\frac{d\sigma}{d\Omega}_T = r_T^2 \left[ \Theta_{\text{scat.}}(r_T) \Theta_{\text{inc.}} \cdot \right. \\
\left. = (k_T/k_T') \left| f_{A_0T_0} \delta_{A0} \delta_{T0} + f_{A1} \right|^2 \right].
\]

The degeneracy among the different angular momentum projections is traditionally handled by averaging over the initial states and summing over the final ones. Thus,

\[
\frac{d\sigma}{d\Omega}_T = (2J_{T0}+1)^{-1} (2J_{A0}+1)^{-1} \sum_{M_{T0},M_{A0},M_{T1},M_{A1}} \left( \frac{d\sigma}{d\Omega}_T \right).
\]

### 3.3 Inelastic Coupling Elements

In order to solve the coupled set of equations (3-4), the matrix elements \( V_{TT}^I(r_T) \) must be parameterized. To this end, the spherical Woods-Saxon potential of section 2.6 is allowed to have small deformations. Thus

\[
\begin{align*}
\hat{r}_{v} &= \hat{r}_{v}(0) + R_v, \\
\hat{r}_{w} &= \hat{r}_{w}(0) + R_w, \quad \text{and} \\
\hat{r}_{c} &= \hat{r}_{c}(0) + R_c,
\end{align*}
\]

in which
The potential becomes

\[
\frac{V}{1 + \exp \left[ \frac{(r_{T} - r_{v})}{a_{v}} \right]} + i \frac{W}{1 + \exp \left[ \frac{(r_{T} - r_{w})}{a_{w}} \right]} + V_{c}(\hat{r}_{T}),
\]

\[
V_{c}(\hat{r}_{T}) = \frac{\mu_{T}^{2}}{\pi^{2}} k T_{\pi} / \mu_{T} \int d\vec{r}_{T} \rho(\hat{\vec{r}}_{T}) / |\hat{\vec{r}}_{T} - \hat{\vec{r}}_{T}^{c}|,
\]

\[
\rho(\hat{\vec{r}}_{T}) = \left[ \frac{3}{4\pi r_{c}^{3}(0)} \right] \theta(r_{c} - r_{T}).
\]

The \( \theta \) function is the step function, so that the charge density is taken to be constant within the deformed volume specified by the Coulomb radius \( r_{c} \).

To the first order\(^3\) in \( r_{v} \) and \( r_{w} \), the nuclear potential is

\[
\frac{V}{1 + e_{v}} + i \frac{W}{1 + e_{w}} +
\]

\[
+ V(e_{v} r_{v} / a_{v}) / (1 + e_{v})^{2} + i W(e_{w} r_{w} / a_{w}) / (1 + e_{w})^{2},
\]

with \( e_{v} = \exp \left[ \left( r_{T} - r_{v}(0) \right) / a_{v} \right] \),

and \( e_{w} = \exp \left[ \left( r_{T} - r_{w}(0) \right) / a_{w} \right] \).

With the usual multipole expansion

\( ^{3}\)The following Taylor expansions are straightforward but tedious. Similar ones are given to second order in (Ta 65) and all orders in (Gl 67).
the Coulomb potential is given to first order in \( R_C \) by

\[
\frac{\mathcal{A}^2 k_T \mu_T}{2 \mu_T r_C(0)} \left( 3 \frac{x_T^2}{r_C^2(0)} + 3 \frac{\mathcal{A}^2 k_T \mu_T}{\mu_T} \sum_{\lambda \mu} \frac{x_T^\lambda r_C^{-(\lambda+2)}(0)}{2 \mu_T r_C(0)} \right) R_C(0) \alpha_\mu \gamma_\lambda^\mu(\hat{r}_T)
\]

if \( r_T < r_C \), and

\[
\frac{2 \mathcal{A}^2 k_T \mu_T}{\mu_T r_T} + \frac{3 \mathcal{A}^2 k_T \mu_T}{\mu_T} \sum_{\lambda \mu} \frac{x_T^\lambda r_C^{-(\lambda+2)}(0)}{2 \mu_T r_C(0)} R_C(0) \alpha_\mu \gamma_\lambda^\mu(\hat{r}_T)
\]

if \( r_T \geq r_C \). These two cases are connected by the approximation that the first expression is valid for \( r_T < r_C(0) \) while the second holds true for \( r_T \geq r_C(0) \).

In this macroscopic approach to the inelastic scattering it is customary to ignore exchange effects in the evaluation of the potential matrix elements. Therefore,

\[
\mathcal{V}_{T T}(r_T) = \int d \mathcal{A} d \mathcal{A} d \mathcal{A} \left[ \left\{ \gamma_{L_T}^* (\hat{r}_T) \psi_T^* (\hat{r}_T) \right\}_{K_A} \otimes \psi_A^* (\hat{r}_T) \right]_I \times \mathcal{V}_{A T}(\mathcal{A}) \left[ \left\{ \gamma_{L_T}^* (\hat{r}_T) \otimes \psi_T^* (\hat{r}_T) \right\}_{K_A} \otimes \psi_A^* (\hat{r}_T) \right]_I.
\]

The interaction between the \( A- \) and \( T- \) systems has the form

\[
\mathcal{V}_{A T}(\mathcal{A}) = v(r_T) + \sum_\lambda v_\lambda(r_T) \left( \mathcal{A} \otimes \gamma_\lambda^\mu(\hat{r}_T) \right), \quad (3-6)
\]

in which the scalar product is given by

Note that this somewhat specialized form excludes projectile excitation.
Equation (3-6) describes the deformed potential if
\[ q_{\lambda}^{\mu}(R) = a_{\lambda}^{\mu}(R), \]
and
\[ n(r_T) = n/(1+e_v) + iW/(1+e_w) + n^{(c)}(r_T), \]
where
\[ \nu^{(c)}(r_T) = \begin{cases} \frac{2k_T n_T}{\mu T r_{C}(0)} (3 - \frac{r_T^2}{r_{C}(0)^2}), & r_T \leq r_{C}(0) \\ \frac{2k_T n_T}{\mu T r_T}, & r_T > r_{C}(0) \end{cases} \]
and
\[ \nu_{\lambda}(r_T) = \nu(e_v r_{V}(0)/a_v)/(1+e_v)^2 + i W(e_w r_{W}(0)/a_w)/(1+e_w)^2 + \nu^{(c)}(r_T), \]
where
\[ \nu^{(c)}(r_T) = \begin{cases} \frac{3k_T n_T}{\mu T} \frac{r_T r_{C}^{-(\lambda+2)}(0)}{2\lambda+1} r_{C}(0), & r_T \leq r_{C}(0) \\ \frac{3k_T n_T}{\mu T} \frac{r_T r_{C}^{-\lambda-1}(0)}{2\lambda+1} r_{C}(0), & r_T > r_{C}(0) \end{cases} \]
The relation
\[ \sum_{\lambda}^{\mu} a_{\lambda}^{\mu} Y_{\lambda}^{\mu}(\hat{r}_T) = \sum_{\lambda} (a_{\lambda}^{\mu} \otimes Y_{\lambda}(\hat{r}_T)) \]
used here follows simply from the property
\[ a_{\lambda}^{\mu*} = (-1)^{\mu} a_{\lambda}^{-\mu}. \]
which is a consequence of the reality of the nuclear radius.

Now with the explicit form of the interaction specified by (3-6), application of the Wigner-Eckart theorem yields:

\[ v_{TT'}(r_T) = \delta_{TT'} \nu(r_T) + \sum_{\lambda} \nu_\lambda(r_T) \left( -1 \right)^{J_A+I+K_A'} \binom{J_A}{J_A',K_A',\lambda} \times \]

\[ \sum \left[ \gamma_{LT}(T) \otimes \psi_{T'}(T) \right] K_A^{-1} \left| \gamma_\lambda(T) \right| \left| \sum \gamma_{LT}(T) \otimes \psi_{T'}(T) \right] K_A^{-1} \left| \psi_{A}^{*}(A) \right| \left| \psi_{A}^{*}(A') \right| \]

The quantity \[ \binom{J_A}{J_A',K_A',\lambda} \] is a \(6\ j\)-symbol.

The first reduced matrix element is easily evaluated. It is nonzero only if the T-system does not change its state. In that case the matrix element becomes a simple geometrical factor:

\[ \left( -1 \right)^{J_T+K_A'} \frac{\lambda}{4\pi} \binom{K_A K_A',L_T L_{T',\lambda}}{L_T, L_{T'}, \lambda} \binom{L_T K_A J_T}{K_A',L_{T'},\lambda} \]

\( K_A \) is defined as \( 2K_A + 1 \), and \( (o\ o\ o) \) represents a \(3\ j\)-symbol.

The second reduced matrix element is evaluated in Appendix C for the weak couplings considered here. The result is given by equation (c-3):

\[ \left( \psi_{A}^{*}(A) \right| a_{\lambda}^{*}(A) \left| \psi_{A'}^{*}(A') \right) = \left( \frac{\lambda}{\lambda} \right)^{1/2} \beta_{\lambda} \]

5Racah's definition of the reduced matrix elements is used throughout this work.
Conventions associated with the potential and deformation parameters are discussed in Appendix D. Simple properties of the inelastic excitation are presented in Section 3.5.

3.4 Solution of the Coupled Equations

The sets of coupled equations (3-5) are solved in a manner similar to that used in section 2.5 for the uncoupled equations (2-17)\(^6\). For the coupled equations it is necessary to generate a set of linearly independent solutions upon which the relative motion solutions are expanded.

\[ U_{\pi T}(r_T) = \sum_{T''} \alpha_{T''}^{\pi I} u_{TT''}(r_T). \]

Substitution into (3-5) yields

\[
\left[ \frac{\mu^2}{2\mu_T} \left( -\frac{d^2}{dr_T^2} + \frac{L_T(L_T+1)}{r_T^2} \right) + V_{TT}(r_T) - E_T \right] u_{TT''}(r_T) = \]

\[ \sum_{T' \neq T} V_{TT'}(r_T) u_{TT'}(r_T). \]

If the functions \( u_{TT''}(r_T) \) on the right side of (3-7) are known, the differential equation can be solved for all values of \( T'' \). In particular, the boundary conditions near the origin

\(^6\)This method is developed in (Gl 67. and As 69) for inelastic excitations and transfers.
\[ u_{TT}^{\pi I}(r_T) \sim (k_T r_T)^{L+1} \delta_{TT}^\pi / (2L+1)! \]

imply that

\[ u_{\pi I}^T(r_T) \sim \alpha_T^\pi I k_T r_T^{L_T} (k_T r_T) \]

which is the proper behavior for small \( r_T \).

In practice the source terms on the right side of (3-7) are not initially known, so that the equations are solved by iteration. First the ground state solutions \( (T=T_0) \) are approximated by the homogeneous solutions. These functions are substituted into the right side of (3-7) to get first order solutions for the excited states. In turn, the functions for an excited state are coupled to those of the ground state and other excited states for higher order solutions. The iteration procedure continues until any desired level of accuracy is reached. (First order is sufficient for weak coupling.)

Once the functions \( u_{TT}^{\pi I}(r_T) \) are known, the coefficients \( \alpha_T^\pi I \) and the S-matrix elements can be found. At some large value of \( r_T = \bar{r}_T \), outside the range of any short range interactions,

\[ u_{\pi I}^T(\bar{r}_T) = \sum_T \alpha_T^\pi I u_{TT}^{\pi I}(\bar{r}_T), \]

\[ = \delta_{TT}^0 L_{T_0} (k_T \bar{r}_T) - (k_T / k_T) \delta_{TT}^I L_T \]

This set of equations, together with their derivatives with respect to \( r_T \), can be solved simultaneously to find the
unknowns $\alpha_{\pi I}$ and $S_{\pi I}^{\pi I}$. These S-matrix elements are used to obtain the scattering amplitudes and cross sections of section 3.2.

3.5 Properties of the Inelastic Excitation

Many of the ideas associated with the systematics of inelastic excitation are applicable to a wide range of phenomena, extending from early analyses of inelastic alpha particle scattering (Ba 62, Ro 62) through various kinds of transfer mechanisms. Although the physical processes occurring in these reactions differ considerably, simple concepts arising from where the processes occur in the nuclear wavefunction can be used to explain the general features of the angular distributions.

Excitations of the nucleus which result from surface deformations correspond, of course, to the surface region of the nuclear wavefunction. This region, delineated by some range in $r_T$, has associated with it a range of $L_T$ in the partial wave expansion. In simple cases, a broad range of terms in $L_T$ results in a relatively narrow angular distribution, while a localization in $L_T$-space yields a broad, somewhat forward-peaked distribution. If the number of terms in the scattering amplitude sum is sufficiently small, forward-angle oscillations appear with a frequency

$^7$The general properties outlined in this section are detailed in (As 74).
which is characteristic of the surface partial waves.

Compared to the typical bell-shaped angular distributions of direct one-nucleon transfer reactions, the inelastic excitation distributions are broad and oscillatory. These properties result from the localization in $L_T$-space of the nuclear excitation; the longer range Coulomb excitation is generally weaker. (Sometimes the destructive interference between the high-$L_T$ Coulomb terms and low-$L_T$ nuclear terms is observed as a dip in the inelastic excitation distributions.)

Finally a generalization can be made. If in conjunction with inelastic excitation, some other process with a larger "$L_T$-window" occurs (e.g. particle transfer), this multistep process is easily distinguished from the corresponding direct one without the inelastic excitation. The multistep process has a broader, more forward-peaked distribution than the purely direct reaction. This fact is illustrated by figures (3-1) and (3-2).

Figure (3-1) is a comparison of the $S$-matrix localization in angular momentum space, between direct proton transfer, and multistep transfer plus inelastic excitation. The reaction of this example is $^{48}\text{Ti}(^{16}\text{O},^{15}\text{N})^{49}\text{V}(3/2^-)$, with a projectile energy of 50 MeV. No Coulomb terms are included in the transfer interaction. Partial waves above $\sim 26\hbar$ tend to be dominated by Coulomb excitation in the multistep route. The magnitudes of the $S$-matrix elements are
Figure (3-1). A comparison of the S-matrix localization in angular momentum space, between direct proton transfer, and multistep transfer followed by inelastic excitation. The direct transfer mechanism yields much wider distributions in $L_T$-space.
Fig. (3-1)

\[ ^{48}\text{Ti}(^{16}\text{O},^{15}\text{N})^{49}\text{V} \]

\[ E = 50 \text{ MeV} \]
\[ \frac{1}{2}^- \]
\[ 0.153 \text{ MeV} \]

\[ |S_{PT0}| \]

\[ I \text{ or } L_{T0} (\text{h}) \]

\[ |S_{PT0}| \]

\[ I \text{ or } L_{T0} (\text{h}) \]
parameterized by channel labels defined in table (3-1).

Figure (3-2) shows the corresponding angular distributions for direct and multistep transfer. Some destructive nuclear-Coulomb interference in the multistep route is evident in the shallow dip around 49°, a few degrees forward of the grazing peak of the direct route.
Figure (3-2). A comparison of typical angular distributions resulting from a direct transfer mechanism, and an indirect mechanism consisting of transfer followed by inelastic excitation. The multistep route is characterized by a broader, more forward-peaked angular distribution.
Fig. (3-2)

$^{48}\text{Ti}^\left(^{16}\text{O},^{15}\text{N}\right)^{49}\text{V}$

E = 50 MeV

$^{-3/2}$

0.153 MeV

$\frac{d\sigma}{d\Omega} \left( \mu b/sr \right)$

$\theta_{cm}$
TABLE (3-1)\textsuperscript{8}

Transfer Channels to $^{49}$\textit{\Psi}(3/2\textsuperscript{-})

a: $I = I_{T_{O}} = I_{p} + 2\hbar = K_{B} + 3/2 \hbar$

b: $I = I_{T_{O}} = I_{p} = K_{B} + 1/2 \hbar$

c: $I = I_{T_{O}} = I_{p} = K_{B} - 1/2 \hbar$

d: $I = I_{T_{O}} = I_{p} - 2\hbar = K_{B} - 3/2 \hbar$

\textsuperscript{8}Particle transfer notation is described in Chapter 4.
CHAPTER 4
THE TRANSFER CHANNELS

4.1 The Coupled Equations

In discussing the channels which are populated by the transfer of particles from or to the projectile (These processes are defined as stripping and pick-up respectively), it is once again necessary to generalize the model wavefunction. If, for example, N particles are stripped from the projectile and captured by the target during the reaction, the outgoing system has P=T-N particles, while the residual nucleus contains B=A+N particles. These new systems have bound-state wavefunctions respectively given by

\[(H(P) - \varepsilon_P) \psi_P(R_P) = 0\]

and

\[(H(B) - \varepsilon_B) \psi_B(R_B) = 0\]

The total wavefunction, previously given by (3-1), now becomes

\[
\psi_C(R) = (C)^{-1} \sum_{AT} \sum_{\nu} (-1)^{\nu_P} \psi_A(R_A) \psi_T(R_T) \phi_{AT}(r) + \\
(C)^{-1} \sum_{BP} \sum_{\nu} (-1)^{\nu_P} \psi_B(R_B) \psi_P(R_P) \phi_{BP}(r).
\]

The permutation operator in the first term interchanges particles between the A- and T- systems, while that of the second term interchanges particles between the B- and
P-systems. The relationships between the spatial coordinates of these different arrangements of particles are given in Appendix E for one- and two-nucleon transfer.

In terms of the total angular momentum-parity basis, equation (4-1) is written

$$\psi_C(\xi) = \sum_{IM} \left\{ \sum_T \mathcal{F}_{TM}(r_T) \psi_{T_M}(T, \xi) + \sum_P \mathcal{F}_{PM}(r_P) \psi_{P_M}(P, \xi) \right\}$$

The T-channel basis functions are again given by

$$\phi_{T_M}(r_T) = \sum_L \sum_J \sum_K \sum_{I'} Y_L^J(r_T) \phi_{K_M}(K, \xi) \phi_{I'M}(I', \xi)$$

while those for the P-channels are

$$\phi_{P_M}(r_P) = \sum_L \sum_J \sum_K \sum_{I'} Y_L^J(r_P) \phi_{K_M}(K, \xi) \phi_{I'M}(I', \xi)$$

The radial functions are related to the relative motion wavefunctions by

$$\mathcal{F}_{TM}(r_T) = C_{M-M_A-M_T} C_{M-M_A-M_T} F_{T_M}(r_T)$$

And

$$\mathcal{F}_{PM}(r_P) = C_{M-M_B-M_P} C_{M-M_B-M_P} F_{P_M}(r_P$$

when the expansions
\[ \phi_{AT}(r_T) = \sum_{L_T m_T} \mathcal{F}_{AT m_T}(r_T) \gamma_{L_T}^{m_T}(\hat{r}_T) \]

and

\[ \phi_{BP}(r_P) = \sum_{L_P m_P} \mathcal{F}_{BP m_P}(r_P) \gamma_{L_P}^{m_P}(\hat{r}_P) \]

are used.

If the total wavefunction (4-2) is substituted into the Schrödinger equation, and the overlap with a basis state of the form \( \phi_{T, \pi I}^{M}(\hat{r}_T, \hat{r}_I, \hat{r}_A) \) is taken, the T-channel portion of the wavefunction (4-2) yields the terms

\[
\left[ \frac{\hbar^2}{2m_T} \left( \frac{1}{r_T^2} \frac{d}{dr_T} \right)^2 + \frac{L_T(L_T+1)}{r_T^2} + V_{TT}^{\pi I}(r_T) - E_T \right] \phi_{TM}^{\pi I}(r_T) +
\]

\[
+ \sum_{T' \neq T} V_{TT'}^{\pi I}(r_T) \phi_{TM'}^{\pi I}(r_T)
\]

as before. These terms, together with the analogous overlaps for the P-channels, must have a sum of zero. The P-channel overlaps are

\[
\int d\hat{r}_T \left\{ (H(C)_E) \phi_{T, \pi I}^{M}(\hat{r}_T, \hat{r}_I, \hat{r}_A) \right\}^* \times
\]

\[
\times \sum_{P} \phi_{PM}^{\pi I}(r_P) \phi_{P, \pi I}^{M}(\hat{r}_P, \hat{r}_I, \hat{r}_A)
\]

\[
= \int d\hat{r}_T \left\{ \left[ \phi_{T, \pi I}^{M}(\hat{r}_T, \hat{r}_I, \hat{r}_A) \right]^* \times
\]

\[
\times \left[ \left[ L_T(\hat{r}_T) \otimes \phi_{T, \pi I}^{M}(\hat{r}_T) \right]_{K_A} \otimes \phi_{A}(\hat{r}_A) \right] \times \sum_{P} \phi_{PM}^{\pi I}(r_P) \phi_{P, \pi I}^{M}(\hat{r}_P, \hat{r}_I, \hat{r}_A)
\]
in which the nonorthogonality term is

\[ \mathcal{J}_{PM}(r_T) = \int d\alpha d\beta \phi^*_T(r_T, \alpha) \int d\gamma \int d\delta \phi^*_M(r_T, \gamma) \mathcal{J}_{PM}(r_T, \gamma, \alpha) \]

and the source term is \( p_{TP}^{\pi I}(r_T) = \)

\[ = \int d\alpha d\beta d\gamma \left( C_T - k_T (-1) p_T (V_{AT}(C_T) - V_{TT}(r_T)) \times \right) \]

\[ \times \left[ \mathcal{Y}_{L_T}(A_T) \hat{\phi}_{T}(A) \right] K_A \hat{\phi}_A(A) \int \mathcal{J}_{PM}(r_T, \gamma, \alpha) \phi^*_T(r_T, \gamma, \alpha) \]

The Schrodinger equation for channel \( T \) is obtained by adding the \( T \)- and \( P \)-overlaps,

\[ \frac{\hbar^2}{2 \mu_T} \left( \frac{1}{r_T} \frac{d}{dr_T} \left( r_T^2 \frac{d}{dr_T} \right) + \frac{L_T(L_T+1)}{r_T^2} + V_{TT}(r_T) - E_T \right) \mathcal{J}_{\pi I}(r_T) = \]

\[ - \mathcal{E} \sum_{T \neq T, M} \mathcal{J}_{\pi I}(r_T) \mathcal{J}_{T,M}(r_T) - p_{TP}^{\pi I}(r_T). \] (4-4)

Similarly, for channel \( P \),

\[ \frac{\hbar^2}{2 \mu_P} \left( \frac{1}{r_P} \frac{d}{dr_P} \left( r_P^2 \frac{d}{dr_P} \right) + \frac{L_P(L_P+1)}{r_P^2} + V_{PP}(r_P) - E_P \right) \mathcal{J}_{\pi I}(r_P) = \]

\[ - \mathcal{E} \sum_{P \neq P, M} \mathcal{J}_{\pi I}(r_P) \mathcal{J}_{P,M}(r_P) - p_{TP}^{\pi I}(r_P). \] (4-5)

The reduced forms of these equations follow from
\[ G_{\gamma \zeta}^{\pi I}(r_{\zeta}) = \begin{cases} \Sigma \begin{cases} L_{TO} & J_{TO} & K_{AO} & K_{AO} & J_{AO} & I \\ L_{TO} & K_{AO} & C & O & M_{TO} & M_{TO} & C & M_{TO} & M_{AO} & M \end{cases} \end{cases} \times \\
i L_{TO} + 1 \left[ 4\pi (2L_{TO} + 1)^{\frac{1}{2}} e^{i\sigma L_{TO} (2K_{TO})^{-1}} \right] U_{\gamma}^{\pi I}(r_{\zeta})/r_{\zeta}, \] (4.6)

in which \( \gamma, \zeta \) each take on the values \( T, P \). The result, for either partition, is

\[ \left[ \frac{\mu^2}{2u_{\gamma}} \frac{(-d^2)}{dr_{\gamma}^2} + \frac{L_{\gamma}(L_{\gamma}+1)}{r_{\gamma}^2} + V_{\gamma \gamma}(r_{\gamma}) - E_{\gamma} \right] (U_{\gamma}^{\pi I}(r_{\gamma}) + U_{\zeta}^{\pi I}(r_{\gamma})) = \]

\[ = - \sum_{\gamma \neq \gamma} V_{\gamma \gamma},(r_{\gamma}) U_{\gamma}^{\pi I}(r_{\gamma}) - \rho_{\gamma \zeta}^{\pi I}(r_{\gamma}) \] (4.7)

The reduced source term is given by

\[ p_{\gamma \zeta}^{\pi I}(r_{\gamma}) = \begin{cases} \Sigma \begin{cases} L_{TO} & J_{TO} & K_{AO} & K_{AO} & J_{AO} & I \\ L_{TO} & K_{AO} & C & O & M_{TO} & M_{TO} & C & M_{TO} & M_{AO} & M \end{cases} \end{cases} \times \\
i L_{TO} + 1 \left[ 4\pi (2L_{TO} + 1)^{\frac{1}{2}} e^{i\sigma L_{TO} (2K_{TO})^{-1}} \right] \rho_{\gamma \zeta}^{\pi I}(r_{\gamma})/r_{\gamma}. \]

Therefore

\[ \rho_{\gamma \zeta}^{\pi I}(r_{\gamma})/r_{\gamma} = \\
= \int dC_{\gamma} d\phi_{\gamma} d^2 r_{\gamma} \left\{ (C_{\gamma})^{-\frac{1}{2}} \Sigma (-1)^{\nu_p} (\nu_{C_{\gamma}, \gamma}(C_{\gamma}) - U_{\gamma \gamma}^{\pi I}(r_{\gamma})) \times \\
\times \left[ \begin{bmatrix} L_{\gamma}(C_{\gamma})^0 \psi_{\gamma}(C_{\gamma}) \\ L_{\gamma} \end{bmatrix} K_{C_{\gamma}} \Theta_{C_{\gamma}} (C_{\gamma}) \right] \right\}^{\ast} \Sigma U_{\gamma}^{\pi I}(r_{\zeta})/r_{\zeta} \phi_{\zeta}^{\pi I}(r_{\zeta}) \zeta \zeta \zeta. \] (4.8)

The notations \( C_{\gamma} \) and \( C_{\zeta} \) each refer to the \( \Lambda \)- and \( B \)- systems.

The nonorthogonality term \( U_{\gamma}^{\pi I}(r_{\gamma}) \) in (4.7) falls to zero very rapidly for large \( r_{\gamma} \) (Go 72). Thus if the equation is solved for the sum \( U_{\gamma}^{\pi I}(r_{\gamma}) \equiv U_{\gamma}^{\pi I}(r_{\gamma}) + U_{\zeta}^{\pi I}(r_{\gamma}) \), the
asymptotic boundary conditions are the same for $U^\Pi_Y(r_\gamma)$ as for the sum. The required conditions are

$$U^\Pi_Y(r_\gamma) = \delta_{\gamma T_0} L_{T_0} (k_{T_0} r_\gamma) - (\mu_\gamma k_{T_0}/(\mu_T k_\gamma)) S^\Pi_{\gamma T_0} O_{\gamma T_0} k_\gamma (r_\gamma).$$

(4-9)

If equation (4-7) is solved subject to (4-9), that part of the total wavefunction corresponding to the T-A partition has the same asymptotic behavior as that of the total wavefunction described in section 3.2. Consequently the present scattering amplitudes and cross sections are given by the same formulas as in 3.2 for the T-channels. The part of the total wavefunction corresponding to the P-B partition has the asymptotic form

$$\sum P^{-\mathcal{M}} L_{T_0} J_{T_0} K_{A_0} J_{A_0} I_{M_{T_0}} M_{T_0} M_{A_0} M^{-\mathcal{M}_L} T_{L_0} S^\Pi_{T_0} O_L (k_{P_{\mathcal{M}_L}}) \phi_{\mathcal{M}_L} (r_{P_{\mathcal{M}_L}}, R_{P_{\mathcal{M}_L}})/(2k_{T_0} r_P),$$

$$\sum P^{-\mathcal{M}} L_{T_0} K_{A_0} M_{T_0} M_{A_0} M^{-\mathcal{M}_L} T_{L_0} S^\Pi_{T_0} O_L (k_{P_{\mathcal{M}_L}}) \phi_{\mathcal{M}_L} (r_{P_{\mathcal{M}_L}}, R_{P_{\mathcal{M}_L}})/(2k_{T_0} r_P),$$

$$\sum P^{-\mathcal{M}} L_{T_0} K_{A_0} M_{T_0} M_{A_0} M^{-\mathcal{M}_L} T_{L_0} S^\Pi_{T_0} O_L (k_{P_{\mathcal{M}_L}}) \phi_{\mathcal{M}_L} (r_{P_{\mathcal{M}_L}}, R_{P_{\mathcal{M}_L}})/(2k_{T_0} r_P),$$

As a result, the corresponding scattering amplitude is
The transfer cross sections are given by

$$f_{BP}(\hat{r}_P) = (2i)^{-1}(\mu_p/(\mu_{T_0}k_Tk_P))^\frac{1}{2} \sum_{L_{T_0}} K_{A_0} L_P K_{B_1} [4\pi(2L_{T_0} + 1)]^\frac{1}{2} \times$$

$$\times i e^{i(\sigma_{L_{T_0}} + \sigma_{L_P})} L_{T_0} J_{T_0} K_{A_0} K_{A_0} J_{A_0} I_{T_0} M_{T_0} M_{T_0} M_{A_0} M_{A_0} M_{T_0} + M_{A_0} \times$$

$$\times S_{P_{T_0}}^{\pi I} L_P J_P K_B K_B J_B I_{T_0} \sum_{m_P} \sum_{m_B} \sum_{m_{T_0}} \sum_{m_{A_0}} \sum_{m_{T_0} + m_{A_0}} m_P L_P (\hat{r}_P).$$

The transfer cross sections are given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{BP} = \frac{k_{P_{T_0}}}{k_{T_0}} |f_{BP}|^2.$$

More generally,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\gamma} = \frac{k_{\gamma_{T_0}}}{k_{T_0}} |f_{A_0 T_0} \delta_{C-\gamma, A_0} \delta_{\gamma, T_0} + f_{C-\gamma, \gamma}|^2$$

and

$$\frac{d\sigma}{d\Omega} = (2J_{T_0} + 1)(2J_{A_0} + 1) \sum_{M_{T_0}} \sum_{M_{A_0}} \sum_{M_{C-\gamma}} \left(\frac{d\sigma}{d\Omega}\right)_{\gamma}$$

for a particular set of states in the final arrangement of particles.

These formulas hold for the present example of stripping, but they must be generalized to incorporate pick-up reactions. It is convenient to retain the relationship $T = P + N$, so that the $T$-channels are always those in which the transferred particles are bound to the $P$ core. With this specification the entrance channels are
denoted $T_o$ for stripping or $P_o$ for pick-up, $\xi_o$ in general.

The cross section for either case is

$$\frac{d\sigma}{d\Omega} = (2J^{\xi_o} + 1)^{-1}(2J^{C-\xi_o} + 1)^{-1} \sum_{M^{C-\xi_o} M^C M^C} \frac{d\sigma}{d\Omega}^C, \quad (4-10)$$

$$\frac{d\sigma}{d\Omega} = \frac{C}{k^{C, \xi_o, \mu, \gamma}} \left| f_{C-\xi_o, \xi_o} \delta_{C, C-\xi_o} \xi_o \delta_{\gamma, \xi_o} + f_{C-\gamma, \gamma} \right|^2,$$

$$f_{C-\xi_o, \xi_o} = -\eta_{\xi_o}/(2k^{C, \xi_o} \sin^2(\theta/2)) \exp \left[ -i\eta_{\xi_o} \sin^2(\theta/2) + 2i\sigma \right],$$

$$f_{C-\gamma, \gamma} = (2i)^{-1}(m_{\gamma}/(\xi_o k)) \left[ L^{C, \xi_o} L^C - I \right] \left[ 4\pi(2L_{\xi_o} + 1) \right] \chi_{\xi_o} C_{\xi_o} M_{\xi_o} M_{\xi_o} M_{\xi_o} M_{\xi_o} M_{\xi_o} M_{\xi_o} M_{\xi_o} M_{\xi_o} M_{\xi_o}$$

$$x S_{\gamma} \left[ L_{\gamma} J_{\gamma} K_{\gamma} M_{\gamma} M_{\gamma} M_{\gamma} M_{\gamma} M_{\gamma} M_{\gamma} M_{\gamma} M_{\gamma} \right] \gamma_{\gamma}(r_{\gamma})$$

4.2 Solution of the Transfer Equations

The solution of equations (4-7)

$$\left[ \frac{m^2}{2\mu_{\gamma}} + \frac{L_{\gamma} L_{\gamma} + 1}{\gamma_{\gamma}} \right] \psi_{\gamma}(r_{\gamma}) = \psi_{\gamma}(r_{\gamma}) - a_{\gamma}^\pi(r_{\gamma})$$

subject to the boundary conditions

$$U^\pi_{\gamma}(r_{\gamma}) \psi_{\gamma}(r_{\gamma}) = L^{C, \xi_o} (k \xi_o r_{\gamma}) - (m_{\gamma}/(\xi_o k))^{k \xi_o} S_{\gamma} \xi_o l_{\gamma} (k r_{\gamma}) \quad (4-12)$$
is usually done in such a way that the transfer source terms, which describe the intrinsically weak couplings between partitions, are computed for one iteration only. The prescription used to obtain such solutions, for two partitions, is outlined in this section.

The first iterations are for the entrance partition \( \xi \) decoupled from the transfer partition. This set of solutions, which are identical to those of Chapter 3, are a consequence of the zeroth order approximation for the transfer channel:

\[
U_{\gamma}^{\pi I}(r_{\gamma}) \approx 0
\]

at least as far as the initial description of the entrance channels is concerned. Thus

\[
U_{\gamma}^{\pi I}(r_{\xi}) \approx 0
\]

and

\[
\rho_{\xi \gamma}^{\pi I}(r_{\xi}) \approx 0,
\]

so that equations (4-11) reduce to (3-5).

Once these initial solutions to \( U_{\xi}^{\pi I}(r_{\xi}) = U_{\xi}^{\pi I}(r_{\xi}) \) are known,

\[1\]

Higher order solutions, which may be associated with more than two partitions, are dealt with in the following chapter.
they are used to generate the source terms \( \gamma_I^\nu(r_\gamma) \),
for use in solving (4-11) for the transfer channels. \(^2\)

These solutions \( \gamma_I^\nu(r_\gamma) \) are computed in a manner similar
to that described in section 3.4, with the exception that
the transfer source terms are now included in the differential
equations. Since \( \gamma_I^\nu(r_\gamma) \) and \( \gamma_I^\nu(r_\gamma) \) are asymptotically
the same, the S-matrices found for \( \gamma_I^\nu(r_\gamma) \) are the same as
those of \( \gamma_I^\nu(r_\gamma) \), which are to be used to obtain the
scattering amplitudes and cross sections of equations
(4-10).

Cross sections so obtained are said to be solved to
all orders for inelastic scattering and to first order for
particle transfers. Higher order transfer solutions may
be found, however, with some additional effort. The source
terms which couple the transfer channels back to the
entrance channels \( \rho^I_\xi(r_\gamma) \) require that the solutions
\( \gamma_I^\nu(r_\gamma) \) be substituted into equation (4-8). The solutions
to (4-11) are in fact \( \gamma_I^\nu(r_\gamma) \), but \( \gamma_I^\nu(r_\gamma) \) can be found
by the explicit subtraction of the nonorthogonality term

\[
\gamma_I^\nu(r_\gamma) = \gamma_I^\nu(r_\gamma) - \gamma_I^\nu(r_\gamma).
\]

The nonorthogonality term is no more difficult to
calculate than the corresponding transfer source term.
Indeed, the formula for the nonorthogonality term is identical

\(^2\)Details on the calculation of the source terms are presented
in following sections.
to that of the source term with the interaction set to unity.

\[ U_{\zeta}^{\pi I}(r) / r = \]

\[ = \int d\chi d\chi d\chi \, \phi_{\pi I} M(\hat{r}, c-\gamma) \epsilon U_{\zeta}^{\pi I}(r) / r \phi_{\zeta} M(\hat{r}, c-\gamma). \]

Second order transfer solutions also may be found through the use of the prior-post interaction form, which is discussed in the following chapter.

4.3 Core Exchange and Indirect Interaction Terms

Examination of the source term equation (4.8) reveals that the number of overlap terms may be quite large due to the permutation sums required by the wavefunction antisymmetrization. The total number of terms is

\[ (C_T)^N (C_B)^N = \frac{C! C!}{N! A! P! N!}, \]

but some of these terms are much smaller than others. There are, for example,

\[ (C_T)^N (C_B)^N = \frac{C!}{A! P! N!} \]

identical "direct" terms, involving the transfer of \( N \) and only \( N \) particles between the T- and B- systems. More generally there are
identical terms involving the exchange of \( X \) pairs of particles between the \( A \)- and \( P \)-cores, in addition to the transfer of \( N \) particles. These exchange terms are composed of less efficient overlaps than those of the corresponding direct terms. Consequently, the exchange effects are usually small and generally neglected. Within this approximation, the source term is given by

\[
\rho_{\gamma\zeta}(r_\gamma)/r_\gamma = \sum d\tilde{C}_\gamma d\tilde{C}_\zeta \left\{ N_\gamma \left[ \left[ Y_{L\gamma}(\vec{r}_\gamma)^{\otimes \psi}(\chi^-) \right] \otimes \psi_{C_{\gamma}=\gamma}(\tilde{C}_{\gamma}) \right] \right\}^M \times \\
\times \left( V_{C_{\gamma}=\gamma,\gamma}(\vec{r}_\gamma,\vec{r}_\zeta) \right) \left( U_{\gamma\zeta}^{\Pi}(r_\gamma)/r_\gamma \right) \times \left( N_\zeta \left[ \left[ Y_{L\zeta}(\vec{r}_\zeta)^{\otimes \psi}(\xi^-) \right] \otimes \psi_{C_{\zeta}=\zeta}(\tilde{C}_{\zeta}) \right] \right) \left( U_{\gamma\zeta}^{\Pi}(r_\gamma)/r_\gamma \right) \times
\]

The statistical factors \( N_\gamma \) and \( N_\zeta \) are chosen to be either \( \left( \begin{array}{c} T \end{array} \right)^{L/2} \) or \( \left( \begin{array}{c} B \end{array} \right)^{L/2} \) for the \( T \)- or \( P \)-partition respectively, so that their product is

\[
\left( \begin{array}{c} T \end{array} \right)^{L/2} \left( \begin{array}{c} B \end{array} \right)^{L/2} = \left( \begin{array}{c} C \end{array} \right)^{-L/2} \left( \begin{array}{c} C \end{array} \right)^{-L/2} \frac{C!}{\left( A!P!N! \right)}
\]

\[
= \left( \begin{array}{c} \gamma \end{array} \right)^{-L/2} \left( \begin{array}{c} \zeta \end{array} \right)^{-L/2} \times \text{number of transfer terms}.
\]

The source term of equation (4-13) is said to include
the post interaction form because the potential difference $V_{C,\gamma,\gamma}(r) - V_{\gamma,\gamma}(r)$ is with respect to the second partition, the $\gamma$-partition, of the two coupled partitions $\zeta$ and $\gamma$. If the coupled equations (4-11) are to be solved without any explicit inelastic excitation couplings, the source term can also be written in terms of a prior interaction. In this situation the entrance channel functions

$$u_\zeta^{\pi I}(r_\zeta) = u_\zeta^{\pi I}(r_\zeta)$$

are homogeneous solutions of (4-11), and the only source terms necessary for a first order transfer solution are those which couple these entrance channel functions to those of the exit channels. I.e. only $y_\zeta^{\pi I}(r_\zeta)$ is needed. The post form of the interaction is

$$V_{C,\gamma,\gamma}(r) - V_{\gamma,\gamma}(r) = H(C) - H(C_{\gamma}) - H(\gamma) - T(\gamma) - V_{\gamma,\gamma}(r),$$

$$= H(C) - E - [T(\gamma) + V_{\gamma,\gamma}(r) - E_{\gamma}],$$

when the bound-state Hamiltonians are applied to the left in the source term. The bracketed terms contribute an overlap term which is the same as the nonorthogonality term on the left side of equation (4-11) (with $\zeta = \xi$). Therefore, if these terms are subtracted from both sides of the equation, the differential equation can be solved for $U_\gamma^{\pi I}(r)$ directly, rather than $u_\gamma^{\pi I}(r)$. 
The required source term interaction, to be used in (4-13) (with \( \zeta = \xi \)), is now

\[
H(\xi) - E = H(C_{\xi}) + H(\xi) + T(r_\xi^+) + V_{C-\xi,\xi}(r_\xi) - E. \tag{4-14}
\]

Since \( U_{\xi}^{\piI}(r_\xi) \) is a homogeneous solution of (4-11) (with \( \gamma = \xi \)),

\( T(r_\xi^+) \) may be replaced with

\[
\left\{ V_{\xi,\xi}(r_\xi) - E_\xi \right\}
\]

in expression (4-14), so that the prior interaction form results.

\[
H(\xi) - E = V_{C-\xi,\xi}(C) - V_{\xi,\xi}(r_\xi)
\]

To first order then, the prior interaction form yields the solutions \( U_{\gamma}^{\piI}(r_\gamma) \), while the post form of the interaction gives \( U_{\gamma}^{\piI}(r_\gamma) \). Since the nonorthogonality terms are asymptotically zero, the same cross section should result for either case. This relationship is known as the post-prior equality.

Whichever form of the interaction is used, the potential difference in the source term is either

\[
V_{AT}(C) - V_{TT}(r_T) \quad \text{or} \quad V_{BP}(C) - V_{PP}(r_p),
\]

depending on whether the transferred particles are stripped from or picked up by the projectile. Since the T- and B-systems are now to be partitioned into P+N or A+N particles, it is useful to partition the above interactions.
\[ V_{AT}(C) = V_{AP}(C) + V_{AN}(B) \]

and

\[ V_{BP}(C) = V_{AP}(C) + V_{PN}(T), \]

in which the definitions

\[ V_{AN}(B) = V(B) - V(A) - V(N) \]

and

\[ V_{PN}(T) = V(T) - V(P) - V(N) \]

are similar to those of \( V_{AT}(C) \) and \( V_{BP}(C) \) (equation (A-2)). The resulting core-core interaction,

\[ V_{AP}(C) = V(C) - V(T) - V(B) + V(N) \]

can also be written

\[ V(A+P) - V(A) - V(P) \]

if the potential energy is a sum of two-body interactions.

Presently there are two conventions used for the inclusion of the interaction terms, both of which involve the approximation that the overlap of the core-core interaction cancels the appropriate optical potential overlap. In terms of the T-partition,\(^3\) the potential difference is

\(^3\)Similar relationships for the P-partition are obvious.
\[ V_{AT}(C) - V_{TT}^{\pi I}(r_T) = V_{AP}(C) + V_{AN}(B) - V_{TT}^{\pi I}(r_T), \]

\[ = V_{AP}^{(N)}(C) + V_{AN}^{(N)}(B) - V_{TT}^{(N)\pi I}(r_T) + \]

\[ + V_{AP}^{(C)}(C) + V_{AN}^{(C)}(B) - V_{TT}^{(C)\pi I}(r_T), \]

in which the superscripts designate the nuclear (N) and Coulomb (C) parts of the interactions. In the first convention, the approximations

\[ V_{AP}^{(N)}(C) - V_{TT}^{(N)\pi I}(r_T) \approx 0 \]

and

\[ V_{AP}^{(C)}(C) + V_{AN}^{(C)}(B) - V_{TT}^{(C)\pi I}(r_T) \approx 0 \]

are made. These terms are relegated to being indirect interaction terms and are neglected. The physical justification for the neglect of the nuclear terms is based on the assumption that the core-core interaction \( V_{AP}^{(N)}(C) \) can be described adequately by an optical potential nearly identical to \( V_{TT}^{(N)\pi I}(r_T) \). Although the omission of the Coulomb terms can produce sizable absolute normalization changes, relative normalizations and angular distributions generally can be reproduced without these terms because their effects tend to concentrate outside of the surface region. (The effect of \( V_{AN}^{(C)}(B) \) can be seen in figure (4-6.))

The remaining direct interaction term \( V_{AN}^{(N)}(B) \) is taken to
be a sum of Woods-Saxon functions of the coordinates $R_1, R_2, \ldots, R_N$ (which connect the $\Lambda$-system center-of-mass to each of the transferred particles) in addition to the corresponding spin-orbit interaction terms.\(^4\)

Recently Devries, Satchler, and Cramer (De 74) have improved upon this convention by including all the Coulomb interaction terms as the potentials associated with uniform spherical charge distributions. While the inclusion of these terms does not appreciably affect the calculated angular distributions, the Coulomb terms are apparently necessary for consistency between the post and prior absolute normalizations.\(^5\)

At this point, there is, in principle, no reason why the indirect nuclear terms cannot also be included as optical model potentials. A typical transfer calculation would then take roughly twice the computer time presently necessary. There are, however, two pertinent considerations which must be faced before such an undertaking. First, the transfer form factors (defined in Section 4.6) would become complex, so their required memory storage area, which may be quite large for some problems, would double.

\(^4\)See also (Sm 71, To 73).

\(^5\)The coordinates are described in Appendix F, and the explicit form of the potential is given in Section 4.7.
Perhaps more important is the fact that the form factors would now be dependent on the optical potential parameters. Data analyses would require a complete regeneration of the form factors for each minor change in the scattering potential.

The analyses described in this work neglect all the indirect interaction terms. Therefore the potential difference in the source term (4-13) is approximated by $V_{AN}(N)$, which is the post form for pick-up and the prior form for stripping, or by $V_{PN}(N)$, which is the post form for stripping and the prior form for pick-up.

4.4 The Parentage Expansions

The source term (4-13) contains the wavefunctions $\Psi_T(T)$ and $\Psi_B(B)$, one of which is operated on by the appropriate potential $V_{PN}(T)$ or $V_{AN}(B)$. In order to calculate the source term, these wavefunctions are expanded in terms of the P- or A-system core and the transferred particles. For one-nucleon or cluster transfer, the wavefunction of the transferred particle in the T-system has the form

$$\Psi_t(\chi) = \phi_t(x, \hat{\sigma}) \chi_{\tau T}^{(0)}$$

(4-15)

in which the spatial and spin dependence is given by

$$\phi_t(x, \hat{\sigma}) = \mathcal{R}_t(x) \mathcal{Y}_i^{(s)} X_{\tau T}^{(0)}$$
and \( \chi_{t} (\hat{\sigma}) \) is the isotopic spin function. \( \mathcal{R}_t (r) \) is the radial part of the wavefunction, while the spin-orbit function is

\[
\mathcal{Y}_{t} s_t j_t (\hat{r}, \hat{\sigma}) = [\mathcal{Y}_t (\hat{r}) \chi_{s_t} (\hat{\sigma})] j_t
\]

Similarly, in the \( B \)-system,

\[
\psi_{b} (\vec{x}) = \phi_{b} (\hat{r}, \hat{\sigma}) \chi_{t_b} (\hat{\sigma}). \tag{4-16}
\]

These wavefunctions are combined with those of the cores according to

\[
\psi_{T} (\vec{x}) = (\frac{T}{1})^{-\frac{1}{2}} \sum_{\mu} \beta_{T} [P, T] \psi_{P} (\vec{r}) \psi_{T} (\vec{x}) \] \tag{4-17}

and

\[
\psi_{B} (\vec{x}) = (\frac{B}{1})^{-\frac{1}{2}} \sum_{\mu} \beta_{B} [A, B] \psi_{A} (\vec{x}) \psi_{B} (\vec{x}) \] \tag{4-18}

The coefficients \( \beta_{T} [P, T] \) and \( \beta_{B} [A, B] \) are spectroscopic amplitudes, discussed in Appendix H. The vector coupling denoted in equations (4-17) and (4-18) includes the coupling of the total angular momenta, e.g. \( J_p \) and \( j_t \) to \( J_T \). The isospin functions are left uncoupled because their projections are fixed throughout the reaction.

For two-nucleon transfer, similar parentage expansions hold,
\[
\psi_{tt'}^{(\mathcal{T})} = (2)^{-{1\over 2}} \sum \sum \beta_{JPT}^{(\mathcal{T})} \left[ P, T \right] [\psi_{P, T}^{(P)} \otimes \psi_{tt'}^{(N)}] J_{PT}^{(\mathcal{T})} (\xi_1, \xi_2) \]

(4-19)

and

\[
\psi_{bB}^{(N)} = (2)^{-{1\over 2}} \sum A_{bB}^{(N)} \sum b' \beta_{bb'}^{(N)} J_{AB}^{(N)} [A, B] [\psi_{A, B}^{(A)} \otimes \psi_{bb'}^{(b)}] J_{AB}^{(N)} (\overline{\xi}_1, \overline{\xi}_2) \]

(4-20)

but now the transferred particles' wavefunctions must be antisymmetrized.

\[
\psi_{tt'}^{(\mathcal{T})} J_{PT}^{(\mathcal{T})} (\xi_1, \xi_2) = \\
\left\{ 2 \left[ 1 - (-1)^{2j_t - J_{PT}} (-1)^{2j_t - J_{PT}} \right] \right\}^{1/2 - {1\over 2}} \\
\times \left\{ \left[ \psi_{t}^{(\xi_1)} \otimes \psi_{t'}^{(\xi_2)} \right] J_{PT}^{(\mathcal{T})} \right\} \\
\left\{ \left[ \psi_{t}^{(\xi_2)} \otimes \psi_{t'}^{(\xi_1)} \right] J_{PT}^{(\mathcal{T})} \right\} 

(4-21)

\[
\psi_{bb'}^{(b)} J_{AB}^{(b)} (\overline{\xi}_1, \overline{\xi}_2) = \\
\left\{ 2 \left[ 1 - (-1)^{2j_b - J_{AB}} (-1)^{2j_b - J_{AB}} \right] \right\}^{1/2 - {1\over 2}} \\
\times \left\{ \left[ \psi_{b}^{(\overline{\xi}_1)} \otimes \psi_{b'}^{(\overline{\xi}_2)} \right] J_{AB}^{(b)} \right\} \\
\left\{ \left[ \psi_{b}^{(\overline{\xi}_2)} \otimes \psi_{b'}^{(\overline{\xi}_1)} \right] J_{AB}^{(b)} \right\} 

(4-22)

The vector coupling of equations (4-21) and (4-22) is that of the isospin formalism; both total angular momenta
and isospins are coupled. I.e.

$$\psi_{t_1}(x_1) \otimes \psi_{t_2}(x_2) |_{M_{pT} M_{pT}}^{M_{pT} M_{pT}}$$

$$= \sum_{m, m'} C_{m m'}^{j t_1} C_{m' m}^{j t_2} \left( X_{t_1} \right)^{\psi} \left( X_{t_2} \right)^{\psi'}$$

In (4-21) the subscript "t" which occurs with the Kronecker delta symbol in the normalization factor refers to all quantum numbers necessary to specify the state of a single particle except for the total angular momentum and isospin projections. (E.g. $t = n, l, j, t$.) The normalization factor itself can be simplified by the fact that

$$j_t = \frac{1}{2} \times \text{(odd integer)},$$

and

$$\tau_t = \frac{1}{2}$$

for any nucleon. Thus,

$$\xi^2 [1-(-1)^{j_t} \left( X_{t} \right)^{\psi} \left( X_{t} \right)^{\psi'}]^{J_{pT} \delta_{t t'}}^{2 \tau_t - \tau_{pT}} =$$

$$= \xi^2 [1-(-1)^{J_{pT} \delta_{t t'}}]^{2 \tau_t - \tau_{pT}}$$

and a similar relation holds for the B-system function.

Typically the isospin formalism is used in nuclear structure calculations. However, for an accurate description of nuclear reactions, the radial functions $\mathcal{R}_t(r_1)$, etc. must have rather different behavior for protons than for
neutrons. This symmetry breaking is easily expressed for the cases in which two protons or two neutrons are transferred. In such cases there is only one term in the isospin coupling sum, and the appropriate Clebsch-Gordon coefficient is unity.

\[ C_{\pm \frac{1}{2} \pm \frac{1}{2} \pm 1} = 1. \]

Therefore, with \( \tau_{PT} = \tau_{AB} = 1 \), equations (4-21) and (4-22) become

\[ \psi_{tt'} J_{PT} \rho_{PT} (x_1, x_2) = \]

\[ = \left\{ 2[1 + (-1)^{J_{PT}\delta_{tt'}}] \right\}^{-\frac{1}{2}} \chi_{\frac{1}{2}}^{\pm \frac{1}{2}} (\sigma_1) \chi_{\frac{1}{2}}^{\pm \frac{1}{2}} (\sigma_2) \times \]

\[ \times \left\{ [\phi_t (\hat{r}_1, \hat{\sigma}_1) \otimes \phi_t, (\hat{r}_2, \hat{\sigma}_2)]_{J_T} - [\phi_t (\hat{r}_2, \hat{\sigma}_2) \otimes \phi_t, (\hat{r}_1, \hat{\sigma}_1)]_{J_T} \right\} \]

and

\[ \psi_{bb'} J_{AB} \rho_{AB} (x_1, x_2) = \]

\[ = \left\{ 2[1 + (-1)^{J_{AB}\delta_{bb'}}] \right\}^{-\frac{1}{2}} \chi_{\frac{1}{2}}^{\pm \frac{1}{2}} (\sigma_1) \chi_{\frac{1}{2}}^{\pm \frac{1}{2}} (\sigma_2) \times \]

\[ \times \left\{ [\phi_b (\hat{r}_1, \hat{\sigma}_1) \otimes \phi_b, (\hat{r}_2, \hat{\sigma}_2)]_{J_B} - [\phi_b (\hat{r}_2, \hat{\sigma}_2) \otimes \phi_b, (\hat{r}_1, \hat{\sigma}_1)]_{J_B} \right\} \]

It is apparent from the form of the wavefunctions (4-23) and (4-24) that as long as the interactions \( V_{PN}^{(N)} (T) \) and \( V_{AN}^{(N)} (R) \) are diagonal in isospin space, the source
term overlap is identical to that obtained by a treatment of protons and neutrons as distinct particles.

It should be remembered that the wavefunctions \( \Psi_T(T) \) which occur in the expansions (4-17) and (4-19) are with respect to the T-system center of mass. However the source term overlap requires that \( \Psi_P(P) \) be expressed with the P-system center of mass as the origin of coordinates. The necessary transformation of coordinates is very similar to that of Appendix A: If \( \Psi_P(P) \) is written in terms of the P-mass center, \( \Psi_T(X) \) is a function of the relative coordinate \( \vec{r} \), while \( \Psi_{P_{PT}}(X_1, X_2) \) is a function of \( \vec{r} \) and the internal coordinate \( \vec{p} \), which are easily transformed to the single particle coordinates \( \vec{r}_1 \) and \( \vec{r}_2 \). Similar transformations for the B-system are implicit in equations (4-18) and (4-20).\(^6\)

In the present section it is assumed that \( \Psi_{P_{PT}}(X_1, X_2) \) and \( \Psi_{AB}(X_1, X_2) \) are known functions of the single particle coordinates \( \vec{r}_1, \vec{r}_2 \) and \( \vec{r}_1, \vec{r}_2 \).\(^7\) These wavefunctions,\(^6\)

---

\(^6\) It is common practice to use spectroscopic amplitudes from shell model calculations which use the T- or B-centers of mass as the coordinate origins. Possible errors due to such small discrepancies are ignored.

\(^7\) Generation of the radial functions \( \rho_t(r_1) \) etc. is discussed in Section 4.7.
and the appropriate potential, must be transformed to functions of $\mathbf{r}, \mathbf{\rho}$ and $\mathbf{\hat{r}}, \mathbf{\hat{\rho}}$. This bipolar expansion is presented explicitly for the T-system orbitals with the potential $V_{PN}^{(N)}(T)$. The expansion of the B-system orbitals without any interaction, and the expansions for both systems when the interaction $V_{AN}^{(N)}(B)$ is used, are to be done similarly.

For two-nucleon transfer

$$V_{PN}^{(N)}(T) \equiv V^{(N)}(\chi_1) + V^{(N)}(\chi_2).$$

Thus

$$V_{PN}^{(N)}(T) \Psi_{\Gamma \Gamma} J_{\Gamma T} \Psi_{\Gamma T} (\chi_1, \chi_2) =$$

$$= \left\{ V^{(N)}(\chi_1) + V^{(N)}(\chi_2) \right\} \left\{ 2[1+(-1)^{J_{\Gamma T} \delta_{\Gamma T}}] \right\}^{-\frac{1}{2}} \chi_{\lambda_1}(\hat{\sigma}_1) \chi_{\lambda_2}(\hat{\sigma}_2) \times$$

$$\times [\phi_t(\mathbf{r}_1, \hat{\sigma}_1) \otimes \phi_t(\mathbf{r}_2, \hat{\sigma}_2)]_{J_T} - [\phi_t(\mathbf{r}_2, \hat{\sigma}_2) \otimes \phi_t(\mathbf{r}_1, \hat{\sigma}_1)]_{J_T}. $$

Application of the potential operators gives the expression

$$\left\{ 2[1+(-1)^{J_{\Gamma T} \delta_{\Gamma T}}] \right\}^{-\frac{1}{2}} \chi_{\lambda_1}(\hat{\sigma}_1) \chi_{\lambda_2}(\hat{\sigma}_2) \times$$

$$\times \left\{ [V_t^{(N)}(r_1) + V_t^{(N)}(r_2)] [\phi_t(\mathbf{r}_1, \hat{\sigma}_1) \otimes \phi_t(\mathbf{r}_2, \hat{\sigma}_2)]_{J_T} -$$

$$- [V_t^{(N)}(r_2) + V_t^{(N)}(r_1)] [\phi_t(\mathbf{r}_2, \hat{\sigma}_2) \otimes \phi_t(\mathbf{r}_1, \hat{\sigma}_1)]_{J_T} \right\}_T,$$

which can be expanded in terms of total orbital angular momentum $l$ and total spin $S$. The transformation yields
\[ \left\{ 2 \left[ 1 + (-1)^{J_{PT}} \delta_{tt'} \right] \right\}^{-1/2} \chi_{t}^{(\hat{\sigma}_{1})} \chi_{t}^{(\hat{\sigma}_{2})} \times \]
\[ \times \sum_{l \geq S} \left( j_{t} \hat{j}_{t'}, \hat{l} S \right)^{1/2} \left\{ \begin{array}{ccc} \ell & k & j_{t} \\
 & \ell & j_{t'} \\
 & l & S \end{array} \right\}_{J_{PT}} \times \]
\[ x \left\{ \chi^{(N)}_{t} (r_{1}) + \chi^{(N)}_{t} (r_{2}) \hat{\rho}_{t} (r_{1}) \hat{\rho}_{t'} (r_{2}) \left[ \gamma_{l_{t}} (\hat{r}_{1}) \otimes \gamma_{l_{t'}} (\hat{r}_{2}) \right]_{l_{t}} \otimes \right. \]
\[ \left. \left[ \chi_{\hat{\sigma}_{t}}^{(\hat{\sigma}_{1})} \otimes \chi_{\hat{\sigma}_{t}}^{(\hat{\sigma}_{2})} \right]_{S} \right\} _{J_{PT}^{2}} - \]
\[ - \left\{ \chi^{(N)}_{t} (r_{2}) + \chi^{(N)}_{t} (r_{1}) \hat{\rho}_{t} (r_{2}) \hat{\rho}_{t'} (r_{1}) \left[ \gamma_{l_{t}} (\hat{r}_{2}) \otimes \gamma_{l_{t'}} (\hat{r}_{1}) \right]_{l_{t}} \otimes \right. \]
\[ \left. \left[ \chi_{\hat{\sigma}_{t}}^{(\hat{\sigma}_{2})} \otimes \chi_{\hat{\sigma}_{t}}^{(\hat{\sigma}_{1})} \right]_{S} \right\} _{J_{PT}^{2}}, \]

in which the bracketed array denotes a \( 9j \)-symbol.

The necessary expansion is defined by the transformation\(^8\)

\[ \left\{ \chi^{(N)}_{t} (r_{1}) + \chi^{(N)}_{t} (r_{2}) \hat{\rho}_{t} (r_{1}) \hat{\rho}_{t'} (r_{2}) \left[ \gamma_{l_{t}} (\hat{r}_{1}) \otimes \gamma_{l_{t'}} (\hat{r}_{2}) \right]_{l_{t}} \otimes \right. \]
\[ \left. \left[ \chi_{\hat{\sigma}_{t}}^{(\hat{\sigma}_{1})} \otimes \chi_{\hat{\sigma}_{t}}^{(\hat{\sigma}_{2})} \right]_{S} \right\} _{J_{PT}^{2}} = \]
\[ \equiv \sum_{\lambda_{r}, \lambda_{r'} \rho, \lambda_{r'}, \lambda_{r}} \mathcal{J}_{tt'}, \lambda_{r} \lambda_{r'} \rho \left[ \gamma_{\lambda_{r}} (\hat{\sigma}) \otimes \gamma_{\lambda_{r'}} (\rho) \right]_{l_{t}}. \]

The expansion for the exchange term differs from the direct one by the factor \((-1)^{l_{t}}\). This fact is easily verified by noting that interchanging \(\hat{r}_{1}\) and \(\hat{r}_{2}\) is equivalent to the transformation \(\hat{\sigma} \rightarrow -\hat{\sigma}\), which gives rise to the phase factor. Similarly the spin functions in the exchange term yield the factor \((-1)^{l_{t}+l_{t'}-S}\) when their vector coupling order is changed. With these transformations incorporated,

\(^8\)See Appendix F for the explicit form of the expansion
\[ V_{PN}^{(N)}(\mathcal{T}) \psi_{tt'}J_{PT}^{\tau_{PT}}(\chi_1, \chi_2) = \]
\[ = \left\{ 2[1+(-1)^{\lambda_P}J_{\delta_{tt'}}] \right\}^{\frac{1}{2}} \chi^x_{\frac{1}{2}}(\hat{\sigma}_1) \chi^x_{\frac{1}{2}}(\hat{\sigma}_2) \times \]
\[ \times \sum_{SS' \lambda} \left( \begin{array}{c} j_{tt'}^x \sim S \frac{1}{2} \end{array} \right) \frac{1}{2} \left( \begin{array}{c} l_t \frac{1}{2} \end{array} j_t \right) \left( \begin{array}{c} l_t' \frac{1}{2} \end{array} j_t' \right) \frac{1}{2} \left( \begin{array}{c} l' \lambda \end{array} \right) \left[ 1+(-1)^{\lambda_P+S} \right] \times \]
\[ \times \tilde{T}_{tt'}^{l} \lambda \lambda \left( r, \rho \right) \left[ \left( \begin{array}{c} j_{tt'}^x \sim S \frac{1}{2} \end{array} \right) \frac{1}{2} \left( \begin{array}{c} l_t \frac{1}{2} \end{array} j_t \right) \frac{1}{2} \left( \begin{array}{c} l_t' \frac{1}{2} \end{array} j_t' \right) \frac{1}{2} \left( \begin{array}{c} l' \lambda \end{array} \right) \left[ 1+(-1)^{\lambda_P+S} \right] \tilde{T}_{tt'}^{l} \lambda \lambda \left( r, \rho \right), \]
\[ (4-25) \]

It is useful to note that the "sum" over total spin S contains only one term. Triangularity constraints imply that S equal zero or one, but the antisymmetrization factor \( 1+(-1)^{\lambda_P+S} \) is nonzero only if \( \lambda_P \) and S are both even or odd. Thus, if \( \lambda_P \) is even, S=0; and if \( \lambda_P \) is odd, S=1.

Finally, with the definition
\[ \tilde{f}_{\lambda}^{l} \lambda \left( r, \rho \right) \equiv \]
\[ \equiv \sum_{tt'} \beta_{J_{PT}}^{l} \left[ \right] \left\{ 2[1+(-1)^{\lambda_P}J_{\delta_{tt'}}] \right\}^{\frac{1}{2}} \left( \begin{array}{c} j_{tt'}^x \sim S \frac{1}{2} \end{array} \right) \frac{1}{2} \left( \begin{array}{c} l_t \frac{1}{2} \end{array} j_t \right) \frac{1}{2} \left( \begin{array}{c} l_t' \frac{1}{2} \end{array} j_t' \right) \frac{1}{2} \left( \begin{array}{c} l' \lambda \end{array} \right) \left[ 1+(-1)^{\lambda_P+S} \right] \tilde{T}_{tt'}^{l} \lambda \lambda \left( r, \rho \right), \]

the T-system wavefunction (4-19) operated on by \( V_{PN}^{(N)}(\mathcal{T}) \) is written.
4.5 Two-Nucleon Transfer Source Terms

In this section formulas are derived for two-nucleon transfer source terms. The formulas for one-nucleon or cluster transfer are presented in Appendix G.

Substitution of expression (4-26) and the analogous expansion for the B-system into the source term (4-13) yields the expressions

\[
V_{PN}^{(N)}(T') \psi_T(T) =
\]

\[
= \left( \frac{T}{2} \right)^{-k} \sum_{J_{PT}} \sum_{\lambda_1 \lambda_2 \rho} \gamma_{\lambda_1 \lambda_2 \rho} (r, P) x_{\lambda_1} (\hat{\sigma}_1) x_{\lambda_2} (\hat{\sigma}_2) x \]

\[
x \left[ \psi_{P}(P) \otimes \left[ \gamma_{\lambda_1 \lambda_2 \rho} (\hat{\sigma}_1) \otimes x_{\lambda_2} (\hat{\sigma}_2) \right] S \right] J_{PT} J_T.\]  

(4-26)

\[\rho_{PT}(r_p)/r_p = \]

\[
= \sum \int \int dBDp dP \sum A_{AB} L^* R^\rho \sum P' J_{PT} \gamma_{\lambda_1 \lambda_2 \rho} (r, P) x \]

\[
x \left[ \gamma_{L_P} (\hat{\sigma}_1) \otimes \left[ \gamma_{R} (\hat{\sigma}_2) \right] S \right] J_{AB} J_B \] 

\[
= \psi_{P}(P) \otimes \left[ \gamma_{\lambda_1 \lambda_2 \rho} (\hat{\sigma}_1) \otimes x_{\lambda_2} (\hat{\sigma}_2) \right] S \right] J_{PT} J_T.\]

\[\psi_A(A) \] 

for stripping and
\[ \rho_{TP}(r_T)/r_T = \]
\[ = \sum_{p} \int d\phi d\theta d\lambda \rho T^{\lambda \rho}_{AB} L_{\lambda \rho} P_{\lambda \rho} U_{\lambda \rho}^{TP}(r_p)/r_p \times \]
\[ \times j_{\rho}^{L \lambda \rho}(R, \rho') \frac{d^L \rho}{\rho} \tau_{\rho}^{\lambda \rho} (r, \rho) \hat{R}_{\lambda \rho}(\sigma_1, \sigma_2) \hat{R}_{\lambda \rho}(\sigma_2, \sigma_1) \hat{R}_{\lambda \rho}(\sigma_2, \sigma_1) \hat{R}_{\lambda \rho}(\sigma_1, \sigma_2) \times \]
\[ \times \left[ (\gamma_{LT}(\hat{R}_T) \otimes \gamma_{LP}(\hat{R}_P)) \otimes \left[ \gamma_{A}(\hat{R}_A) \otimes \gamma_{B}(\hat{R}_B) \right] \right] \frac{d^M \rho}{\rho} \tau_{\rho}^{M \rho} \]
\[ \times \left[ (\psi_{LT}(\hat{R}_T) \otimes \psi_{LP}(\hat{R}_P)) \otimes \psi_{A}(\hat{R}_A) \otimes \psi_{B}(\hat{R}_B) \right] \frac{d^M \rho}{\rho} \tau_{\rho}^{M \rho} \]
\[ \times \left[ (\hat{\chi}_{\lambda \rho}(\sigma_1) \otimes \hat{\chi}_{\lambda \rho}(\sigma_2)) \right] \frac{d^M \rho}{\rho} \tau_{\rho}^{M \rho} \]

for pick-up. In these expressions

\[ d^3_R = d\phi d\theta d\lambda \rho \hat{R}_{\lambda \rho} d\sigma_1 d\sigma_2 d\sigma_1 d\sigma_2, \]

and

\[ d^3_T = d\phi d\theta d\lambda \rho \hat{R}_{\lambda \rho} d\sigma_1 d\sigma_2 d\sigma_1 d\sigma_2. \]

It is convenient to integrate in terms of the channel radii \( \hat{r}_T \) and \( \hat{r}_P \) rather than the coordinates \( \hat{R} \) and \( \hat{r} \). Therefore, the transformations

\[ d^3_R = u^3 d^3 \hat{r}_T = u^3 r_T^2 dr_T d\hat{r}_T \]
\[ d^3_T = u^3 d^3 \hat{r}_P = u^3 r_P^2 dr_P d\hat{r}_P \]

are employed. In each case the Jacobian is \( u^3 = (B T/(NC)) \)

by (E-12). The radial functions \( j_{\rho}^{L \lambda \rho}(R, \rho) \) and \( j_{\rho}^{L \lambda \rho}(r, \rho) \),
generated by the bipolar expansion, are both real. (They describe bound states.) Moreover, when all of the overlaps except the radial integral over $r_T(r_P)$ for stripping (pick-up) are done, the angular-momentum coupled quantities yield a real-valued factor in the remaining integrand. (The matrix element is real because the interaction operator is Hermitian.) Thus, the separate expressions for stripping and pick-up can be combined.

$$
\rho_{\gamma_L}(r_\gamma) = \frac{\pi I}{\zeta} \left[ \sum_0^3 \int d_0 d_1 d_2 r_P d_T r_\gamma U_{\zeta}^I(r_\gamma) \right] x
$$

$$
= \sum_0^3 \sum_{\alpha, \lambda} J_{AB} L_{\alpha} R_{\lambda} \rho_P J_{PT} \rho_P \sum_{\alpha, \lambda} J_{AB} L_{\alpha} R_{\lambda} \rho_P (r_\gamma) \frac{L_{\alpha} R_{\lambda}}{L_{\alpha} R_{\lambda} \rho_P} \left[ \mathcal{Y}_{L_0} (r_\gamma) \otimes [\mathcal{Y}_{L_0} (r_\gamma) \otimes \mathcal{Y}_{L_0} (r_\gamma)] \right] x
$$

Equation (4-27) must be simplified in order to obtain a readily computable form for the source term. First the core wavefunctions $\psi_A (A)$ and $\psi_P (P)$ are eliminated. Some rearrangement is necessary:
The symbol \( W(K_B J_{AB} I J_A ; K_A, J_B) \) denotes a Racah recoupling coefficient. After integration over the \( A \)-system coordinates, (4-27) becomes

\[
\rho_{\gamma^i} (r_{\gamma}) = \sum_k \frac{4}{\gamma^i} \int \! d^2 \rho d^3 \rho d^2 d^2 \rho d^2 \rho d^2 \rho d^2 \rho r_{\gamma} U_{\gamma^i} (r_{\gamma}) \times
\]

\[
\sum_{m_A^J_{AB}} \frac{\lambda^R_{\gamma^i}}{L \lambda \lambda \rho} P^l_{\gamma^i} (R, \rho) F_{\lambda \lambda \rho} (R, \rho) \times
\]

\[
x (-1)^{J_A^+ J_{AB} - J_B} \left( \frac{\lambda^R_{\gamma^i}}{L \lambda \lambda \rho} \right) \times
\]

\[
W(K_B J_{AB} I J_A ; K_A, J_B) \times
\]

\[
x \left[ [\gamma_{L_P} (\hat{p} P_P) \otimes \gamma_{P_P} (P_P) \right]_{K_B} \otimes [\gamma_{R_L} (\hat{p} R_L) \otimes \gamma_{R_L} (\hat{p} R_L) \right]_{L} \otimes \left[ \chi_{\lambda} (\hat{p} \lambda \rho) \otimes \chi_{\lambda} (\hat{p} \lambda \rho) \right]_{S} \right]_{J_{AB} J_A K_A}^{m_A^*} \times
\]

\[
x \left[ [\gamma_{L_L} (\hat{p} L_L) \otimes \gamma_{P_P} (P_P) \right]_{K_B} \otimes [\gamma_{R_R} (\hat{p} R_R) \otimes \gamma_{R_R} (\hat{p} R_R) \right]_{L} \otimes \left[ \chi_{\lambda} (\hat{p} \lambda \rho) \otimes \chi_{\lambda} (\hat{p} \lambda \rho) \right]_{S} \right]_{J_{PT} J_T K_A}^{m_A^*} \times
\]

(4-28)

Equation (4-28) is readily obtained after noting that the source term is independent of the total angular momentum projection \( m \) (e.g. by the Wigner-Eckart theorem). Thus the source term can be summed over \( m \) and divided by \( I \).
to get a form which can be simplified by the orthogonality properties of the coupling coefficients,

\[ C_{\mathbf{m}_A}^{\mathbf{M}_A \mathbf{M}_A} = (-1)^{K_A-J_A+M} \mathbf{I}_{\mathbf{M}_A - \mathbf{M}_A}^{\mathbf{M}_A} \mathbf{I}^{\mathbf{M}_A} \mathbf{I}^{\mathbf{K}_A} \]

etc.

Before the P-system coordinates are integrated, some more recoupling is done.

\[ [\gamma_{\rho_L} (\hat{r}_P) \otimes \psi_p (\rho)]_{K_B} = \mathbf{I}_{\mathbf{M}_A - \mathbf{M}_A}^{\mathbf{M}_A} \mathbf{I}^{\mathbf{M}_A} \mathbf{I}^{\mathbf{K}_A} \]

Similarly,

\[ [\gamma_{\rho_L} (\hat{r}_T) \otimes \psi_p (\rho)]_{J_T} = \mathbf{I}_{\mathbf{M}_A - \mathbf{M}_A}^{\mathbf{M}_A} \mathbf{I}^{\mathbf{M}_A} \mathbf{I}^{\mathbf{K}_A} \]

Now the source term is given by
At this point it is convenient to rewrite two of the Racah coefficients in terms of the product of another Racah coefficient and 9j-symbol summed over the angular momentum transfer $J$.

\[
\begin{align*}
W(J_{AB} L T P A J P; K_B) & = W(L_P J_{AB} L_T J_{PT}; K_B) \\
& = \sum_J \left( \begin{array}{ccc}
K_B & K_A & J_{AB} \\
J_T & J_P & J_{PT} \\
L_T & L_P & J
\end{array} \right) \sum_{\lambda_R \lambda_T \lambda_P} \lambda_R \lambda_T \lambda_P \langle \lambda_R \lambda_T \lambda_P | J_{AB} J_{PT} \rangle \langle \lambda_R \lambda_T \lambda_P | J \rangle
\end{align*}
\]

A two-dimensional form factor can now be introduced by the relation

\[
\hat{\mathcal{J}}_J(r_T, r_P) = \sum_{\lambda_R \lambda_T \lambda_P} \lambda_R \lambda_T \lambda_P (r_T, r_P)
\]
In terms of the form factor, the source term is expressed

\[ \rho \gamma^\pi I (r_Y) / r_Y = \]

\[ = \sum_{\zeta} \int d^3 r \zeta r \zeta \gamma^\pi I (r_\zeta) J_{AB} J_{PT} J_{T} \left[ \zeta^- (-1) \rho_{L,R}^+ \right] \]

\[ \times (K_{AB} J_{AB} J_{T}) \Lambda W(K_{AB} J_{AB} J_{T} L_{T} L_{P}) \left\{ \begin{array}{ccc} K_{A} & K_{B} & J_{AB} \\ J_{T} & J_{P} & J_{PT} \end{array} \right\} \left( \zeta^{L,R} \right) \left( r_\zeta, r_\zeta \right). \]

(4-31)

Once the form factor is known, the source term may be calculated using equation (4-31).

### 4.6 Two-Nucleon Transfer Form Factors

In this section formulas are derived for two-nucleon transfer form factors. The formulas for one-nucleon or cluster transfer are presented in Appendix G.

In simplifying equation (4-30) it is convenient to reexpress the complex conjugates of the spherical harmonics and spin wavefunctions according to the properties
\[
y_{L_p}^{m_p} (\hat{r}_p) = (-1)^{m_p} y_{L_p}^{-m_p} (\hat{r}_p)
\]

and

\[
\chi_{\frac{1}{2}}^{\mu_1} (\hat{r}_1) = (-1)^{\frac{1}{2} - \mu_1} \chi_{\frac{1}{2}}^{-\mu_1} (\hat{r}_1)
\]

With the Clebsch-Gordan coupling coefficients rewritten in terms of their counterparts with opposite angular momentum projections, the wavefunctions can be recoupled.

\[
[[ \chi_{R}^{(R)} (\hat{r}) \otimes \chi_{\rho}^{(\rho)} (\hat{r}) ]_{L} \otimes [ \chi_{k}^{(1)} (\hat{r}_1) \otimes \chi_{k}^{(2)} (\hat{r}_2) ]_{S, J} ]_{J_{AB}} \otimes y_{L_p}^{(\hat{r}_p)} ]_{\kappa}^{\mu_1} = (-1)^{\lambda_R^{+} + \lambda_{p}^{+} + L_p} \chi_{k}^{(1)} (\hat{r}_1) \otimes \chi_{k}^{(2)} (\hat{r}_2) \right]
\]

Since \( C_{\mu - \mu}^{\kappa} = (-1)^{\kappa - \mu} \),

the form factor \((4-30)\) is

\[
\mathcal{F}_{R L_T L_p}^{(r_T, r_p)} = \sum_{\lambda} \sum_{\rho} \sum_{\lambda_R \lambda_{p}^{+} + L_p} \sum_{\lambda_{\rho}^{+} + L_p} \sum_{\kappa} \sum_{\mu} x L_p^{+} \sum_{\lambda R \lambda_{\rho}^{+} + L_p} \sum_{\lambda_{\rho}^{+} + L_p} \sum_{\kappa} \sum_{\mu}
\]

\[
\mathcal{F}_{R L_T L_p}^{(r_T, r_p)} (r_T, r_p) = \chi_{\frac{1}{2}}^{\mu_1} (\hat{r}_1) \otimes \chi_{\frac{1}{2}}^{-\mu_1} (\hat{r}_1) \otimes \chi_{\frac{1}{2}}^{\mu_1} (\hat{r}_1) \otimes \chi_{\frac{1}{2}}^{-\mu_1} (\hat{r}_1) \right]
\]

Some simplification results from the recoupling.

\[
(4-32)
\]
because the $9j$-symbol reduces to a Racah coefficient whose orthogonality property can be exploited:

\[
\sum_{\kappa} \gamma_{L_{T} J_{T} PT}^{\kappa} \gamma_{L_{P} J_{A} B}^{\kappa} = (-1)^{J_{T} + J_{A} - \kappa} (J_{T}'; J_{A}'), \delta_{JJ'} W(L_{P} J_{A} B_{T} J_{P} T; \kappa J'),
\]

and

\[
\sum_{\kappa} \hat{\gamma}_{J}^{\kappa} W(L_{P} J_{A} B_{T} J_{P} T; \kappa J) W(L_{P} J_{A} B_{T} J_{P} T; \kappa J') = \delta_{JJ'}.
\]

Thus, the form factor becomes

\[
\mathcal{F}^{J}_{\lambda_{R} \lambda_{L} T}^{J_{P} L_{T} L_{P}}(r_{T}, r_{P}) =
\]

\[
= \int d^{3} \rho d \hat{\rho}_{1} d \hat{\rho}_{2} d_{T}^{L_{P}} \sum_{\rho} \mathcal{F}^{L_{T}}_{\lambda_{R} \lambda_{L} T}^{L_{P}}(r_{T}, r_{P}) (-1)^{L_{T} + \lambda_{R} + \lambda_{L} + L_{P}} \delta_{JJ'}.
\]

(4-33)
This expression is further simplified by the integration over spin coordinates.

\[ \mathcal{F}_{\lambda R L_T L_P}^{J}(r_T, r_P) = \]

\[ = \int d^3 \rho d_1 d_2 \hat{r}_P d\hat{r}_T L_{\lambda \rho}, \ell \lambda \rho \mathcal{F}_{\lambda R}^{L}(r, \rho) \mathcal{F}_{\lambda R}^{L}(r, \rho) (-1)^{L_T^* \lambda R + \lambda \rho} \]

\[ \times \mathcal{L}_J \left( \hat{J}_{PT \lambda \rho} \right) \frac{1}{2} \left\{ \begin{array}{ccc} \ell & S & \hat{J}_{PT} \\ L & S' & \hat{J}_{\lambda \rho} \end{array} \right\} \left( SS'_1 S'_2 \right) \frac{1}{2} \left\{ \begin{array}{ccc} \ell & \ell & S \\ S_1 & S_2 & J \end{array} \right\} \]

\[ \times (\hat{J}) \frac{1}{2} \left( \mathcal{L}_J \left( \hat{J}_{PT \lambda \rho} \right) \right) \left( \gamma_{L_T}^{(r)} \otimes \gamma_{L_P}^{(r)} \right) \left( \hat{J}_{\lambda \rho} \right) \]

\[ \otimes \left[ \gamma_{\lambda R}^{(\hat{r})} \otimes \gamma_{\lambda \rho}^{(\hat{r})} \right] \frac{1}{2} \left[ \gamma_{\lambda R}^{(\hat{r})} \otimes \gamma_{\lambda \rho}^{(\hat{r})} \right] \frac{1}{2} \left( \mathcal{L}_J \right) \]

\[ \otimes \left[ x_{\lambda}^{(\hat{\sigma}_2)} \otimes x_{\lambda}^{(\hat{\sigma}_2)} \right] S_1 \otimes \left[ x_{\lambda}^{(\hat{\sigma}_2)} \otimes x_{\lambda}^{(\hat{\sigma}_2)} \right] S_2 \mathcal{L}_J \frac{1}{2} \frac{1}{2}. \]
Since

\[ \int D_{\mu_1} D_{\mu_2} \left[ \chi_{x_1}(\hat{\sigma}_1) \otimes \chi_{x_2}(\hat{\sigma}_2) \right] S_1 \otimes \left[ \chi_{x_1}(\hat{\sigma}_2) \otimes \chi_{x_2}(\hat{\sigma}_1) \right] S_2 \]

= \sum_{m_1 m_2} \frac{S_1 S_2 \int \delta_{\mu_{11} \mu_{11}} \epsilon_{\mu_{12} \mu_{21}} \epsilon_{\mu_{22} \mu_{12}} \delta_{m_1 m_2} \delta_{m_1 m_2}}{m_1 m_2}

\times \left( -1 \right)^{\frac{1}{2} + \mu_1} \left( -1 \right)^{\frac{1}{2} + \mu_2} \delta_{\mu_1 \mu_1} \delta_{\mu_2 \mu_2} \delta_{m_1 m_2} \delta_{m_1 m_2}

= \sum_{m_1 m_2} \frac{S_1 S_2 \int \delta_{\mu_{11} \mu_{11}} \epsilon_{\mu_{12} \mu_{21}} \epsilon_{\mu_{22} \mu_{12}} \delta_{m_1 m_2} \delta_{m_1 m_2}}{m_1 m_2}

\times \left( -1 \right)^{\frac{1}{2} + \mu_1} \left( -1 \right)^{\frac{1}{2} + \mu_2} \delta_{\mu_1 \mu_1} \delta_{\mu_2 \mu_2} \delta_{m_1 m_2} \delta_{m_1 m_2}

= \left( -1 \right)^{\frac{1}{2} + \mu_1} \left( -1 \right)^{\frac{1}{2} + \mu_2} \sum_{m_1 m_2} \frac{S_1 S_2 \int \delta_{\mu_1 \mu_1} \delta_{\mu_2 \mu_2} \delta_{m_1 m_2} \delta_{m_1 m_2}}{m_1 m_2}

= \left( -1 \right)^{\frac{1}{2} + \mu_1} \left( -1 \right)^{\frac{1}{2} + \mu_2} \sum_{m_1 m_2} \frac{S_1 S_2 \int \delta_{\mu_1 \mu_1} \delta_{\mu_2 \mu_2} \delta_{m_1 m_2} \delta_{m_1 m_2}}{m_1 m_2}

= \left( -1 \right)^{\frac{1}{2} + \mu_1} \left( -1 \right)^{\frac{1}{2} + \mu_2} \sum_{m_1 m_2} \frac{S_1 S_2 \int \delta_{\mu_1 \mu_1} \delta_{\mu_2 \mu_2} \delta_{m_1 m_2} \delta_{m_1 m_2}}{m_1 m_2}

the form factor reduces to

\[ \frac{\hat{\omega}}{\lambda_l \lambda_R L_T L_P} (r_T, r_P) = \]

= \int d^3 \rho d^3 \rho' d^3 \rho'' d^3 \rho''' \left( -1 \right)^{\frac{1}{2} + \mu_1} \left( -1 \right)^{\frac{1}{2} + \mu_2} \sum_{m_1 m_2} \frac{S_1 S_2 \int \delta_{\mu_1 \mu_1} \delta_{\mu_2 \mu_2} \delta_{m_1 m_2} \delta_{m_1 m_2}}{m_1 m_2}

\times \left( -1 \right)^{\frac{1}{2} + \mu_1} \left( -1 \right)^{\frac{1}{2} + \mu_2} \sum_{m_1 m_2} \frac{S_1 S_2 \int \delta_{\mu_1 \mu_1} \delta_{\mu_2 \mu_2} \delta_{m_1 m_2} \delta_{m_1 m_2}}{m_1 m_2}

\times \left( -1 \right)^{\frac{1}{2} + \mu_1} \left( -1 \right)^{\frac{1}{2} + \mu_2} \sum_{m_1 m_2} \frac{S_1 S_2 \int \delta_{\mu_1 \mu_1} \delta_{\mu_2 \mu_2} \delta_{m_1 m_2} \delta_{m_1 m_2}}{m_1 m_2}

\times \left( -1 \right)^{\frac{1}{2} + \mu_1} \left( -1 \right)^{\frac{1}{2} + \mu_2} \sum_{m_1 m_2} \frac{S_1 S_2 \int \delta_{\mu_1 \mu_1} \delta_{\mu_2 \mu_2} \delta_{m_1 m_2} \delta_{m_1 m_2}}{m_1 m_2}

(4-34)
In a similar manner the angular dependence on the relative coordinate $\mathbf{P}$ can be isolated and integrated. The result is

$$J^+_{\lambda_R \lambda_T L_T L_P}(r_T, r_P) =$$

$$= \int d\rho d\rho' \mathcal{G}_{\lambda_T \lambda_P}^{L_T} \mathcal{G}_{\lambda_R \rho}^{L} (r, \rho) \mathcal{G}_{\lambda_{P_T} \lambda_{P_L}}^{L} (r, \rho) (-1)^{L_T} (-1)^{L_R} \lambda_R^{+\lambda_P + L_P} \times$$

$$\times \left[ \mathcal{J}^{P_T+L-S-J}_{\rho} \mathcal{J}^{L_T}_{\rho} \right]_{J^0} W(J_P T \hat{J}_{AB}^{2L}; S J) W(J_P T \hat{J}_{AB}^{2L}; S J) x$$

$$x \left[ \mathcal{J}^{P_T+L-S-J}_{\rho} \mathcal{J}^{L_T}_{\rho} \right]_{J^0} W(J_P T \hat{J}_{AB}^{2L}; S J) W(J_P T \hat{J}_{AB}^{2L}; S J) x$$

(The factor $\delta_{SS}$ can be dropped because it is implied by the fact that the relative angular momentum of the transferred nucleons is the same for either the $T$- or $B$-system; $\lambda_{\rho} = \lambda_{\rho}$.)

Now it is necessary to transform the functions of the center of mass coordinates $\mathbf{r}, \mathbf{R}$ to functions of the channel radii $\mathbf{r}_T, \mathbf{r}_P$. Once again the expansion of Appendix F is employed. The definition of the center of mass function,

$$\mathcal{J}^J_{\lambda_T \lambda_R}(r, R) \equiv$$

$$\sum L \lambda_{\rho}^{+J + L - S - \lambda} \mathcal{J}_{PT}^{\hat{J}_{AB}^{2L}} \mathcal{J}_{PT}^{\hat{J}_{AB}^{2L}} W(J_P T \hat{J}_{AB}^{2L}; S J) W(J_P T \hat{J}_{AB}^{2L}; S J) \times$$

$$\times \int d\rho d\rho' \mathcal{G}_{\lambda_T \lambda_P}^{L_T} \mathcal{G}_{\lambda_R \rho}^{L} (r, \rho) \mathcal{G}_{\lambda_{P_T} \lambda_{P_L}}^{L} (r, \rho),$$

together with its transformation,
allows the form factor (4-35) to be written

\[
\tilde{F}_{\lambda R}^J (r_T, r_P) = \sum_{T \leq P} \tilde{F}_{\lambda R}^J (r_T, r_P) \left[ \gamma_{\ell_T} (\hat{r}_T) \otimes \gamma_{\ell_P} (\hat{r}_P) \right]_J^0
\]

This expression is easily simplified by recoupling and integration. Finally

\[
\tilde{F}_{\lambda R}^J (r_T, r_P) = \sum_{T \leq P} \tilde{F}_{\lambda R}^J (r_T, r_P) \left[ \gamma_{\ell_T} (\hat{r}_T) \otimes \gamma_{\ell_P} (\hat{r}_P) \right]_J^0
\]

That is, the form factor is given by the bipolar expansion of the center of mass function (4-36).

These formulas are summarized in Appendix G. Appendix I contains information about the numerical techniques used for the calculation of the form factors.
4.7 Bound-State Wavefunctions for the Transferred Particles

In order to compute the transfer form factors, the bound-state radial functions (e.g. $R_t(r)$) must be specified. These functions are found by a numerical integration of the Schrödinger equation for the transferred particle bound in a potential well. A real Woods-Saxon potential, a spin-orbit potential, plus a Coulomb interaction (for protons) compose the model potential. The forms of the Woods-Saxon and Coulomb potentials are as given in Section 2.6, but for the bound states, the radius conventions are

$$
\begin{align*}
\begin{cases}
 r_v = r_{vo}^{p1/3} \\
r_c = r_{co}^{p1/3}
\end{cases} & \quad \text{T-System}
\end{align*}
$$

and

$$
\begin{align*}
\begin{cases}
 r_v = r_{vo}^{A1/3} \\
r_c = r_{co}^{A1/3}
\end{cases} & \quad \text{B-System}
\end{align*}
$$

The depth of the Woods-Saxon well is varied to reproduce the observed separation energy for the state, whose wavefunction has the correct number of nodes according to the shell model (De 63). All wavefunctions are asymptotically positive.

The spin-orbit interaction term is parameterized as

$$-\chi^2 \, v_s \, \hat{r}_t \cdot \hat{s}_t \, r^{-1} \, \frac{d}{dr} \left(1 + e^{(r-r_v)/a v}ight)^{-1}$$

in which $\chi^2$, the square of the pion Compton wavelength,
is taken to be 2.0 \( F^2 \). In this work \( V_S \) and \( V \) are related by

\[ \alpha^2 V_S/a_V = (32V/45.2) \text{ fm}. \]

For two-nucleon transfer, the binding energy for each particle is approximated by one-half the two-nucleon separation energy. The potential radii are given by

\[
\begin{align*}
    r_V &= r_{VO}(P+1)^{1/3} \\
    r_C &= r_{CO}(P+1)^{1/3}
\end{align*}
\tag{**T-system**}
\]

and

\[
\begin{align*}
    r_V &= r_{VO}(A+1)^{1/3} \\
    r_C &= r_{CO}(A+1)^{1/3}
\end{align*}
\tag{**B-system**}
\]

For cluster transfer, the numbers of nodes in the wavefunctions and the center-of-mass angular momenta, \( l_t \) and \( l_b \), can be deduced from those of the cluster constituents. Here the oscillator convention is used: The cluster wavefunction is assumed to have the properties it would have if its constituents were bound in an harmonic oscillator potential (Mo 59, Br 60, Gl 65).

4.8 Form Factor Properties

Despite the complexity of the form factor formulas, certain simple generalizations can be made about the form factors by considering the coordinate transformation employed in their generation. Thus, equations (F-12) are
The inverse transformation is given by
\[ \hat{r}_T = -\left(\frac{t}{u}\right)\hat{r} + \hat{R} = -(P/T)\hat{r} + \hat{R}, \]
\[ \hat{r}_p = -\hat{r} + \left(\frac{s}{u}\right)\hat{R} = -\hat{r} + (A/B)\hat{R}. \]

These transformations yield two distinct asymptotic limits for any transfer reaction.

In a stripping reaction, the entrance channel has associated with it the asymptotic limit in which \( \hat{r} \approx 0 \), compared to the other coordinates involved. In this T-channel limit a simple scaling relationship results for the channel radii.

\[ \hat{r}_p + (A/B)\hat{r}_T. \]

Figure (4-1) illustrates the limit. This scaling of the channel radii is, of course, reminiscent of the no-recoil scaling, which has had a very wide application. However there is additionally an exit channel limit in which \( \hat{R} \approx 0 \). The P-channel limit is characterized by the scaling

\[ \hat{r}_p + (T/P)\hat{r}_T. \]
For heavy-ion reactions in which the transferred mass is small compared to that of the ionic cores, neither of the scale factors is very different from unity.

\[ \frac{A}{B} \approx \frac{T}{P} \]

Consequently the form factors tend to be peaked near the diagonal \( r_p = r_T \) in this situation. It is convenient, therefore, to parameterize the form factors in terms of \( r_T \) and \( r_p - r_T \) rather than \( r_T \) and \( r_p \). This convention serves to make the features associated with the channel scaling limits more apparent and, more importantly, saves computer memory space. Figure (4-3) relates the two parameterizations. It is important to note that the scaling lines as applied to the radial form factor coordinates

\[ r_p = (A/B)r_T \]

and

\[ r_p = (T/P)r_T \]

correspond only asymptotically to the T- and P-channel limits, which also imply that the vectors \( \hat{r}_p \) and \( \hat{r}_T \) are aligned.

Figure (4-4) contains contour plots for three typical transfer form factors. The shaded regions are negative; the unshaded ones are positive. The contours represent decades, the smallest one being \( 10^{-9} \).

The scaling lines, which correspond to the channel
Figure (4-2). Relationships of transfer coordinates near the asymptotic P-channel limit.
\[ \text{P-Channel Limit} \]

\[ \mathbf{R} \rightarrow \mathbf{O} \]
\[ \mathbf{r}_P \rightarrow \mathbf{r} \]
\[ \mathbf{r}_P \rightarrow \frac{T}{P} \mathbf{r}_T \]
Figure (4-3). Comparison of two parameterizations of the form factors. In this work form factors are expressed in terms of $r_T$ and $r_p - r_T$. 
Form Factor
Coordinates

\[ r_p = \frac{T}{P} r_T \]

\[ r_p = \frac{A}{B} r_T \]

\[ r_p - r_T \]

\[ 0 \]

\[ 0 \]

\[ 0 \]

\[ r_T \]
Figure (4-4). Typical examples of transfer form factors for one-nucleon, two-nucleon, and alpha-cluster transfer reactions. Contours represent decades, and shaded regions are negative. Dashed lines above and below diagonals represent P- and T-channel scaling lines respectively. Mass and binding energy systematics are discussed in text.
Typical Transfer Form Factors

$^{48}\text{Ca}(^{28}\text{Si},^{28}\text{Si})^{47}\text{Ca}$

$L_T = 35, L_p = 36$

$J = 3$

$^{90}\text{Zr}(^{16}\text{O},^{14}\text{C})^{92}\text{Mo}$

$L_T = L_p = 32$

$J = 0$

$^{28}\text{Si}(^{16}\text{O},^{12}\text{C})^{32}\text{S}$

$L_T = L_p = 20$

$J = 0$
limits, are denoted by the dashed lines. These lines
give a good indication of the "amount of recoil" in a
reaction, i.e., the width of the form factor across
the diagonal. The form factors get progressively wider
going from one-neutron transfer to two-proton transfer
to alpha-cluster transfer.

The length of a form factor (i.e. its extent in the
r_T or r_p direction) is largely determined by the binding
energies which hold the transferred particle(s) to each
of the A- and P-cores. Specifically, the extent of the
form factor along the T-channel scaling line is determined
by the binding energy of the B-system, since r_n^b. Similarly
the extent of the form factor along the P-channel scaling
line is determined by the binding energy of the T-system,
since r_T^b. This fact becomes apparent when one of the
systems is very loosely bound. If, for example, the
B-system has a very small separation energy compared to
the T-system, the extent of the form factor is characterized
by the loose binding in all directions except along the
P-channel scaling line. Here the form factor appears to
be "sucked in," as shown in figure (4-5).

One other generalization can be made about the form
factor fall-off. In the vicinity of either scaling line,
a given contour line tends to parallel the other scaling
line if the variations near the nodal lines are ignored.
For example, in the 28^Si(16O, 12C)32^S form factor of figure
(4-4), the five lobes forming the 10^{-9} contour near the T-scaling line, if connected, would have nearly the same slope as the P-scaling line. This feature is simply explained by the fact that if $\dot{r}=0$ asymptotically on the T-scaling line, the only remaining degree of freedom is the variation in $R$. As the distance from the P-scaling line ($R\approx 0$) increases, $R$ increases, and the form factor reflects the falling off of the B-system wavefunction. This asymptotic nature of the contour lines near the scaling lines is also seen in the plots of the other form factors, as well as in figure (4-6).

In general, the occurrence of the nodal lines is somewhat more difficult to explain. For cases in which the geometry is simple, however, a correspondence can be made between the bound-state wavefunction nodes and the form factor nodal lines. Figure (4-6) contains cluster model form factors for the $^{48}\text{Ca}(^{16}\text{O},^{14}\text{C})^{50}\text{Ti}$ reaction. In this simple model the two $1P_{\frac{1}{2}}$ protons of the projectile are described by a $2S_0$ wavefunction (one node), according to the oscillator convention. Similarly the $1f_{7/2}$ protons of the residual nucleus are assumed to have a $4S_0$ wavefunction (with three nodes). These nodes are immediately apparent in the unit interaction (actually $-1$) form factor. On either side of the T-scaling line, there is one node. On either side of the P-scaling line, there are three nodes. These nodal lines correspond respectively to the nodes encountered when increasing $r$ and $R$ from zero in the
Figure (4-5). Form factor for the transfer from a very tightly bound system (-10.55 MeV) to a very loosely bound one (-1.86 MeV). Mismatch is evident near P-scaling line (dashed line above diagonal).
Fig. (4-5)
Figure (4-6). Two-proton-cluster transfer form factors calculated with unit, post, and prior interactions. A simple correspondence between form factor nodal lines and bound-state wavefunction nodes is shown. Coulomb interactions are included to illustrate the form factor sensitivity to various regions of the bound-state wavefunctions. Dashed lines above and below diagonals represent P- and T-channel scaling lines respectively.
$^{48}$Ca($^{16}$O, $^{14}$C)$^{50}$Ti Form Factors
$L_T = L_P = 3$, $J = 0$

Fig. (4-6)
bound-state wavefunctions.

The remaining form factors in figure (4-6) show the relationship between the post and prior interactions. The post interaction\(^9\) \(V_{PN}(T)\) emphasizes the region below the diagonal, the area of small \(r\), and introduces an additional node on either side of the \(T\)-scaling line. These nodal lines correspond to the node in the interaction where the nuclear and Coulomb potentials cancel. Similarly, the prior interaction \(V_{AN}(R)\) emphasizes the region above the diagonal, the area of small \(R\). The prior interaction also introduces two nodal lines, one of which is evident below the \(P\)-scaling line.

Extensions of these simple observations to examples with more complicated geometry must be done with a great deal of care. One might, for example, incorrectly reason that any non-\(S\) state (whose wavefunction must be zero at \(r\) or \(R=0\)) must have a nodal line following the appropriate scaling line. However examination of the integrand in equation (F-8) yields inverse radial factors sufficient to keep the integrand, and hence the form factor, from going

\[^9\text{These interactions represent a departure from the convention of Section 4.3 in that the Coulomb interactions } V_{PN}(C)(T) \text{ and } V_{AN}(C)(R) \text{ are included.}\]
to zero when the relevant limit is taken. On the other
hand, there are instances in which nodes do tend to lie on the scaling lines. The systematics for these cases have not been explored as yet.

4.9 Transfer Systematics

The essential ingredients for a discussion of the angular distributions of transfer reactions are contained in Sections 3.5 and 4.8. It is noted in Section 3.5 that the relatively narrow bell-shaped angular distribution associated with a typical one-nucleon transfer reaction stems from the wide distribution of S-matrix elements in angular momentum space. It turns out that this $L_p$-space distribution is wide compared to those of not only transfers involving inelastic excitation, but also transfers of more than one particle. The widths of the angular momentum space distributions for the direct transfer of one or more particles are, in fact, simply related to the extent of the transfer form factors in the $r_p$(or $r_T$) direction. Typically groups of two nucleons are more tightly bound than single nucleons in a given nucleus. Therefore the form factors for two-nucleon transfer fall off more rapidly than those for one-nucleon transfer, as shown in Figure (4-4). If the lower cut off in angular momentum space, governed by the strong absorption radius, is approximately the same for either transfer process, the width of the
one-nucleon transfer S-matrix distribution is usually larger than that of the two-nucleon transfer distribution. As a consequence, two-nucleon transfer reactions tend to have somewhat broader, more forward-peaked angular distributions, other parameters being equal.

The S-matrix distributions for one- and two-proton cluster stripping reactions are given in figure (5-2) by the curves labeled "Direct" and "(1)" respectively. The resulting angular distributions are narrow and bell-shaped for the one-proton transfer (upper curve in figure (5-4)) but oscillatory and forward-peaked for the two-proton transfer (figure (5-3)).

Of course geometrical properties can also affect the shapes of the angular distributions, but usually these effects are more subtle. A useful generalization in this regard is that angular distributions for final rearrangements with a relatively large number of channels (e.g. nonzero $J_p$ and $J_Q$) tend to have shallower, more washed out oscillations, if any. This result is due to the spreading of the S-matrix distribution about what would otherwise be only a very few surface terms. Additionally, the population of a relatively large number of final channels can imply a larger cross section. This situation is indicated by relatively large (but still well matched) angular momentum transfers or the population of states with a few units of spin. Finally, particle transfers to states with angular momentum larger than that of the initial states must obtain
the angular momentum from the relative motion. This damping effect is illustrated in figure (3-1) and table (3-1), where the largest channel is characterized by the maximum orbital angular momentum decrease: \( L_p = L_{T_0} - 2 \hbar \).

Section IV B of (Ha 75a) contains a lucid discussion of semi-classical aspects of heavy-ion transfer reactions as well as the kinematical considerations concerning Q-values and angular momentum transfer. For the purposes of this thesis, it is sufficient to note that observable transfer reactions are well described by the Born approximation only if the magnitudes of the Q-values are relatively small \(|Q| = |E_y - E_{c0}| \lesssim \text{a few MeV.}\), and the center-of-mass energy is at least a few MeV above the Coulomb barrier. Other suggested references are (Br 72, Be 74, and Ga 75).
CHAPTER 5
SEQUENTIAL TRANSFERS

5.1 Equations for the Additional Partitions

In order to consider the couplings of three (or more) two-body partitions, the model wavefunction must be generalized once again. If the bound-state wavefunctions for one of these arrangements are given by

\[(H(\gamma) - \epsilon_\gamma) \psi_\gamma(\gamma') = 0\]

and

\[(H(C-\gamma) - \epsilon_{c-\gamma}) \psi_{c-\gamma}(C-\gamma) = 0,\]

the total wavefunction, previously given by equation (4-1), can be expressed as

\[\psi_C(\gamma) = \sum_{\gamma'\gamma} \left( -1 \right)^{\nu} \psi_{c-\gamma}(C-\gamma) \psi_\gamma(\gamma') \phi_{c-\gamma,\gamma}(\gamma') \]  (5-1)

The sum over \(\Gamma\) denotes the sum of the various transfer partitions.

In terms of the total angular momentum-parity basis, equation (5-1) is written

\[\psi_C(\gamma) = \sum_{\Gamma\gamma\gamma'} \left( -1 \right)^{\nu} \psi_{c-\gamma}(C-\gamma) \phi_{c-\gamma,\gamma}(\gamma') \]  (5-2)

The \(\Gamma\)-partition basis functions are given by
and the radial functions are related to the relative motion wavefunctions by

$$
\phi_{\pi}^M (\{ t_{x, y}, \gamma, C, \gamma \}) = \left( C \right)^{-\frac{1}{2}} \sum_{\gamma} (-1)^{\gamma} \left[ \gamma \psi_{x, y} (\gamma) \right]_{K_{C, \gamma}} \left( C_{-\gamma} \right)_I \left( C_{-\gamma} \right)_I
$$

and

$$
\psi_{\pi}^M (\{ t_{x, y}, \gamma, C, \gamma \}) = \left( C \right)^{-\frac{1}{2}} \sum_{\gamma} (-1)^{\gamma} \left[ \gamma \psi_{x, y} (\gamma) \right]_{K_{C, \gamma}} \left( C_{-\gamma} \right)_I \left( C_{-\gamma} \right)_I
$$

when the partial wave expansion

$$
\phi_{\gamma, \gamma}^M (\{ t_{x, y}, \gamma, C, \gamma \}) = \left( C \right)^{-\frac{1}{2}} \sum_{\gamma} (-1)^{\gamma} \left[ \gamma \psi_{x, y} (\gamma) \right]_{K_{C, \gamma}} \left( C_{-\gamma} \right)_I \left( C_{-\gamma} \right)_I
$$

is used.

If the total wavefunction (5-2) is substituted into the Schrödinger equation and the overlap with a basis state of the form \( \phi_{\gamma}^M (\{ t_{x, y}, \gamma, C, \gamma \}) \) is taken, then

$$
\phi_{\gamma}^M (\{ t_{x, y}, \gamma, C, \gamma \}) = \left( C \right)^{-\frac{1}{2}} \sum_{\gamma} (-1)^{\gamma} \left[ \gamma \psi_{x, y} (\gamma) \right]_{K_{C, \gamma}} \left( C_{-\gamma} \right)_I \left( C_{-\gamma} \right)_I
$$

The unreduced functions \( \psi_{\gamma}^M (\{ t_{x, y}, \gamma, C, \gamma \}) \) are defined by

$$
\psi_{\gamma}^M (\{ t_{x, y}, \gamma, C, \gamma \}) = \left( C \right)^{-\frac{1}{2}} \sum_{\gamma} (-1)^{\gamma} \left[ \gamma \psi_{x, y} (\gamma) \right]_{K_{C, \gamma}} \left( C_{-\gamma} \right)_I \left( C_{-\gamma} \right)_I
$$

If the total wavefunction (5-2) is substituted into the Schrödinger equation and the overlap with a basis state of the form \( \phi_{\gamma}^M (\{ t_{x, y}, \gamma, C, \gamma \}) \) is taken, then
if $Z \neq \Gamma$, so that both the diagonal radial function and the nonorthogonality terms appear on the left side of (5-3).

The unreduced source terms are as given in Chapter 4:

$$p_{\gamma Z}(r_\gamma) =$$

$$\int d\Gamma d\gamma dr_\gamma \left\{ \left( C_{\gamma} \right)^{-\frac{\chi}{2}} \left( -1 \right)^{v} p_{\nu}(V_{C-\gamma, \gamma}(\zeta) - V_{\gamma \gamma}^{\pi}(r_\gamma)) \right\} \times$$

$$\left[ \left[ Y_{L\gamma}(r_\gamma) Y_{\gamma}(r_\gamma) \right]_{C-\gamma, \gamma} Y_{\gamma}(r_\gamma) \right]_{M \gamma M} \sum_{\xi} \int d\zeta \left( \hat{r}_z \right) \phi_{\gamma M}(r_\gamma, \zeta).$$

The reduced form of equation (5-3) is

$$\left[ \frac{\hat{r}^2}{2u_\gamma} - \frac{L_{\gamma}(L_{\gamma}+1)}{r_\gamma^2} + V_{\gamma \gamma}^{\pi}(r_\gamma) - \sum_{\xi} U_{\gamma Z}^{\pi}(r_\gamma) \right] =$$

$$= - \sum_{\gamma, \gamma} \Sigma_{\gamma} U_{\gamma Z}^{\pi}(r_\gamma) U_{\gamma Z}^{\pi}(r_\gamma) - \sum_{Z} p_{\gamma Z}(r_\gamma),$$

(5-4)

in which the reduced functions are related to their unreduced counterparts by the amplitude

$$\left( \begin{array}{cccccc}
L_{\xi_o} & J_{\xi_o} & K_{C-\xi_o} & K_{C-\xi_o} & J_{C-\xi_o} & I \\
L_{C-\xi_o} & C_{\xi_o} & M_{\xi_o} & M_{\xi_o} & C_{\xi_o} & M_{C-\xi_o} & M \\
 & L_{\xi_o}+1 & 0 & 0 & 0 & 0 & 0
\end{array} \right) \times$$

$$\left[ 4\pi(2L_{\xi_o}+1) \right]^{\frac{1}{2}} e^{-i(2\xi_o + 1)(2k_{\xi_o} r_\gamma)}.$$

($\xi_o$ denotes the entrance channel.)

The set of coupled equations (5-4) are to be solved according to the boundary conditions
Thus, the cross sections are once again given by equations (4-10).

5.2 Second Order Solution of the Transfer Equations

It is evident from a comparison of equations (4-7) and (5-4) that the forms of the inelastic coupling elements and the transfer source terms are unchanged by the generalization to more transfer partitions. What does change are the number of coupling elements and the degree to which the coupled equations (5-4) must be solved. Typically these equations are solved in such a way that terms involving more than two transfers are dropped. It is also conventional to choose the transfer interactions so that the nonorthogonality terms do not need to be explicitly calculated. To this end, inelastic excitation couplings are not considered in this section.

The first iteration is, of course, that of the entrance partition $\Xi$ decoupled from any of the transfer partitions. This zeroth order approximation is expressed as

$$U_{\gamma}^{\Pi}(r_\gamma) \approx 0$$

for $\Gamma \neq \Xi$. Thus the nonorthogonality terms $U_{\gamma}^{\Pi}(r_\gamma)$ and the source terms $\gamma_{\gamma}^{\Pi}(r_\gamma)$ are zero initially. The solution of
(5-4), which is now homogeneous, yields the distorted waves, \( \bar{U}_\xi^\pi(r_\xi) = U_\xi^\pi(r_\xi) \), to be used in calculating the first order transfers.

At this point, the question of whether to use the post or prior form for the transfer interaction arises. As far as the first order solutions (of the second iteration) are concerned, the post form yields the solutions

\[
\bar{U}_{\gamma}^\pi(r_\gamma) \equiv \sum \bar{U}_Z^\pi(r_\gamma) \equiv U_{\gamma}^\pi(r_\gamma) + \sum_{Z \neq \gamma} \bar{U}_Z^\pi(r_\gamma)
\]

for the population of the state \( \gamma \), while the prior form results in \( U_{\gamma}^\pi(r_\gamma) \) directly, because the entrance channel solution is homogeneous.\(^1\) Although these solutions should give the same cross section because of the post-prior equality, the prior interaction form is traditionally preferred because with its use, it is unnecessary to subtract the nonorthogonality terms to obtain \( U_{\gamma}^\pi(r_\gamma) \), for use in the third iteration.

The third iteration necessarily employs the post interaction to obtain the second order solutions. This choice is required by the fact that the first order solutions are not homogeneous solutions, so the prior form should not be used. This combination of interaction forms for two successive transfers is called the prior-post interaction form (Ka 74, Ud 73). Higher order

\(^1\)These assertions are explained in Section 4.3
solutions, if desired, must make use of the post interaction and the consequent explicit calculation and subtraction of the nonorthogonality terms.

5.3 Properties of Sequential Transfers

Sequential transfer mechanisms can be simply classified according to their relationships with any corresponding direct mechanisms. Specifically the sequential processes can be characterized according to whether they are "similar" or "dissimilar" to a direct, i.e. lower order, population process. A third kind of sequential process has no conventional corresponding direct population mechanism. Some examples of the grouping of these processes are presented in table (5-1).

The simplest, and most often studied, kind of sequential transfers is the group which are similar to direct routes. This nomenclature is based on the well known fact that the sequential stripping or pick-up of two nucleons is calculated to have a comparable differential cross section to that of the corresponding simultaneous two-nucleon transfer mechanism (Go 74, Ka 74, Ko 75, Fe 76, Ka 76, Tu 77). The qualitative likeness of the angular distributions can be understood in terms of the S-matrix magnitude distributions, which are similar for these mechanisms: The strong absorption radii for the various partitions are all about the same
Table (5-1)

Kinds of Sequential Transfers

1. Similar to Direct Route

\[ ^{48}\text{Ca}(^{28}\text{Si},^{27}\text{Al})^{49}\text{Sc}(^{27}\text{Al},^{26}\text{Mg})^{50}\text{Ti} \]
\[ ^{48}\text{Ca}(^{28}\text{Si},^{29}\text{Si})^{47}\text{Ca}(^{29}\text{Si},^{30}\text{Si})^{46}\text{Ca} \]

2. Dissimilar to Direct Route

\[ ^{48}\text{Ca}(^{28}\text{Si},^{24}\text{Mg})^{52}\text{Ti}(^{24}\text{Mg},^{28}\text{Si})^{48}\text{Ca} \]
\[ ^{48}\text{Ca}(^{28}\text{Si},^{24}\text{Mg})^{52}\text{Ti}(^{24}\text{Mg},^{25}\text{Mg})^{51}\text{Ti} \]
\[ ^{48}\text{Ca}(^{28}\text{Si},^{26}\text{Mg})^{50}\text{Ti}(^{26}\text{Mg},^{27}\text{Al})^{49}\text{Sc} \]
\[ ^{48}\text{Ca}(^{28}\text{Si},^{29}\text{Si})^{47}\text{Ca}(^{29}\text{Si},^{28}\text{Al})^{48}\text{Sc} \]

3. No Direct Route

\[ ^{48}\text{Ca}(^{28}\text{Si},^{27}\text{Al})^{49}\text{Sc}(^{27}\text{Al},^{29}\text{Al})^{47}\text{Sc} \]
\[ ^{48}\text{Ca}(^{28}\text{Si},^{29}\text{Si})^{47}\text{Ca}(^{29}\text{Si},^{27}\text{Mg})^{49}\text{Ti} \]
regardless of the transfer process; in either case, the binding energies of the transferred particles are equal. Consequently the lower and upper limits on the angular momentum space distributions are roughly the same, so the angular distributions are similar. A more elaborate argument for the similarity of sequential and simultaneous processes in two-nucleon transfer reactions is given in Section 4 of (Fe 76)\(^2\). Deviations from this general rule are studied in (Tu 77).

The second kind of sequential transfer mechanisms is that group which are dissimilar to the direct population routes. In this context, "dissimilar" means "involving the transfer of more particles than those of the corresponding direct route." Typical dissimilar processes which have been studied are those which include both the stripping and pick-up of a nucleon. If the transferred particles are both protons or neutrons, the sequential transfer mechanism contributes to the elastic channels, which would have been populated anyway in the zeroth order calculation. (See, for example, (Wo 75).) If a proton is stripped and a neutron picked up, or vice versa, the sequential process may compete with a first order charge exchange reaction

\(^2\)The T-matrix formalism is developed in (Ta 74).
which involves an isospin flip mechanism rather than particle transfer (Ga 74, Ku 74, Os 76). Other dissimilar sequential transfer mechanisms compete with lower order transfers of fewer particles. An example of this sort of mechanism is the stripping of two particles followed by the pick-up of one, to populate the two-particle-one-hole components of the residual nucleus. Any single-particle components of these states would also be populated by the direct stripping of one particle (Sh 76). These processes are diagramed in figure (5-1).

In general, the dissimilar sequential transfer mechanisms have rather different angular distributions than the associated direct mechanisms. This assertion is obvious as far as competition with the elastic scattering is concerned. It is also apparent that the transfer of three particles has a broader, more forward-peaked angular distribution than the transfer of one. Figures (5-2) through (5-5) illustrate this fact for the $^{48}\text{Ca}(^{16}\text{O},^{15}\text{N})^{49}\text{Sc}(^{5+},3/2^+)$ reactions induced by 56 MeV $^{16}\text{O}$ (Sh 76).

A comparison of the localization in angular momentum space of S-matrix magnitudes is shown in figure (5-2). (The only channel graphed for the $^{15}\text{N}$-partition is that for which the orbital angular momentum is one unit less than the total: $L_{\gamma}+\Delta l=I$). The transfers comprising the
Figure (5-1). Level diagram for a sequential transfer process which is dissimilar to the direct transfer mechanism.
sequential mechanism have been normalized to the $^{14}$C data of (He 74). Figures (5-3) and (5-4) present the resulting angular distributions. The Cross section for the direct one-proton stripping, in figure (5-4), has a spectroscopic factor of two inserted to describe the population of the $1p_{\frac{3}{2}}$ ground state of $^{15}$N. The target system is expected to have much less single-particle strength but is calculated with unit amplitude. For the second step of the sequential route, the factors are one and two respectively for the $^{15}$N and $^{49}$Sc($\frac{3}{2}^+$) systems. Figure (5-5) shows similar calculations for the $^{49}$Sc($3/2^+$) state.$^{3}$

$^{3}$The magnitudes of the distributions in figures (5-4) and (5-5) are, perhaps, best interpreted as lower limits to experimental cross sections, due to the very absorptive optical potential used; i.e. $^{49}$Sc ground state data are about an order of magnitude larger than these calculated direct routes. In this case, the relative magnitude of the $\frac{1}{2}^+$ and $3/2^+$ cross sections is attributable to the additional node in the $^{49}$Sc($\frac{3}{2}^+$) bound-state wavefunction, which is larger than that of $^{49}$Sc($3/2^+$) by a factor of almost three at large radii. Also note that the interaction $V_{PN}(T)$ is used for transfers from both $^{16}$O and $^{14}$C.
Figure (5-2). A comparison of the S-matrix magnitude distributions in angular momentum space for direct and sequential transfer processes. The direct one-proton transfer distribution, denoted by the dashed curve, falls off more slowly than that of either the direct two-proton transfer (1) or the sequential transfer mechanism (1+2).
Fig. (5-2)

S-matrix Magnitude

Direct $^{48}\text{Ca}(^{16}\text{O}, ^{15}\text{N})^{49}\text{Sc}(^{1/2}^+)$

(1) $^{48}\text{Ca}(^{16}\text{O}, ^{14}\text{C})^{50}\text{Ti}(\text{g.s.})$

(1 + 2) $^{48}\text{Ca}(^{16}\text{O}, ^{14}\text{C})^{50}\text{Ti}(\text{g.s.})(^{14}\text{C}, ^{15}\text{N})^{49}\text{Sc}(^{1/2}^+)$
Figure (5-3). Forward-peaked, oscillatory angular distribution for two-proton transfer. This distribution is characteristic of the population of $0^+$ states via the transfer of tightly bound particles ($\sim 20$ MeV in each system) and indicates that only a few S-matrix elements are contributing significantly to the angular distribution.
\( \frac{d\sigma}{d\Omega} (\mu b/sr) \)

\[ ^{48}\text{Ca}(^{16}\text{O}, ^{14}\text{C})^{50}\text{Ti}(g.s) \]

\( E = 56 \text{ MeV} \)

\( \theta_{\text{c.m.}} \)
Figure (5-4). A comparison of angular distributions for the population of $^{49}\text{Sc}(\frac{1}{2}^+, 2.229 \text{ MeV})$ by direct and sequential transfer mechanisms. The sequential process distribution is typically broader and more forward-peaked than that of the corresponding direct transfer process. $E = 56 \text{ MeV}$. 
Figure (5-5). A comparison of angular distributions for the population of $^{49}\text{Sc}(3/2^+, 2.372 \text{ MeV})$ by direct and sequential transfer mechanisms. The sequential process distribution is typically broader and more forward-peaked than that of the corresponding direct transfer process. $E = 56 \text{ MeV}$. 
Potential Parameters (Ei 76)

<table>
<thead>
<tr>
<th></th>
<th>V (MeV)</th>
<th>W (MeV)</th>
<th>$r_{vo}$ (fm)</th>
<th>$r_{wo}$ (fm)</th>
<th>$r_{co}$ (fm)</th>
<th>$a_v$ (fm)</th>
<th>$a_w$ (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{16}_0^{48}$Ca</td>
<td>-33.9</td>
<td>-110.2</td>
<td>1.344</td>
<td>1.274</td>
<td>1.2</td>
<td>0.424</td>
<td>0.28</td>
</tr>
<tr>
<td>$^{15}_N^{49}$Sc</td>
<td>-33.9</td>
<td>-110.2</td>
<td>1.344</td>
<td>1.274</td>
<td>1.2</td>
<td>0.424</td>
<td>0.28</td>
</tr>
<tr>
<td>$^{14}_C^{50}$Ti</td>
<td>-33.9</td>
<td>-110.2</td>
<td>1.344</td>
<td>1.274</td>
<td>1.2</td>
<td>0.85</td>
<td>0.28</td>
</tr>
<tr>
<td>Bound States</td>
<td>1.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.65</td>
</tr>
</tbody>
</table>
It is obvious from these figures that this dissimilar sequential mechanism yields not only different angular distributions compared to the direct transfer but also much smaller transfer amplitudes. This relative weakness of the dissimilar sequential process is a general property of all the dissimilar sequential mechanisms discussed in this section. Those processes which contribute to the elastic channels are negligible except, perhaps, at back angles. The sequential reactions which compete with charge exchange reactions are just as weak, and the simple isospin flip mechanisms tend to dominate (Os 76).

The third kind of sequential transfers is the group which have no conventional corresponding direct population processes. The two examples listed in this group in table (5-1) involve respectively the pick-up of a unit of mass with the stripping of a unit of charge, and the stripping of a unit of mass with two units of charge. Neither of these reactions can be the result of a single transfer of a known nuclear constituent. The indicated sequential mechanisms are composed of three nucleon transfers and hence are characterized by broad, forward-peaked angular distributions, and cross sections on the order of a few μb/sr.

There is a qualification regarding this classification scheme which should be mentioned. Kovar, et al. (Ko 74)
note that it is conceivable to treat reactions of this third kind in first order if particles are allowed to transfer in opposite directions simultaneously. In principle, charge exchange type reactions could also proceed via such a mechanism. There are, however, apparently no calculations for such processes. Moreover the two-step analyses (Ud 75 and Os 76) evidently achieve satisfactory data reproduction without the inclusion of such effects. This unconventional transfer process is not considered in this work.
CHAPTER 6
ANALYSIS OF THE $^{48}$Ti($^{16}$O,$^{15}$N)$^{49}$V REACTION

6.1 The Nuclear Model

Considerable interest has developed in the study of nuclear structure in the $1f_{7/2}$ shell during the last decade. Analyses reported in these studies have applied a variety of nuclear models (Mc 64, Gi 66, Ma 66, Sc 67, Ha 76, St 76) and light-ion induced reactions (Yn 67, Ba 68, Be 68, Pu 68, Cu 69, Mo 70, Bl 71, Sa 73, Fa 74, Ta 74a, Ha 75, Wh 76) in the determination of the properties of $^{48}$Ti, $^{49}$V, and other nuclides in this region. With the development of heavy-ion beams, other reactions have been utilized toward this end (Ha 70, De 71, Mo 72, Ma 73), and the question of whether or not multistep inelastic processes must be explicitly included in the descriptions of the ($^{16}$O,$^{15}$N) reactions has arisen (Ko 73, Ma 73).

While many of the experimental angular distributions have offered no clear evidence supporting the importance of such effects, the anomalous distributions resulting from the $^{48}$Ti($^{16}$O,$^{15}$N)$^{49}$V reactions have shapes characteristic of the interference between competing direct and indirect transfer processes (Er 76, Er 74, As 74). Subsequent analysis has confirmed this interpretation and indicated why evidence for such processes
is not more prevalent in other related reactions (Sh 77, Sh 77a). Theoretical and experimental differential cross sections are compared for $E=50$ MeV in figure (6-1).

A relatively simple nuclear model is employed, which clearly illustrates the reaction dynamics involved. The $^{48}$Ti ground state is assumed to be both a particle vacuum and a core phonon vacuum state. The first excited 2+ level in $^{48}$Ti is a quadrupole core phonon vibrational state built upon the ground state vacuum. (See figure (6-2).) The $^{49}$V ground state is taken to be the single-particle state of a $7/2^−$ proton bound to the inert $^{48}$Ti core. Near the energy level of the 2+ state in $^{48}$Ti, there are two negative parity states in $^{49}$V, the $11/2^−$ and $9/2^−$ states near 1 MeV. In the weak-coupling model (De 61), these states are considered to be those of $7/2^−$ protons coupled to quadrupole core phonons in $^{48}$Ti, and can be populated only by an indirect transfer process including both inelastic excitation and transfer. (See Sections H.1.1 and H.1.2) It turns out that the two mechanisms by which such a particle-phonon state can be populated from the $^{48}$Ti ground state, i.e. $^{48}$Ti($^{16}_O,^{16}_O′$)$^{48}$Ti(2+)($^{16}_O′,^{15}_N$) $^{49}$V($11/2^−$) and $^{48}$Ti($^{16}_O,^{15}_N$)$^{49}$V($7/2^−$)($^{15}_N,^{15}_N′$)$^{49}$V($11/2^−$), interfere constructively. Consequently the weak-coupling model cannot reproduce the observed destructive interference in the $11/2^−$ differential cross section. In order to
Figure (6-1). A comparison of theoretical and experimental differential cross sections for the reaction $^{48}\text{Ti}(^{16}\text{O},^{15}\text{N})^{49}\text{V}$. A full-recoil coupled channels analysis is employed. Theoretical angular distributions have the same overall normalization for all four states. The experimental cross sections were measured at Brookhaven National Laboratory and are the result of a collaboration between K.A. Erb and D.L. Hanson of Yale and J.J. Kolata of Brookhaven.
Fig. (6-1)

\[ \frac{d\sigma}{d\Omega} (\mu b/\text{sr}) \]

- $^{48}\text{Ti}(^{16}\text{O},^{15}\text{N})^{49}\text{V}$
- $E = 50 \text{ MeV}$
- $\frac{3}{2}^-$ $0.153 \text{ MeV}$
- $\frac{1}{2}^-$ $1.021 \text{ MeV}$
- $\frac{9}{2}^-$ $1.155 \text{ MeV}$

\( \theta_{cm} \)

-135-
Figure (6-2). Low-lying energy levels for $^{48}$Ti and $^{49}$V. Within the weak-coupling model the $9/2^-$ and $11/2^-$ states in $^{49}$V are particle-phonon states. This analysis treats these states, and the lowest $3/2^-$ state, as predominantly particle-phonon states, with small single-particle admixtures.
**Low-Lying Energy Levels**

<table>
<thead>
<tr>
<th>E (KeV)</th>
<th>$J^\pi_A$</th>
<th>E (KeV)</th>
<th>$J^\pi_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2295.1</td>
<td>4$^+$</td>
<td>2182.7</td>
<td>$\frac{7}{2}^-$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2178.5</td>
<td>$\frac{9}{2}^+$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1994.9</td>
<td>$\frac{3}{2}^+$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>16614</td>
<td>$\frac{3}{2}^-$ (\frac{1}{2}^-)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1643.1</td>
<td>$\frac{7}{2}^+$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1603.1</td>
<td>$\frac{5}{2}^-$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1514.4</td>
<td>$\frac{1}{2}^+$</td>
</tr>
<tr>
<td>983.4</td>
<td>$2^+$</td>
<td>1155.4</td>
<td>$\frac{9}{2}^-$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1140.9</td>
<td>$\frac{5}{2}^+$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1021.6</td>
<td>$\frac{11}{2}^-$</td>
</tr>
<tr>
<td>748.3</td>
<td>$3/2^+$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.</td>
<td>$0^+$</td>
<td>152.9</td>
<td>$\frac{3}{2}^- 5/2^-$</td>
</tr>
<tr>
<td>$^{48}_{\text{Ti}}$</td>
<td></td>
<td>90.7</td>
<td>$\frac{7}{2}^-$</td>
</tr>
<tr>
<td>0.</td>
<td>$7/2^-$</td>
<td>$^{49}_{\text{V}}$</td>
<td></td>
</tr>
</tbody>
</table>
populate the excited states in $^{49}\text{V}$ from the $^{48}\text{Ti}$ ground state directly as well as indirectly, and thus allow the possibility of destructive interference, each of the $^{49}\text{V}$ excited states is considered to be a coherent sum of two components. The larger of these components is the particle-phonon part, which can be populated only by two-step processes; additionally a small admixture of the appropriate single-particle state (e.g. $11/2^-$ or $9/2^-$) can be populated only by the direct, one-step transfer process. This model is summarized in table (6-1).

The two-step transfer processes tend to populate the excited states of the residual nucleus more weakly than the direct transfer mechanism. Consequently, the particle-phonon component of the excited state is expected to be larger than the single-particle component ($C_{p.p} > C_{s.p}$), if the two kinds of mechanisms are to interfere efficiently. The ratio of the coefficients $C_{s.p.}/C_{p.p.}$ is taken to be a free parameter, and the coefficients are normalized to one.

$$2C_{p.p.} + 2C_{s.p.} = 1$$

If a $7/2^-$ particle and a $2^+$ core phonon are vector coupled, a five-membered multiplet of states can result, with spins ranging from $3/2^-$ to $11/2^-$. It is clear from figure (6-2) that such a multiplet does not exist in $^{49}\text{V}$ (and that the weak-coupling model should not be assigned
TABLE (6-1)

Nuclear Model for the A- and B- Systems

\[
\begin{align*}
|^{48}_{\text{Ti}}(2^+)\rangle &= \nu_2 |^{48}_{\text{Ti}}(0^+)\rangle \\
|^{49}_{\text{V}}(J_B^-)\rangle &= (\text{Cp.p.}[\nu_{1f}^+ \otimes \nu_{2}^+]_{J_B} + \text{Cs.p.} \nu_{Bj}^+) |^{48}_{\text{Ti}}(0^+)\rangle \\
|^{48}_{\text{Ti}}(0^+)\rangle &= \nu_{1f}^+ |^{48}_{\text{Ti}}(0^+)\rangle \\
|^{49}_{\text{V}}(7/2^-)\rangle &= \nu_{1f}^+ |^{48}_{\text{Ti}}(0^+)\rangle
\end{align*}
\]
too much validity under these circumstances). However, the lowest 3/2^- state in ^{49}\text{V} does have an enhanced reduced quadrupole transition probability to the ground state. (See table (6-2).) The collective behavior of the 3/2^- state indicates that the model which is employed for the analysis of the cross sections of the 11/2^- and 9/2^- states should be applicable to the 3/2^- state also.

Experimental cross section points for the lowest 5/2^- state were not resolved from those of the ground state, and thus were not subject to analysis within this model.

Figure (6-3) contains a qualitative description of the interference effects between the one- and two-step processes if the coefficients C_p,p. and C_s,p. have the same sign. The signs labelling the different reaction mechanisms denote the signs of the appropriate reduced matrix elements and geometrical factors. The signs of the inelastic transitions are those corresponding to the attractive nuclear potential. The Coulomb excitation is, of course, important, but it is unnecessary for these qualitative diagrams.

Each of the two-step routes populating the 11/2^- state has a net plus sign. These mechanisms interfere constructively with each other but destructively with the direct-transfer route. For the 9/2^- state all three routes have a net minus sign and interfere constructively. If C_p,p. and C_s,p. have the same sign for the 3/2^- state,
Table (6-2)

Experimental Reduced Quadrupole Transition Probabilities for Deexcitation to Ground States

<table>
<thead>
<tr>
<th>$J^\pi_A$</th>
<th>$B(E2) (e^2fm^4)$</th>
<th>$J^\pi_B$</th>
<th>$B(E2) (e^2fm^4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2$^+$</td>
<td>138 ± 12 a</td>
<td>9/2$^-$</td>
<td>58 ± 33 c</td>
</tr>
<tr>
<td></td>
<td>144 ± 8 b</td>
<td>11/2$^-$</td>
<td>106 ± 28 d</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3/2$^-$</td>
<td>144 ± 28 c</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>172 ± 59 d</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>197 ± 20 c</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>197 ± 3 d</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>197 ± 3 e</td>
</tr>
</tbody>
</table>

$48_{\text{Ti}}$

$49_{\text{V}}$

a (Ha 70)
b (De 71)
c (Ha 75)
d (Ta 74a)
e (Wh 76) (18.5 ± 0.3 W.U.)

See (Dh 77) for other experimental results and derived effective charges.
Figure (6-3). Relative signs of competing routes in the $^{48}\text{Ti}(^{16}\text{O},^{15}\text{N})^{49}\text{V}$ reaction. A qualitative description of the interference effects between one- and two-step processes if Cp.p. and Cs.p. have the same sign. Since angular distributions for the $11/2^-$ and $9/2^-$ states exhibit destructive and constructive interference effects respectively, Cp.p. and Cs.p. do have the same sign for these states. The diagram for the $3/2^-$ state is identical to that of the $11/2^-$ state. Since the angular distribution for the $3/2^-$ state exhibits constructive interference, Cp.p. and Cs.p. have opposite signs for this state.
Relative Signs of Competing Routes

Fig. (6-3)

\[ \begin{array}{c}
2^+ & \rightarrow & \frac{11}{2}^- \\
0^+ & \rightarrow & \frac{7}{2}^- \\
^{48}\text{Ti} & \rightarrow & ^{49}\text{V}
\end{array} \]

Destructive Interference

\[ \begin{array}{c}
2^+ & \rightarrow & \frac{9}{2}^- \\
0^+ & \rightarrow & \frac{7}{2}^- \\
^{48}\text{Ti} & \rightarrow & ^{49}\text{V}
\end{array} \]

Constructive Interference
a destructive interference diagram identical to that of the 11/2^- state results. Since constructive interference is experimentally observed for the 3/2^- and 9/2^- states, and destructive interference is observed for the 11/2^- state, the coefficients C_p.p. and C_s.p. have the same sign for the 11/2^- and 9/2^- states and opposite signs for the 3/2^- state.

The deformation parameters used in this analysis are given in table (6-3). For simplicity the ^{49}_V system is assumed to have the same deformations as the ^{48}_Ti system. The nuclear deformation parameter is consistent with that derived from ^{4}_He scattering on ^{48}_Ti by Yntema and Satchler (Yn 67). The charge deformation parameter is obtained from the B(E2) of ^{48}_Ti measured by Häusser, et al. (Ha 70). (See Appendix D for a discussion of the relevant conventions.)

6.2 The ^{49}_V Ground State Angular Distribution

The optical potential parameters used in this analysis (table (6-4)) are determined by fitting the experimental angular distribution for the ^{49}_V ground state. Three theoretical angular distributions, employing different potential parameters, are shown in figure (6-4). The dashed curve corresponds to the four-parameter potential which has been used to fit elastic scattering data at 48 MeV (Ma 73, Mo 72). (See also (Ma 74).) This calculated
<table>
<thead>
<tr>
<th></th>
<th>( \beta_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nuclear</td>
<td>0.230 a</td>
</tr>
<tr>
<td>Charge</td>
<td>0.241 b</td>
</tr>
</tbody>
</table>

\( a \) (Yn 67)  
\( b \) (Ha 70)
curve misses the experimental grazing peak by a few degrees and underestimates the data at forward angles. One procedure used to fit the grazing peak is to increase the real radius slightly. A 3% increase in the real radius parameter for both the entrance and exit channels does help to fit the peak, but the forward angle points are still underestimated. Another method which can be used to fit the grazing peak is to increase the exit channel diffuseness by 0.1 fm. While this procedure has the drawback of exacerbating the neglect of the indirect interaction terms (in principle at least), it does fit the experimental peak and continues to underestimate the forward angle points. If the distribution fitting procedures are restricted to those in which only one parameter is varied, the one which is apparently best to change is the imaginary radius. A reduction of 25% in both the entrance and exit channels reproduces the ground state angular distribution and gives the best fits to the excited states. The normalization factor for the solid curve in figure (6-4) and for the calculated angular distributions for the $^{49}$V excited states is 0.9, assuming unit spectroscopic strength for the T-system.

6.3 The $^{49}$V (3/2$^-$, 0.153 MeV) Angular Distribution

In Section 3.5 the characteristic differences in angular distributions resulting from the one- and two-step processes populating the 3/2$^-$ state of $^{49}$V are discussed.
Table (6-4)

Potential Parameters for $^{48}\text{Ti}(^{16}\text{O},^{15}\text{N})^{49}\text{V}$ at 50 MeV

<table>
<thead>
<tr>
<th>V(MeV)</th>
<th>W(MeV)</th>
<th>$r_{vO}$(fm)</th>
<th>$r_{wO}$(fm)</th>
<th>$r_{CO}$(fm)</th>
<th>$a_{v}$(fm)</th>
<th>$a_{w}$(fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scattering</td>
<td>-100. -40.</td>
<td>1.22</td>
<td>0.915</td>
<td>1.2</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Bound States</td>
<td></td>
<td>1.2</td>
<td>1.2</td>
<td>0.65</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure (6-4). A comparison of calculated angular distributions for the reaction $^{48}\text{Ti}(^{16}\text{O},^{15}\text{N})^{49}\text{V}(7/2^-,\text{g.s.})$ at $E=50$ MeV using different optical potential parameters. The dashed curve corresponds to a four-parameter optical potential which fits elastic scattering data at $E=48$ MeV. The dotted curve reflects the improved fit resulting from increasing the real radius parameter by 3% in the entrance and exit channels. The solid curve corresponds to the potential parameters used in this analysis. This potential has an imaginary radius 25% less than the original four-parameter one.
Fig. (6-4)

$^{48}\text{Ti} \left(^{16}\text{O}, ^{15}\text{N}\right)^{49}\text{V}$

$E = 50 \text{ MeV}$

$\frac{7}{2} \text{ g.s.}$

$\frac{d\sigma}{d\Omega} \left(\mu b/\text{sr}\right)$

- Elastic Optical Potential Parameters
- Real Radius Increased 3%
- Imaginary Radius Decreased 25%

$\theta_{\text{c.m.}}$
The bell-shaped distribution which typically results from the direct transfer mechanism and the broad, forward-peaked angular distribution of the indirect process are shown in figure (3-2). The experimental data points, shown in figure (6-5), are not fitted well by either of these processes taken individually. The pure direct process would underestimate the forward angle points compared to the grazing peak, while the pure two-step processes would do the opposite. In fact, these data form an angular distribution which is characteristic of the indirect mechanisms at forward angles but also exhibits the direct transfer bell shape at back angles. Such a distribution can be reproduced by the coherent constructive interference of both processes, as is shown in figure (6-5). The fitting procedure yields a value for the ratio Cs.p./Cp.p. of about -0.13 and indicates a small but important directly populated component in the 3/2^- state.

6.4 The ^49V(11/2^-,1.021 MeV) Angular Distribution

Figure (6-6) shows the various angular distributions resulting from the destructive interference between one- and two-step transfer processes. The curve labelled by Cs.p./Cp.p. = 0. exhibits the broad, forward-peaked shape expected from a pure two-step transfer. Its magnitude is correct at forward angles, but the shallow dip near 47^o, which is the result of destructive nuclear-Coulomb interference within the inelastic excitation, is not deep
Figure (6-5). A comparison of calculated angular distributions for the reaction $^{48}\text{Ti}(^{16}\text{O},^{15}\text{N})^{49}\text{V}(3/2^-, 0.153\text{ MeV})$ at $E=50\text{MeV}$ using various single-particle/particle phonon mixtures. The experimental angular distribution indicates constructive interference between the direct and indirect processes. Curves are labelled by the ratio $C_{s.p.}/C_{p.p.}$.
Fig. (6-5)

$^{48}{\text{Ti}}(^{16}{\text{O}}^{15}{\text{N}})^{49}{\text{V}}$

$E = 50\text{ MeV}$

$\frac{d\sigma}{d\Omega}$ (µb/sr)

DISTRIBUTIONS FOR VARIOUS $\frac{C_{\text{s.p.}}}{C_{\text{p.p.}}}$ MIXTURES

$\theta_{\text{c.m.}}$
enough to reproduce the minimum in the experimental distribution. As Cs.p. is slightly increased, the required minimum emerges in the calculated distributions. If Cs.p. is increased further, the bell-shaped distribution of a direct transfer appears. The angular position of the minimum in a theoretical distribution is not very sensitive to the Cs.p./Cp.p. ratio or to the chosen optical potential parameters. However, the minimum can be moved somewhat by the artificial introduction of a complex phase between Cs.p. and Cp.p. For example, the dashed curve indicates that the calculated minimum can be moved toward the experimental one if Cs.p. is allowed to have a small negative imaginary part. This observation is not meant to suggest that Cp.p. or Cs.p. should be complex. On the contrary, these coefficients should be real in order to describe a bound state. One might interpret this sensitivity as an indication that the discrepancy of a few degrees in positioning the minimum is not necessarily due to a structural inadequacy of the nuclear model, but may reflect a deficiency in the description of the reaction dynamics.

In this regard, it is interesting to compare the angular distributions resulting from different sets of partial waves. In Section (6-2) it is stated that the transfer differential cross sections are fitted best with a set of optical potential parameters which has a reduced imaginary
Figure (6-6). A comparison of calculated angular distributions for the reaction $^{48}$Ti($^{16}$O,$^{15}$N)$^{49}$V($11/2^-$,1.021 MeV) at $E = 50$ MeV using various single-particle/particle-phonon mixtures. The experimental angular distribution indicates destructive interference between the direct and indirect processes. The angular position of the minimum is insensitive to the mixing ratio and the optical potential parameters. The dashed curve shows the sensitivity to an artificially introduced complex phase between Cs.p. and Cp.p.
Fig. (6-6)

$^{48}\text{Ti}(^{16}\text{O},^{15}\text{N})^{49}\text{V}$

$E = 50 \text{ MeV}$

$\frac{d\sigma}{d\Omega} (\mu\text{b}/\text{sr})$

$\theta_{c.m.}$

Distributions For Various $\frac{C_{s.p.}}{C_{p.p.}}$ Mixtures

$11^{-}$/2

1.021 MeV

-0.4

0.6

0.4

0.1

0.15

0.21

$\phi = -14^\circ$
radius compared to that which fits the elastic scattering cross section. This modification is apparently necessary to decrease the effect of scattering from small impact parameters compared to that from the surface region. However, there is a rather dramatic confirmation of the elastic scattering optical potential parameters, at least as far as the surface partial waves are concerned. In figure (6-7) the dashed curve corresponds to a full calculation, using elastic scattering parameters, of the angular distribution for the $1\frac{1}{2}^-$ state. The calculated destructive interference minimum is positioned a few degrees back from the minimum in the data, in roughly the same position it would be if the imaginary radius were reduced. The fit at forward angles is unsatisfactory, as it is for the ground state data in figure (6-4). However, if only those partial waves which include the largest contributions from both the two-step and direct mechanisms are considered, the interference minimum is positioned correctly, as shown by the solid curve. Moreover, deviations from these elastic scattering parameters tend to wash out the minimum within this truncated angular momentum space.

Under these circumstances one might hypothesize one of the ad hoc remedies which have been applied to analyses of elastic scattering data at back angles. In this situation also, the effects of the surface partial
Figure (6-7). A comparison of destructive interference angular distributions for the reaction $^{48}$Ti($^{16}$O, $^{15}$N)$^{49}$V $(11/2^-, 1.021$ MeV) at $E = 50$ MeV for different sets of partial waves. Elastic scattering potential parameters are used. ($r_{VO} = r_{WO} = 1.22$ fm) The dashed curve corresponds to the full calculation; the solid curve shows the effects of only the surface partial waves. The theoretical angular distributions are individually normalized to the data.
$^{48}\text{Ti}(^{16}\text{O},^{15}\text{N})^{49}\text{V}$

$E = 50\text{ MeV}$

Elastic Optical Potential Parameters

- $0 \leq L_{T_0} \leq 58$
- $24 \leq L_{T_0} \leq 29$

$\frac{1}{2}^{11}$

$1.021\text{ MeV}$

$d\sigma/d\Omega (\mu b/sr)$

$\theta_{c.m.}$

-158-
waves are apparently underestimated by the usual application of the optical model potential. Consequently a generalization such as an angular momentum dependent potential (Ro 71, Au 71) may be in order. (See also Section 2.7.)

6.5 The $^{49}\text{V}(9/2^-, 1.155 \text{ MeV})$ Angular Distribution

Figure (6-8) shows the angular distributions for the mechanisms populating the $9/2^-$ state in $^{49}\text{V}$. The two indirect transfer routes, indicated by the dotted and dashed curves, are added coherently. The resulting pure two-step angular distribution is broad, forward-peaked and oscillatory, as shown by the solid curve labelled by $\text{Cs.p.}/\text{Cp.p.} = 0$. This distribution tends to underestimate the data points at back angles, and a small single-particle component in the $9/2^-$ state helps to reproduce the experimental points.

However, this change in the calculated angular distribution is somewhat smaller than those for the $3/2^-$ and $11/2^-$ states, and there are uncertainties in the calculation which could affect the pure two-step distribution in such a way as to make a single-particle component in the $9/2^-$ state questionable. For example, the magnitude of the dashed route can be changed by a few percent depending on whether the transfer form factor includes
Figure (6-8). A comparison of calculated angular distributions for the reaction $^{48}$Ti$(^{16}$O,$^{15}$N)$^{49}$V$(9/2^-,1.155$ MeV) at E = 50 MeV. Angular distributions for the individual two-step processes are shown, as is the distribution for their coherent sum. The experimental angular distribution indicates constructive interference between the direct and indirect processes. Solid curves are labelled by the ratio Cs.p./Cp.p.
Fig. (6-8)
the $1f_{7/2}$ wavefunction bound with the proton separation energy from the ground state or the excited state in $^{49}V$. These calculations employ the same transfer form factors for the dotted and dashed routes; i.e. the separation energy from the ground state is used. In addition, the magnitude of the dotted route is uncertain to the extent that the deformation parameters for $^{49}V$ are different from those of $^{48}Ti$. Recently charge deformation parameters have been deduced for $^{49}V$ (St 76, Wh 76). These values are a few percent larger than the one given in table (6-3).

6.6 Interpretation

The single-particle and particle-phonon coefficients for the calculated differential cross sections of figure (6-1) are listed in table (6-5). In each case, the anomalous angular distributions for the excited states have contributions from very small single-particle components (yielding $\lesssim 2\%$ of the full spectroscopic strength), which interfere more or less efficiently with the much larger collective components. Similar excited states in other nuclei may not exhibit these effects if the single-particle components are appreciably larger, because the direct transfer processes may dominate even if the states have collective properties.

While these multistep transfer processes may not be
Table (6-5)

Single-Particle and Particle-Phonon Coefficients

<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>$3/2^-$</td>
<td>-0.13</td>
<td>0.992</td>
<td>-0.129</td>
</tr>
<tr>
<td>$11/2^-$</td>
<td>0.15</td>
<td>0.989</td>
<td>0.148</td>
</tr>
<tr>
<td>$9/2^-$</td>
<td>0.10</td>
<td>0.995</td>
<td>0.100</td>
</tr>
</tbody>
</table>
easily observable in other similar reactions, their presence and importance in the $^{48}\text{Ti}(^{16}\text{O},^{15}\text{N})^{49}\text{V}$ reactions are confirmed. The fitting of the anomalous excited state angular distributions, with the proper relative normalizations at forward angles, is unambiguous, despite the simple nature of the nuclear model. The success of this analysis is not a confirmation of the model, but is rather an indication of the dynamic properties of the actual, more complicated nuclear states.
CHAPTER 7

A SUGGESTED EXPERIMENT FOR OBSERVING SEQUENTIAL TRANSFERS

7.1 Motivation

During the last few years the theoretical study of heavy-ion induced one-nucleon transfer reactions has led to successful reproductions of both the shapes and magnitudes of the experimental differential cross sections. However the extensions to two-nucleon transfer reactions generally have yielded large underestimations of the observed absolute cross sections, while the experimental angular distributions have been more easily reproduced. These underestimations have been characterized by normalization factors of from one to three orders of magnitude. (See, for example, (Ba 74a, Ei 76, Le 77).) The list of possible deficiencies in current reaction models includes, among other items, the omission of the indirect interaction terms, the lack of a realistic interaction between the transferred particles, and the usual neglect of the sequential transfer mode (Ba 74, De 74, Ka 74, Ta 75, Fe 76). The model generalizations necessary to remedy one or more of these possible deficiencies are very likely to produce absolute magnitude changes in the calculated differential cross sections, without significantly and unambiguously altering the shapes of the calculated angular distributions.
In the study of one-nucleon transfer reactions, it is known that large changes in calculated absolute cross sections together with insignificant angular distribution shape changes may result from the inclusion of some of the indirect interaction terms. A similar situation may occur for two-nucleon transfer reactions. Of the reaction models describing the simultaneous transfer of two nucleons, the cluster model implies a rather strong interaction between the transferred particles, while the present microscopic model employs no explicit interaction between the transferred nucleons. These two models, however, generally predict similar angular distributions with different absolute normalizations. Moreover, the corresponding sequential transfer mechanism produces angular distributions much like the above simultaneous ones. It is clear that the model generalizations to remedy the normalization problem are likely to have similar effects on the predicted cross sections, and consequently are not individually subject to experimental verification through the study of two-nucleon transfer reactions.

It is desirable, therefore, to supplement such studies with examples which more clearly illustrate the individual processes composing these model generalizations. For example, experimental evidence which can be interpreted to indicate directly the sequential transfer of particles
can be a valuable test of the sequential reaction mechanism, as it is applied to two-nucleon transfer or other reactions. Based on the considerations of Section 5.3, the kind of sequential transfer reaction which is most likely to offer clear-cut evidence is the kind to which there corresponds no conventional direct population mechanism. There is at present one published example of such an analysis (Ko 74, Ud 75), for the $^{48}$Ca($^{16}\text{O},^{15}\text{C}$)$^{49}$Ti reaction. In this case the magnitudes of the theoretical cross sections are in agreement with the data. However, the evidence supplied by the experimental angular distributions may be strengthened by the study of other, better O-matched reactions. Experimental angular distributions with more clearly defined shapes would offer a stricter test of the reaction theory and could lead to a more cohesive confirmation of the sequential transfer model (Ei 75).

7.2 Choice of Analysis

Analyses of reactions such as $^{48}$Ca($^{16}\text{O},^{15}\text{C}$)$^{49}$Ti involve the transfer of at least three particles, or the transfer of one particle together with a charge exchange process. These reactions are, of course, ultimately more complicated than two-nucleon transfer reactions. Thus, it is important to enunciate clearly how the desired information can be extracted from experimental data with-
out directly assaulting the complexities already inherent in two-nucleon transfer reactions.

Feng's recent analyses (Fe 76) provide excellent examples of just how complicated the two-nucleon transfer problem has become and how similar the distributions of the competing reaction processes can be. It would be frivolous (if at all possible with present computation facilities) to incorporate such intricate detail in the analyses of higher order reaction processes when such analyses are intended to illuminate only one process, the sequential transfer of particles. Instead, it is recommended that a more empirical approach, similar to that employed in (Ud 75), be adopted. This more economical approach circumvents possible normalization problems not associated with the sequential transfer mechanism, and consequently provides a more direct test of that process.

Specifically, this empirical approach involves the normalization of the calculations for the intermediate transfer steps to the experimental differential cross sections. For example, the reaction $^{48}\text{Ca}(^{16}\text{O}, ^{15}\text{C})^{49}\text{Ti}$ can proceed via the following second and third order transfer mechanisms.

$^{48}\text{Ca}(^{16}\text{O}, ^{17}\text{O})^{46}\text{Ca}(^{17}\text{O}, ^{15}\text{C})^{49}\text{Ti},$

$^{48}\text{Ca}(^{16}\text{O}, ^{17}\text{O})^{47}\text{Ca}(^{17}\text{O}, ^{16}\text{N})^{48}\text{Sc}(^{16}\text{N}, ^{15}\text{C})^{49}\text{Ti},$

$^{48}\text{Ca}(^{16}\text{O}, ^{14}\text{C})^{50}\text{Ti}(^{14}\text{C}, ^{15}\text{C})^{49}\text{Ti},$

$^{48}\text{Ca}(^{16}\text{O}, ^{15}\text{N})^{49}\text{Sc}(^{15}\text{N}, ^{14}\text{C})^{50}\text{Ti}(^{14}\text{C}, ^{15}\text{C})^{49}\text{Ti}.$
The third order processes are not calculated explicitly. Their effects are included implicitly by the renormalization of the calculations for the remaining second order transfer mechanisms. The $^{48}\text{Ca}(^{16}\text{O},^{17}\text{O})^{47}\text{Ca}$ step and the $^{50}\text{Ti}(^{14}\text{C},^{15}\text{C})^{49}\text{Ti}$ step are each renormalized by the factor necessary to bring the calculated and experimental angular distributions for the $^{48}\text{Ca}(^{16}\text{O},^{17}\text{O})^{47}\text{Ca}$ reaction into agreement. (No data exist for the $^{50}\text{Ti}(^{14}\text{C},^{15}\text{C})^{49}\text{Ti}$ reaction.) Likewise, the calculations for the $^{47}\text{Ca}(^{17}\text{O},^{15}\text{C})^{49}\text{Ti}$ step and the $^{48}\text{Ca}(^{16}\text{O},^{14}\text{C})^{50}\text{Ti}$ step are each renormalized by the factor resulting from the $^{48}\text{Ca}(^{16}\text{O},^{14}\text{C})^{50}\text{Ti}$ data. The transition amplitudes for the two sequential transfer processes are then added coherently to yield the predicted differential cross section.

In such an analysis there is no advantage in employing a microscopic model to describe the two-nucleon transfer steps: The corresponding sequential transfer of two nucleons is expected to contribute significantly. Under these circumstances it is more economical to employ the cluster model; it generally yields angular distribution shapes very similar to those of the microscopic and sequential mechanisms, and its magnitude is empirically set anyway.

Of course there are situations in which the cluster model may be insufficient for an adequate description of the transfer processes. For example, it has been suggested...
by Ascuitto, Körner (Ko 75), and others that calculated and measured ratios of cross sections be compared to determine the effects of the different competing transfer mechanisms. The results of this sort of an analysis are very sensitive to the model employed, and consequently a detailed microscopic model is imperative. (Tung (Tu 77) has reported a study of the effects of simultaneous and sequential transfer processes on two-nucleon transfer cross section ratios using the computer codes developed during the course of this thesis work.) Nevertheless, the simple cluster model is expected to be sufficient for the analysis described above.

As far as the $^{48}\text{Ca}(^{16}\text{O},^{15}\text{C})^{49}\text{Ti}$ reaction is concerned, it should be remarked that there is one third order transfer mechanism which cannot be incorporated into the renormalization of the second order processes. This transfer route is given by $^{48}\text{Ca}(^{16}\text{O},^{15}\text{N})^{49}\text{Sc}(^{15}\text{N},^{16}\text{N})^{48}\text{Sc}(^{16}\text{N},^{15}\text{C})^{49}\text{Ti}$. It is also conceivable that the final states may be populated by processes consisting of a charge exchange mechanism together with a proton transfer. E.g.

$^{48}\text{Ca}(^{16}\text{O},^{16}\text{N})^{48}\text{Sc}(^{16}\text{N},^{15}\text{C})^{49}\text{Ti},$

$^{48}\text{C}(^{16}\text{O},^{15}\text{N})^{49}\text{Sc}(^{15}\text{N},^{15}\text{C})^{49}\text{Ti}.$

However, these processes are apparently inhibited by the fact that the usual isospin flip process cannot occur in these reactions without a neutron being promoted above the closed lp shell to form either the $^{16}\text{N}$ or $^{15}\text{C}$ system.
These processes are not included in the analysis of (Ud 75) and presumably do not significantly affect the theoretical and experimental agreement.

7.3 Choice of Reactions

One of the theoretical considerations which must be addressed for the assessment of possible experiments directed toward the observation of sequential processes is that the constituent transfers composing the "sequential" steps be well matched in energy and angular momentum transfer. It is desirable that the actual and optimum Q-values not differ significantly in order that each of the transfer steps be well defined theoretically. Moreover, it is important that each of the transfer Q-values has a relatively small magnitude, so that the sequential reaction products yield cross sections large enough to be experimentally observable. It is also desirable that the nuclei to be studies consist of closed or nearly closed shells in order that the reaction dynamics can be described simply, with as little recourse to spectator particles or core excitations as possible. Reactions satisfying these criteria and involving stable projectiles and targets are very rare.

In an effort to find such reactions, the Wright Nuclear Structure Laboratory IBM360/44 computer system was used to search the Oak Ridge mass table (Oa 72) for
favorable Q-values (|Q| \leq 2 \text{ MeV}) in the vicinity of closed shells. This initial attempt was unsuccessful in finding a nuclear system worthy of serious experimental study. However, a second search, including nuclear systems near closed subshells, resulted in finding a phenomenally well matched system.

The suggested experiment (Sh 77b) consists of reactions induced by $^{28}\text{Si}$ on $^{48}\text{Ca}$. Table (7-1) lists the possible reaction products resulting from transfers of \leq 4 mass units and \leq 2 charge units. Only the partitions for which |Q| < 5 MeV are contained in the table. Perhaps the most striking aspect of this table is that there are over a dozen different well matched rearrangements of particles which can result from the initial system. Because of this fact, there are a large number of sequential transfer processes which may play a significant role in the population of these reaction channels.

Before an examination in the following sections of the apparently clearest examples of sequential transfer processes in this system, it is worthwhile to enumerate the other sequential processes which may be expected to occur, but which would require more complex analyses.

First of all the two-proton stripping reaction $^{48}\text{Ca}(^{28}\text{Si},^{26}\text{Mg})^{50}\text{Ti}$ is expected to have significant contributions from the second order process $^{48}\text{Ca}(^{28}\text{Si},^{27}\text{Al})^{49}\text{Sc}(^{27}\text{Al},^{26}\text{Mg})^{50}\text{Ti}$. Likewise the two-neutron pick-up
Table (7-1)

Possible Reaction Products from $^{28}$Si+$^{48}$Ca at 100 MeV

<table>
<thead>
<tr>
<th></th>
<th>$\Omega$ (MeV)</th>
<th>$\Omega_{opt}$ (MeV)</th>
<th>$J_{opt}$ (\AA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{24}$Mg$^{52}$Ti</td>
<td>-2.31</td>
<td>-3.61</td>
<td>2</td>
</tr>
<tr>
<td>$^{25}$Mg$^{51}$Ti</td>
<td>-2.79</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$^{26}$Mg$^{50}$Ti</td>
<td>1.93</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>$^{27}$Mg$^{49}$Ti</td>
<td>-2.57</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$^{28}$Mg$^{48}$Ti</td>
<td>-2.21</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$^{27}$Al$^{49}$Sc</td>
<td>-1.96</td>
<td>-1.58</td>
<td>1</td>
</tr>
<tr>
<td>$^{28}$Al$^{48}$Sc</td>
<td>-4.37</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$^{29}$Al$^{47}$Sc</td>
<td>-3.17</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$^{29}$Si$^{47}$Ca</td>
<td>-1.47</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$^{30}$Si$^{46}$Ca</td>
<td>1.86</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$^{31}$Si$^{45}$Ca</td>
<td>-1.96</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$^{32}$Si$^{44}$Ca</td>
<td>-0.16</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$^{31}$P$^{45}$K</td>
<td>-4.66</td>
<td>1.13</td>
<td>5</td>
</tr>
</tbody>
</table>
reaction (and the three- and four-neutron reactions) can have contributions from higher order transfers. Analyses of such reactions may warrant the explicit inclusion of inelastic excitation effects in the P- and T-systems (La 74, So 74).

Low has expressed interest in the possibility that $^3$He stripping reactions may have important contributions from second order processes consisting of $^4$He stripping and neutron pick-up. (See Section 5.2 of (Lo 76), (Co 74 and Ma 74a).) The sequential routes composing the $^{48}\text{Ca}(^{28}\text{Si}, ^{25}\text{Mg})^{51}\text{Ti}$ reaction are very well matched and could provide useful tests for such hypotheses. The $^3$H pick-up reaction $^{48}\text{Ca}(^{28}\text{Si}, ^{31}\text{P})^{45}\text{K}$ is not quite so well matched, but it also can occur via sequential transfer processes, among which is the second order mechanism $^{48}\text{Ca}(^{28}\text{Si}, ^{27}\text{Al})^{49}\text{Sc}(^{27}\text{Al}, ^{31}\text{P})^{45}\text{K}$ (proton stripping followed by $^4$He pick-up).

7.4 The Mass Exchange Type Reactions

7.4.1 The $^{48}\text{Ca}(^{28}\text{Si}, ^{27}\text{Mg})^{49}\text{Ti}$ Reaction

The $^{48}\text{Ca}(^{28}\text{Si}, ^{27}\text{Mg})^{49}\text{Ti}$ reaction involves the stripping of two units of charge and one unit of mass, and therefore cannot proceed as the direct transfer of a known nuclear component. This reaction is analogous to the $^{48}\text{Ca}(^{16}\text{O}, ^{15}\text{C})^{49}\text{Ti}$ reaction of (Ko 74 and Ud 75). However the $^{28}\text{Si}$ projectile
leads to generally smaller Q-values for the constituent transfers of the sequential processes. I.e. the values are respectively -1.47 MeV and -1.09 MeV for the first and second transfers of $^{48}\text{Ca}(^{28}\text{Si}, ^{29}\text{Si})^{47}\text{Ca}(^{29}\text{Si}, ^{27}\text{Mg})^{49}\text{Ti}$, and 1.93 MeV and -4.50 MeV for those of $^{48}\text{Ca}(^{28}\text{Si}, ^{26}\text{Mg})^{50}\text{Ti}(^{26}\text{Mg}, ^{27}\text{Mg})^{49}\text{Ti}$.

A model calculation for the first of these sequential transfer mechanisms has been done for a beam energy of 100 MeV. A characteristic angular distribution is shown in figure (7-1). The broad, forward-peaked shape of the distribution is consistent with the general properties discussed in Section 5.3. In the calculation, the prior-post interaction form is used to eliminate the nonorthogonality terms. The second transfer step, $^{47}\text{Ca}(^{29}\text{Si}, ^{27}\text{Mg})^{49}\text{Ti}$, employs a microscopic description of the two-proton stripping, although the cluster model is expected to result in a very similar angular distribution and be entirely adequate for the suggested analysis.

The optical potential parameters for this calculation (table (7-2)) are extracted from elastic scattering data of $^{28}\text{Si}$ on $^{40}\text{Ca}$ (Wh 77). These data are slightly better fit with a very shallow imaginary potential ($W \approx -5$ MeV), but the more conventional parameters shown are employed in the $^{28}\text{Si} + ^{48}\text{Ca}$ calculation.
Figure (7-1). An angular distribution for the sequential transfer process $^{48}\text{Ca}(^{28}\text{Si},^{29}\text{Si})^{47}\text{Ca}(^{29}\text{Si},^{27}\text{Mq})^{49}\text{Ti}$. The prior-post interaction form is used to eliminate the non-orthogonality terms. Recoil is treated exactly, and the microscopic description of the two-proton transfer contains all terms with up to three units of relative angular momentum. $E = 100$ MeV.
Fig. (7-1)

\[ \frac{d\sigma}{d\Omega} \text{(arb. units)} \]

\[ ^{48}\text{Ca}(^{28}\text{Si},^{29}\text{Si})^{47}\text{Ca}(^{29}\text{Si},^{27}\text{Mg})^{49}\text{Ti} \]

\[ E = 100 \text{ MeV} \]

\[ \theta_{\text{c.m.}} \]
Table (7-2)

Potential Parameters for $^{40}\text{Ca}(^{28}\text{Si},^{28}\text{Si})^{40}\text{Ca}$ at 87.6 MeV (Wh 77)

$V = -98.4$ MeV

$W = -25.1$ MeV

$r_{VO} = 1.36$ fm

$r_{WO} = 1.33$ fm

$a_V = 0.296$ fm

$a_w = 0.195$ fm
7.4.2 The $^{48}\text{Ca}(^{28}\text{Si}, ^{29}\text{Al})^{47}\text{Sc}$ Reaction

The $^{48}\text{Ca}(^{28}\text{Si}, ^{29}\text{Al})^{47}\text{Sc}$ reaction requires the stripping of one unit of charge and the pick-up of one unit of mass. This reaction cannot be described by a conventional first order mechanism, but it can occur via two well matched second order transfer mechanisms:

\[ ^{48}\text{Ca}(^{28}\text{Si}, ^{27}\text{Al})^{49}\text{Sc}(^{27}\text{Al}, ^{29}\text{Al})^{47}\text{Sc} \]
\[ (Q = -1.96 \text{ MeV and } -1.20 \text{ MeV respectively}) \]

and

\[ ^{48}\text{Ca}(^{28}\text{Si}, ^{30}\text{Si})^{46}\text{Ca}(^{30}\text{Si}, ^{29}\text{Al})^{47}\text{Sc} \]
\[ (Q = 1.86 \text{ MeV and } -5.03 \text{ MeV respectively}) \]

These reaction processes are expected to have angular distribution shapes similar to that of figure (7-1). It is suggested that calculations for these second order transfers be empirically normalized also, in order to include some of the higher order effects.

7.5 The Charge Exchange Type Reactions

7.5.1 The $^{48}\text{Ca}(^{28}\text{Si}, ^{28}\text{Al})^{48}\text{Sc}$ Reaction

In Section 7.2 it is mentioned that the charge exchange process $^{48}\text{Ca}(^{16}\text{O}, ^{16}\text{N})^{48}\text{Sc}$ cannot proceed via the simple
isospin flip mechanism because the additional neutron in $^{16}_N$ cannot be placed in the closed lp shell. A similar situation exists for the $^{28}_Si$ projectile because of its zero isospin and complete ld$_{5/2}$ subshells. Consequently the direct charge exchange process in $^{48}_Ca(^{28}_Si, ^{28}_Al)^{48}_Sc$ may be somewhat inhibited. In any case, the competing second order transfer reactions are well matched and expected to populate the $^{28}_Al$ partition:

$$^{48}_Ca(^{28}_Si, ^{29}_Si)^{47}_Ca(^{29}_Si, ^{28}_Al)^{48}_Sc$$

($Q = -1.47$ MeV and $-2.89$ MeV respectively)

and

$$^{48}_Ca(^{28}_Si, ^{27}_Al)^{49}_Sc(^{27}_Al, ^{28}_Al)^{48}_Sc$$

($Q = -1.96$ MeV and $-2.40$ MeV respectively).

See references (Ku 74, Os 76, and Lo 76).

7.5.2 The $^{48}_Ca(^{28}_Si, ^{28}_Mg)^{48}_Ti$ Reaction

In the $^{48}_Ca(^{28}_Si, ^{28}_Mg)^{48}_Ti$ reaction, two units of charge are stripped. A charge exchange component of this reaction would involve the isospin flip (and promotion) of two particles. The reaction can also proceed via several multistep transfer reactions. Four of the second order reactions are
\[ ^{48}\text{Ca}(^{28}\text{Si},^{30}\text{Si}) ^{46}\text{Ca}(^{30}\text{Si},^{28}\text{Mg}) ^{48}\text{Ti} \]

\( Q = 1.86 \text{ MeV and } -40.7 \text{ MeV respectively}, \)

\[ ^{48}\text{Ca}(^{28}\text{Si},^{26}\text{Mg}) ^{50}\text{Ti}(^{26}\text{Mg},^{28}\text{Mg}) ^{48}\text{Ti} \]

\( Q = 1.93 \text{ MeV and } -4.14 \text{ MeV respectively}, \)

\[ ^{48}\text{Ca}(^{28}\text{Si},^{32}\text{Si}) ^{44}\text{Ca}(^{32}\text{Si},^{28}\text{Mg}) ^{48}\text{Ti}, \]

\( Q = -0.16 \text{ MeV and } -2.05 \text{ MeV respectively}, \)

and

\[ ^{48}\text{Ca}(^{28}\text{Si},^{24}\text{Mg}) ^{52}\text{Ti}(^{24}\text{Mg},^{28}\text{Mg}) ^{48}\text{Ti} \]

\( Q = -2.31 \text{ MeV and } 0.11 \text{ MeV respectively}. \)

The first two mechanisms involve the transfer of four particles, two neutrons and two protons. The second two mechanisms involve the transfer of eight particles, but since four of them are contained in a \(^4\text{He}\) stripping process, which may be rather strong, these mechanisms may possibly compete with the first two. The particularly well matched neutron pick-up processes may provide insight into the redistribution of the \(^{48}\text{Ca}\) neutron excess.
CHAPTER 8

CONCLUSION

The two examples of heavy-ion induced transfer reactions discussed in this thesis are chosen to illuminate higher order processes which may significantly affect the observation of differential cross sections. In each case the applicable nuclear model is chosen to illustrate simply the relevant reaction dynamics. Under these circumstances, characteristic angular distributions can provide unambiguous interpretations of the reaction mechanisms, which then can provide useful information about the nuclear states involved.

While the overall perspective is one of simplicity of interpretation, this work has necessitated extensive development of computer codes to describe adequately the first and second order transfer mechanisms. The inclusion of the recoil effects, which is imperative for these examples, does not obscure the simple concepts associated with the angular distribution features. However, the full-recoil analyses do tend to be more formidable numerical problems.

The analyses of the differential cross sections for the \(^{48}\text{Ti}(^{16}\text{O},^{15}\text{N})^{49}\text{V}\) reaction at \(E = 50\) MeV are described in Chapter 6. Here the application of a relatively simple structural model leads to a good fit of the experimental data and the conclusion that multistep inelastic
processes are important and evident in this transfer reaction. The selection of this particular reaction for study is based on the experimental history of its concomitant anomalous angular distributions (Ma 73, Er 76), in addition to theoretical considerations.

It is hoped that a similar rationale will be useful in the study of sequential transfer mechanisms. In particular, those processes which cannot occur by the direct, one-step transfer of a nuclear constituent may offer a simple confirmation of the description of higher order mechanisms, if the individual transfer steps can be empirically normalized. Well defined experimental angular distributions would be very useful in this regard.

The choice of reactions for such a study (those induced by $^{28}\text{Si}$ on $^{48}\text{Ca}$) is based on a wide ranging search for favorable transfer $Q$-values. The energy and angular momentum matching criteria are expected to be particularly important for the experimental observation of the appropriate differential cross sections.

The suggested analyses complement the two-nucleon transfer reactions, in which first and second order transfer mechanisms can compete. In the suggested reactions, the testing of the second order transfer processes, separate from any conventional first order process, could provide valuable insight into the description of second order transfers in two-nucleon or other transfer reactions.
APPENDIX A
PARTITION OF THE HAMILTONIAN

A.1 Potential Energy Operator

The Hamiltonians can be divided into their kinetic and potential energy operators.

\[ H(\text{A}) = T(\text{A}) + V(\text{A}) \]
\[ H(\text{T}) = T(\text{T}) + V(\text{T}) \]
\[ H(\text{C}) = T(\text{C}) + V(\text{C}) \]  \hspace{1cm} (A-1)

The interaction between the A-system and the T-system is defined by

\[ V_{\text{AT}}(\text{C}) = V(\text{C}) - V(\text{A}) - V(\text{T}) \]

\[ V(\text{A}) \text{ and } V(\text{T}) \text{ include all interactions internal to systems A and T respectively. Thus,} \]

\[ V(\text{C}) = V_{\text{AT}}(\text{C}) + V(\text{A}) + V(\text{T}) \]  \hspace{1cm} (A-2)

A.2 Kinetic Energy Operator

The mass of the A-system is the sum of the masses of the constituent particles.

\[ m_A = \sum_{k=1}^{A} m_k \]

Similarly,
\[ C \]
\[ m_T = \sum_{K=A+1}^{C} m_K \]

and

\[ m_C = m_A + m_T. \]

The choice of the C-system space coordinate origin as the center of mass can be expressed by the constraint

\[ m_C^{-1} \sum_{K=1}^{C} m_K \dot{r}_{OK} = 0 \quad \text{(A-3)} \]

The A- and T-system coordinates \( \dot{r}_K \) are related to the C-system coordinates \( \dot{r}_{OK} \) by the definitions of the centers of mass of the A- and T-systems in the C-system coordinates.

\[ \dot{r}_K = \begin{cases} \dot{r}_{OK} - \dot{r}_{OA}, & K = 1, \ldots, A, \\ \dot{r}_{OK} - \dot{r}_{OT}, & K = A+1, \ldots, C, \end{cases} \]

\[ \dot{r}_{OA} = m_A^{-1} \sum_{\ell=1}^{A} m_{\ell} \dot{r}_{O\ell}, \]

\[ \dot{r}_{OT} = m_T^{-1} \sum_{\ell=A+1}^{C} m_{\ell} \dot{r}_{O\ell}. \]

Equation (A-3) is used to eliminate \( \dot{r}_{OA} \).

\[ \dot{r}_K = \begin{cases} \dot{r}_{OK} + (m_T/m_A) \dot{r}_{OT}, & K = 1, \ldots, A, \\ \dot{r}_{OK} - \dot{r}_{OT}, & K = A+1, \ldots, C. \end{cases} \quad \text{(A-4)} \]

To partition the kinetic energy operator, it is necessary to relate the time derivatives of the coordinate operators to the conjugate momentum operators. The equation of motion
\[ \dot{r}_{OK} = -(i/\hbar) \left[ \hat{r}_{OK}, H(\zeta) \right] \]

together with the assumption that

\[ [\hat{r}_{OK}, \mathcal{V}(\zeta)] = 0 \]

leads to

\[ \dot{r}_{OK} = -(i/\hbar) \sum_{\ell=1}^{C} \left[ \hat{r}_{OK}, \hat{p}_{0\ell}^2/(2m_{\ell}) \right], \quad (A-5) \]

in which the explicit form of the kinetic energy operator is used:

\[ T(\zeta) = \sum_{\ell=1}^{C} \hat{p}_{0\ell}^2/(2m_{\ell}). \quad (A-6) \]

Since

\[ [r_{OK\alpha}, \hat{p}_{0\ell\beta}] = i\hbar \delta_{KL} \delta_{\alpha\beta}, \]

with \( \alpha, \beta = 1, 2, 3 \) referring to Cartesian components, equation (A-5) reduces to

\[ \dot{r}_{OK} = \hat{p}_{OK}/m_K. \quad (A-7) \]

The commutators with the isolated Hamiltonians

\[ [\hat{r}_K, H(\zeta)] \text{ for } K = 1, \ldots, A \text{ and } [\hat{r}_K, H(\tau)] \text{ for } K = A+1, \ldots, C \]

give the similar connection

\[ \dot{r}_K = \hat{p}_K/m_K. \]

Equations (A-6) and (A-7) imply

\[ T(\zeta) = \sum_{K=1}^{C} \frac{m_K^2}{m_K} \frac{\hat{r}_{OK}^2}{(2m_K)} , \]
by the transformation (A-4). The dot products

\[ - \sum_{K=1}^{A} \left( m_K T/m_A \right) \mathbf{r}_K \cdot \mathbf{r}_{OT} = \sum_{K=A+1}^{C} m_K \mathbf{r}_K \cdot \mathbf{r}_{OT} = 0 \]

so that

\[ T(\mathbf{C}) = \sum_{K=1}^{C} \frac{p_K^2}{2m_K} + \frac{1}{2} m_T (m_T/m_A + 1) \dot{r}_{OT}^2. \]

Now with the definition of the relative coordinate

\[ \dot{r}_T = \dot{r}_{OT} - \dot{r}_{OA} = (1 + m_T/m_A) \dot{r}_{OT}, \]

the kinetic energy operator becomes

\[ T(\mathbf{C}) = T(\mathbf{A}) + T(\mathbf{T}) + \frac{1}{2} (m_A m_T/m_C) \dot{r}_T^2. \]

The momentum conjugate to the relative coordinate is found by differentiation of the Lagrangian.

\[ p_{Ta} = \frac{\partial}{\partial \dot{r}_{Ta}} L(\mathbf{C}) = (m_A m_T/m_C) \dot{r}_T. \]

Finally

\[ T(\mathbf{C}) = T(\mathbf{A}) + T(\mathbf{T}) + T(\dot{r}_T), \quad (A-8) \]

with

\[ T(\dot{r}_T) = p_T^2/(2m_A m_T/m_C) \]
Equation (A-1), (A-2), and (A-8) are now combined for the result

\[ H(C) = H(A) + H(T) + T(r_\eta) + V_{AT}(C). \]
APPENDIX B

PARTITION OF THE WAVEFUNCTION

The antisymmetrization of the total wavefunction is explicitly expressed through the use of a Slater determinant. For elastic scattering,

$$\psi_C(C) = (C!)^{-\frac{1}{2}} \det[CC],$$

$$[CC] = \begin{bmatrix}
AA & AT \\
TT & TA \\
\end{bmatrix}$$

The partitioned matrix $CC$ is composed of the four matrices

$$[AA] = \begin{bmatrix}
\psi_1(1) & \psi_1(2) & \ldots & \psi_1(A) \\
\psi_2(1) & \psi_2(2) & \ldots & \psi_2(A) \\
\vdots & \vdots & \ddots & \vdots \\
\psi_A(1) & \psi_A(2) & \ldots & \psi_A(A) \\
\end{bmatrix},$$

$$[AT] = \begin{bmatrix}
\psi_1(A+1) & \ldots & \psi_1(C) \\
\vdots & \ddots & \vdots \\
\psi_A(A+1) & \ldots & \psi_A(C) \\
\end{bmatrix},$$

$$[TA] = \begin{bmatrix}
\psi_{A+1}(1) & \ldots & \psi_{A+1}(A) \\
\vdots & \ddots & \vdots \\
\psi_C(1) & \ldots & \psi_C(A) \\
\end{bmatrix},$$
The matrix element \( \psi_j(K) \) is the single particle wavefunction for particle \( K \) in quantum state \( j \).

The \( A- \) and \( T- \) partition wavefunctions are also expressible in terms of determinants.

\[
\psi_A(A_0) = (A!)^{-\frac{1}{2}} \det[A],
\]

\[
\psi_T(T_0) = (T!)^{-\frac{1}{2}} \det[TT].
\]

The subscript \( 0 \) indicates that the space coordinates are with respect to the \( C \)-system origin.

Since \( [CC] \) is partitioned, the relation

\[
\det[CC] = \det[AA] \det[TT] - [TA][AA]^{-1}[AT]T
\]

is used to obtain

\[
\psi_C(C) = (C_T)^{-\frac{1}{2}} \det[1-[TA][AA]^{-1}[AT][TT]^{-1}] \psi_A(A_0) \psi_T(T_0).
\]

This determinant permutes particles between the \( A- \) and \( T- \) partitions. Thus the total wavefunction can be written

\[\text{(Cu 66).}\]
in terms of a permutation operator which interchanges $\nu$ particle pairs between the partitions.

$$\psi_C(C) = (C_T)^{-l} \sum_{v} (-1)^v P_v \psi_A(A_O) \psi_T(T_O).$$

Now the coordinate transformation of Appendix A yields the separation of the relative coordinates.

$$\psi_C(C) = (C_T)^{-l} \sum_{v} (-1)^v P_v \psi_A(A) \psi_T(T) \phi_A(T_T).$$
APPENDIX C
REDUCED COUPLING ELEMENTS

C.1 Even-even Nuclei

The evaluation of the reduced matrix element

\[ \langle \psi_A (\vec{\alpha}) | \alpha_\lambda^* (\vec{\alpha}) | \psi_{A'} (\vec{\alpha}) \rangle \]

is conveniently done with the operator formalism of second quantization. The usual (not reduced) matrix element is

\[ \langle A, M | A^\prime, M^\prime \rangle = \int d\vec{\alpha} \langle \psi_A (\vec{\alpha}) | \alpha_\lambda^* (\vec{\alpha}) \rangle \psi_{A'} (\vec{\alpha}) \]

which becomes

\[ \int d\vec{\alpha} \langle A, M | A^\prime, M^\prime \rangle \]

with the wavefunctions for the A and A' states in the coordinate representation. The relationship between the function of the A-system coordinates \( \alpha_\lambda^* (\vec{\alpha}) \) and the more general second quantization operator \( \alpha_\lambda^* \) is given by

\[ \alpha_\lambda^* | A \rangle = \alpha_\lambda^* (\vec{\alpha}) | A \rangle . \]

Thus
Since the unreduced matrix elements are equal, the reduced matrix elements, defined by the Wigner-Eckart theorem, are equal also.

\[
\langle \psi_{A', M_A} | \alpha_{\lambda}^* | \psi_A, M_A \rangle = \int d\lambda \langle A' | J_{A'} | A \rangle \langle A | \alpha_{\lambda}^* | \psi_A, M_A \rangle,
\]

\[
= \langle A' | J_{A'} | A \rangle \langle A | \alpha_{\lambda}^* | \psi_A, M_A \rangle.
\]

The relevant matrix elements for the weak-coupling model are those in which \( A' \) represents the ground state of the A-nucleus, and \( A \) represents a low-lying state related to \( A' \) through the excitation of a core phonon. (Because the states are weakly coupled, it is unnecessary to calculate higher order deexcitation elements.) For the case in which the A-nucleus is an even-even nucleus, the \( A' \) state is a phonon vacuum state with zero spin. The \( A \) state is a one-phonon state, which must have spin \( \lambda \) for the coupling element to be nonzero.

The generalized coordinate operator \( \alpha_{\lambda}^u \) is expressed in terms of creation and annihilation operators for phonon excitations.
\[ \tilde{\alpha}_\lambda^\mu = \beta_\lambda \hat{\Lambda}^{-\frac{1}{2}} (\hat{b}_\lambda^\mu + \hat{b}_\lambda^{\mu+}) \]  

(C-2)

\( \beta_\lambda \) is the root-mean-square deformation of the A-nucleus in its ground state.

\[ \beta_\lambda = \langle \omega | (\hat{\alpha}_\lambda \cdot \hat{\alpha}_\lambda^* ) | \omega \rangle^{\frac{1}{2}} \]

in which the ground state of the A-system

\[ |A'\rangle = |A_0\rangle = |\omega\rangle \]

is explicitly a phonon Vacuum state. \( \hat{b}_\lambda^{\mu+} \) and \( \hat{b}_\lambda^\mu \) are respectively phonon creation and destruction operators, while the operator

\[ \hat{\tilde{b}}\lambda^\mu + = (-1)^{\mu\nu-\mu+} \]

creates a time reversed phonon. The transformation (C-2) is analogous to that which factors the Hamiltonian in the operator solution of the harmonic oscillator.

With \( |A'\rangle = |\omega\rangle \), and \( |A\rangle = \hat{\tilde{b}}\lambda^\mu |\omega\rangle \), the required matrix element becomes

\[ \langle A | \tilde{\alpha}_\lambda^\mu | A' \rangle = \langle \omega | \hat{b}_\lambda^{\mu+} \beta_\lambda \hat{\Lambda}^{-\frac{1}{2}} (\hat{b}_\lambda^{\mu+} + \hat{b}_\lambda^\mu) | \omega \rangle. \]

Since \( \hat{b}_\lambda^\mu |\omega\rangle = 0 \), and

\[ \hat{b}_\lambda^\mu \hat{b}_\lambda^\mu' + \hat{b}_\lambda^{\mu+} \hat{b}_\lambda^{\mu+} = \delta_\lambda \lambda' \delta_\mu \mu' \]

the matrix element is simply
\[
\beta_\lambda \lambda^{-\frac{1}{2}} \delta_{J,0} \delta_{M,0},
\]

which is also equal to

\[
(-1)^{J'} M J' \lambda' 0 \langle A | | \alpha^*_\lambda | | A' \rangle = \]

\[
(-1)^{2\lambda+1} \delta_{J,0} \delta_{M,0} \langle A | | \alpha^*_\lambda | | A' \rangle
\]

Therefore, with (C-1), the reduced matrix element\(^1\) has the value

\[
\langle \psi_A \alpha_\lambda \alpha_\lambda \rangle = (-1)^{2\lambda} \beta_\lambda = \beta_\lambda.
\]

---

\(^1\)Sometimes the coupling elements \(\nu_{TT}(r_T)\) are written in terms of the reduced matrix elements \(\langle A'| | \alpha^*_\lambda | | A \rangle\), which are simply related to the present ones by a phase.

\[
\langle A'| | \alpha^*_\lambda | | A \rangle = (-1)^{J_A-J_A'} \delta_{J,0} \delta_{M,0} \langle A | | \alpha^*_\lambda | | A' \rangle.
\]
C.2 Odd Nuclei (Weak Coupling)

For odd nuclei, the ground state is assumed to be a single-particle state.

\[ |A'> = \hat{\varphi}_{A'} J_{A'} |\omega> \]

The operator \( \hat{\varphi}_{A'} J_{A'} \) creates a particle in the \( A' \) state.

The excited state consists of a phonon excitation of the ground state defined by

\[ |A> = [\hat{\varphi}_{A'}^+ \otimes \hat{\varphi}_{\lambda}^+] M_A |\omega> \]

Thus the matrix element is

\[ <A|\hat{\alpha}_{\lambda}^*|A'> = \omega|\left[ \hat{\alpha}_{A'}^+ \otimes \hat{\varphi}_{\lambda}^+ M_{A'}^+ \hat{\varphi}_{\mu}^* \hat{\varphi}_{A''}^+ \right]|\omega> \]

\[ = \omega|\left[ \hat{\alpha}_{A'}^+ \otimes \hat{\varphi}_{\lambda}^+ M_{A'}^+ \right] b_{\lambda}^\dagger (b_{\lambda}^\dagger + b_{\lambda}) \hat{\varphi}_{A''}^+ |\omega> . \]

Because of the fact that the particle and phonon operators commute, and the particle operators have the anticommutator

\[ \hat{\varphi}_{A''}^+ \hat{\varphi}_{A'}^+ + \hat{\varphi}_{A'}^+ \hat{\varphi}_{A''}^+ = \delta_{A'A''} \delta_{M_A,M_A''}, \]

the matrix element is simply

\[ <A|\hat{\alpha}_{\lambda}^*|A'> = \beta_{\lambda} \lambda^{-\frac{1}{2}} C_{M_A} \lambda M_{A''}, \]

\[ \cdot \]
which is also equal to

\[ (-1)^{J_A - M_A} \frac{J_A}{M_A} | J_{A'} \rangle \langle \alpha^*_\lambda | | A' \rangle = \]

\[ = (-1)^{2J_A} \frac{J_A}{J_{A'}} \frac{J_A}{J_{A'}} \frac{J_A}{J_{A'}} \frac{J_A}{J_{A'}} C_{M_A}^{\lambda \mu} M_A \langle A | \alpha^*_\lambda | A' \rangle. \]

Finally, (C-1) implies

\[ \langle \psi_{\lambda} (A) | \alpha^*_\lambda (A) | \psi_{\lambda} (A) \rangle = \left( \frac{J_A}{\lambda} \right)^{\frac{\lambda}{2}} \beta_\lambda. \]  

\[ \text{(C-3)} \]

The expression (C-3) is also valid for the one-phonon excitation of an even-even nucleus.  \( (J_A = \lambda) \).
APPENDIX D
DEFORMATION CONVENTIONS

D.1 Nuclear Deformations

The expression for the radius of the deformed potential given in Section 3.3

\[ r_v = r_v(0) + R_v, \]

must in the absence of deformations go over to the expression for the spherical potential radius from Section 2.6,

\[ r_v = r_{vo}(A^{1/3} + T^{1/3}). \]

Therefore,

\[ r_v(0) = r_{vo}(A^{1/3} + T^{1/3}). \]

The radius of the diagonal part of the potential is characterized by the sum of the radii of the A- and T-nuclei.

The off-diagonal potential elements, however, reflect the collective excitation of only one of these nuclei (e.g. the A-nucleus), and consequently these matrix elements are assumed to be independent of the radius of the other nuclear system (the T-nucleus). Specifically, the strength of the coupling interaction, which is given
by the product\(^1\) \(R_v(0)\beta_\lambda\), is parameterized
\[
R_v(0)\beta_\lambda = R_{vo} A^{1/3} \beta_\lambda(v) .
\]

Thus, if the deformations associated with different projectile systems or different optical potential parameters are related by
\[
R_v(0)\beta_\lambda(v) = R_{vo} \beta_\lambda(v),
\]
then the same change in the nuclear radius, \(r_v - r_v(0)\), results for either case. This relationship is particularly useful for relating the parameters of different reactions involving the same nucleus.

For the imaginary part of the nuclear potential, the relationships
\[
r_w(0) = r_{wo}(A^{1/3} + \omega^{1/3}),
\]
\[
r_w(0)\beta_\lambda = R_{wo} A^{1/3} \beta_\lambda(w)
\]
hold. It is conventional to take
\[
R_{vo} = R_{wo} = 1.2 \text{ fm}, \text{ and}
\]
\[
\beta_\lambda(v) = \beta_\lambda(w) \equiv \beta_\lambda(N),
\]

\(^1\)In the inelastic coupling elements, the parameters \(\beta_\lambda, R_v(0), R_w(0), R_c(0)\), always occur as one of the products \(R_v(0)\beta_\lambda, R_w(0)\beta_\lambda, \text{ or } R_c(0)\beta_\lambda\). See also (As 72).
so that the \textbf{nuclear deformation} \( \beta_{\lambda}^{(N)} \) specifies the nuclear coupling strength.

\section*{D.2 Charge Deformations}

The Coulomb radius can be parameterized in the same manner as is the nuclear radius in the preceding section.

\[
    r_c(0) = r_{co} \left(A^{1/3} + T^{1/3}\right),
\]

\[
    R_c(0) \beta_\lambda = R_{co} A^{1/3} \beta_\lambda^{(C)}
\]

In practice the diagonal part of the interaction is very insensitive to the Coulomb radius parameter \( r_c(0) \). Consequently, no generality is lost by the simplification

\[
    r_c(0) = R_{co} A^{1/3}
\]

which allows the products \( R_c(0) \beta_\lambda \) to be replaced by \( r_c(0) \beta_\lambda^{(C)} \) in the inelastic coupling elements. It is assumed that

\[
    R_{co} = 1.2 \text{ fm},
\]

so that the \textbf{Coulomb deformation} \( \beta_\lambda^{(C)} \) determines the strength of the Coulomb excitation.

The deformation parameters \( \beta_\lambda^{(C)} \) can be extracted from the experimentally determined \textbf{reduced transition probabilities}. In this work only quadrupole deformations are considered.
That is,

$$\alpha^\mu_{\lambda}(A) = 0$$

unless $\lambda=2$, so that

$$\beta^\mu_{\lambda}(C) = 0$$

for $\lambda \neq 2$. $\beta^\mu_{\lambda}(C)$ is assumed to be the same throughout the reaction in the center-of-mass frame as in the intrinsic frame of the $A$-nucleus. The deformations are taken to be axially symmetric about the intrinsic $Z$-axis. Thus, with $\hat{A}_I$ denoting intrinsic coordinates,

$$r_c = r_c(0) + R_c(0)\sum_\mu \alpha^\mu_{\lambda}(A) Y^\mu_{2}(\hat{r}_T),$$

$$= r_c(0) + R_c(0)\alpha^O_{\lambda}(\hat{A}_I) Y^O_{2}(\hat{r}_{TI}).$$

($\hat{r}_{TI}$ is related to $\hat{r}_T$ by a rotation.) The intrinsic parameters

$$\alpha^\mu_{\lambda}(\hat{A}_I) = 0$$

for $\mu \neq 0$ because $r_c$ is symmetric about the intrinsic $Z$-axis. Now

$$\beta_2 = \langle \omega | \Sigma_\mu \alpha^{\mu*}_{\lambda} \alpha^\mu_{\lambda} | \omega \rangle^1$$

$$= (\psi_{AO}^{\lambda}(\hat{A}_I) | \Sigma_\mu \alpha^{\mu*}_{\lambda}(\hat{A}_I) \alpha^\mu_{\lambda}(\hat{A}_I) | \psi_{AO}^{\lambda}(\hat{A}_I) \rangle^1$$

$$= (\psi_{AO}^{\lambda}(\hat{A}_I) | \alpha^{O*}_{\lambda}(\hat{A}_I) \alpha^O_{\lambda}(\hat{A}_I) | \psi_{AO}^{\lambda}(\hat{A}_I) \rangle^1$$
The quantities $a^{\mu}_{\lambda} (A_{\lambda})$ are taken to be parameters independent of $A_{\lambda}$ during the reaction, so that, 

$$\beta_2 = a_2^0,$$

$$r_c(0)\beta_2^{(C)} = r_c(0)\beta_2 = r_c(0)a_2^0$$

Hence

$$r_c = r_c(0)(1+\beta_2^{(C)}Y_2^0(\hat{r}_{TI})).$$

The reduced quadrupole transition probability\(^2\) is given by

$$B(E2;J_A^+,J_A^-) = \langle \hat{J}/J_A^- \rangle |a_2^0|^2$$

$\hat{J}$ is smaller of $J_A$ and $J_A^-$. The quadrupole moment is defined

$$q_2^0 = z_A \int d\hat{r}_{TI} r_{TI}^2 Y_2^0(\hat{r}_{TI}) \rho(\hat{r}_{TI}),$$

with

$$\rho(\hat{r}_{TI}) = v_D^{-1} \theta(r_c-r_{TI}),$$

and

$$v_D = \int d\hat{r}_{TI} \theta(r_c-r_{TI}).$$

Calculation of the integrals yields

\(^2\)Sections 1.2.1 and 3.2.2 of (Ei 70) discuss the reduced transition probability and the multipole moments in terms of both single particle and collective models.
\[
q_2^0 = \frac{4Z_A r_c^5(0)}{V_D} \left( \frac{\pi}{5} \right) \left( \frac{1}{4} + \frac{8}{7} \gamma^2 + \frac{24}{7} \gamma^3 + \frac{60}{77} \gamma^4 + \frac{848}{1001} \gamma^5 \right),
\]

\[
\gamma = \left( \frac{5}{16 \pi} \right)^{1/2} \beta_2^{(C)}
\]

\[
V_D = (\frac{4\pi}{3} + \beta_2^{(C)2} + \frac{1}{21} (\frac{5}{\pi})^{1/2} \beta_2^{(C)3}) r_c^3(0).
\]

These expressions can be used to calculate \( B(E2; J_A' \to J_A) \). The \( \beta_2^{(C)} \) for which the experimentally determined \( B(E2; 0^+ \to 2^+) \) results is taken to be the correct Coulomb deformation parameter.
APPENDIX E
COORDINATES FOR TRANSFER¹

In figure (E-1) the points A and P refer to the centers of mass of the target and projectile cores respectively. Point N refers to the center of mass of the \( N_1+N_2 \) system for two-nucleon transfer. In the cases of one-nucleon or cluster transfer, the relative coordinate \( \dot{\rho} \equiv N_1N_2 = 0 \), and the point N still refers to the center of mass of the transferred particle(s). (See figures (E-2) and (E-3).) Point T is the center of mass of the \( T=P+N \) system, while point B is the center of mass of the \( B=A+N \) system. Point 0 is the center of mass of all the particles \( T+A \) or \( P+B \).

Based on these definitions, there are five relations among the masses and coordinates.

¹In this section the notation is chosen so that a single letter can designate a system of particle(s), its center of mass point, or the magnitude of its mass. The particular meaning is clear from the context.
Figure (E-1). Coordinates for two-nucleon transfer. Mass centers for various systems are labeled.
Coordinates for Two-Nucleon Transfer
Figure (E-2). Coordinates for two-nucleon transfer. Relationships among coordinates between various systems are shown.
\[ T = P + N_1 + N_2 \]

\[ B = A + N_1 + N_2 \]
Figure (E-3). Coordinates for one-nucleon or cluster transfer. Relationships among coordinates between various systems are shown.
Coordinates for One-Nucleon or Cluster Transfer

\[ T = P + N \]
\[ B = A + N \]
The T-channel coordinate, which describes the relative motion of the \( A^- \) and \( T \)-systems, is defined to be

\[
\begin{align*}
\mathbf{r}_T^+ &\equiv \mathbf{O}_T^+ - \mathbf{O}_A^- \\
\mathbf{O}_T^+ &\equiv \mathbf{O}_T^+ - \mathbf{O}_B^+ \\
\mathbf{O}_T^+ &\equiv \mathbf{O}_T^+ - \mathbf{O}_N^- \\
N_1 \mathbf{N}_1^+ + N_2 \mathbf{N}_2^+ &= 0
\end{align*}
\]

Elimination of \( \mathbf{O}_A^- \) by equation (E-1) yields

\[
\mathbf{O}_T^+ = \frac{\mathbf{A}^-}{\mathbf{A}_T+} \mathbf{r}_T^+
\]

Similarly the P-channel coordinate is

\[
\mathbf{r}_P^+ \equiv \mathbf{O}_P^+ - \mathbf{O}_B^+ \\
\mathbf{O}_P^+ \equiv \mathbf{O}_P^+ - \mathbf{O}_N^- \\
\mathbf{O}_P^+ \equiv \mathbf{O}_B^+ - \mathbf{O}_N^- \\
\]

and equation (E-2) gives

\[
\mathbf{O}_P^+ = \frac{\mathbf{B}^+}{\mathbf{B}_P+} \mathbf{r}_P^+
\]

Now with equations (E-6) and (E-7),

\[
\mathbf{r}_T^+ = \mathbf{O}_T^+ - \mathbf{O}_T^+ \\
= \frac{\mathbf{B}^+}{\mathbf{B}_P+} \mathbf{r}_P^+ - \frac{\mathbf{A}^-}{\mathbf{A}_T+} \mathbf{r}_T^+.
\]
Similarly equations (E-1) and (E-2) give

\[ \vec{B}_A = \vec{O}_A - \vec{O}_B = - \frac{T}{A} \vec{O}_T + \frac{P}{B} \vec{O}_P, \]

so that equations (E-6) and (E-7) yield

\[ \vec{B}_A = - \frac{T}{A+T} \vec{r}_T + \frac{P}{B+P} \vec{r}_P \quad \text{(E-9)} \]

With the help of equations (E-3) and (E-8), the center of mass coordinate for the T-system can now be expressed in terms of the channel coordinates.

\[ \vec{r} = \vec{T}_N - \vec{T}_P = - \frac{P+N}{N} \vec{T}_P \]

\[ = \frac{(P+N)A}{N(A+T)} \vec{r}_T - \frac{(P+N)B}{N(B+P)} \vec{r}_P \quad \text{(E-10)} \]

And finally, for the B-system center of mass coordinate, equations (E-4) and (E-9) give

\[ \vec{R} = \vec{B}_N - \vec{B}_A = \]

\[ = - \frac{A+N}{N} \vec{B}_A \]

\[ = \frac{(A+N)T}{N(A+T)} \vec{r}_T - \frac{(A+N)P}{N(B+P)} \vec{r}_P. \quad \text{(E-11)} \]

Nonrelativistically, the four denominators in (E-10) and (E-11) are the same. The equations are rewritten
\[
\dot{r} = s\dot{r}_T - u\dot{r}_p, \\
\dot{R} = u\dot{r}_T - t\dot{r}_p,
\]
\[s = AT/(NC),\]
\[t = BP/(NC),\]
\[u = BT/(NC).\]

For one-nucleon or cluster transfer, equations (E-12) give the transformation between particle coordinates and channel coordinates. For the transfer of two identical particles, the usual transformation to center of mass and relative coordinates is also needed. The particle coordinates \(r_1, r_2, R_1, R_2\) are defined by \(P_{N_1}, P_{N_2}, A_{N_1}, A_{N_2}\) respectively. With equal masses \(N_1\) and \(N_2\), (E-5) implies

\[
\dot{r}_1 = \dot{r} - \frac{1}{2} \dot{\rho}, \\
\dot{r}_2 = \dot{r} + \frac{1}{2} \dot{\rho}, \\
\dot{R}_1 = \dot{R} - \frac{1}{2} \dot{\rho}, \\
\dot{R}_2 = \dot{R} + \frac{1}{2} \dot{\rho}.
\]

The core-core coordinate, if needed, is given by
\[ \dot{r}_{AP} = \dot{r}_T + \dot{r}_P, \]

\[ = \frac{T}{C} \dot{r}_T + \frac{B}{C} \dot{r}_P, \]

(See (E-8).)
APPENDIX F

THE BIPOLAR EXPANSION

The following Moshinsky expansion derivation (Mo 59) is a prerequisite to the actual bipolar expansion. If a vector \( \hat{r} \) can be decomposed by \( \hat{r} = \hat{r}_1 + \hat{r}_2 \), for two arbitrary vectors \( \hat{r}_1 \) and \( \hat{r}_2 \), then the following multipole expansion holds.

\[
\mathcal{Y}_m^l(\hat{r}) = \sum_{l_1 l_2} G_{l_1 l_2 l} r_1^{l_1} r_2^{l_2} [\mathcal{Y}_{l_1}^l(\hat{r}_1) \otimes \mathcal{Y}_{l_2}^l(\hat{r}_2)]^m_l
\] (F-1)

To find the geometrical factor \( G_{l_1 l_2 l} \), it is sufficient to consider the special case in which \( \hat{r}_1 = \hat{r}_2 = \hat{r} \). In that case,

\[
[\mathcal{Y}_{l_1}^l(\hat{r}_1) \otimes \mathcal{Y}_{l_2}^l(\hat{r}_2)]^m_l = (-1)^{l_1 - l_2} \left( \frac{\hat{r} \cdot \hat{r}_1}{(4\pi)} \right)^{l_1} \left( \frac{\hat{r} \cdot \hat{r}_2}{(4\pi)} \right)^{l_2} \mathcal{Y}_m^0(\hat{r}).
\]

\[
r^l = (r_1 + r_2)^l,
\]

\[
= \sum_{l_2} \binom{l}{l_2} r_1^{l-l_2} r_2^{l_2}
\]

\[
= \sum_{l_2} G_{l_1 l_2 l} r_1^{l_1} r_2^{l_2} (-1)^{l_1 - l_2} \left( \frac{\hat{r} \cdot \hat{r}_1}{(4\pi)} \right)^{l_1} \left( \frac{\hat{r} \cdot \hat{r}_2}{(4\pi)} \right)^{l_2}.
\]

Since \( r_1 \) and \( r_2 \) are arbitrary, \( l_1 = l - l_2 \) and

\[
\binom{l}{l_2} = G_{l_1 l_2 l} (-1)^{l_1 - l_2} \left( \frac{\hat{r} \cdot \hat{r}_1}{(4\pi)} \right)^{l_1} \left( \frac{\hat{r} \cdot \hat{r}_2}{(4\pi)} \right)^{l_2}. 
\]
The value of the 3j-symbol is

\[
\begin{pmatrix} l_1 - l_2 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix} = (-1)^l \frac{\Gamma(2l_2)!}{\Gamma(l_2^2 - l_2)!} \frac{\Gamma(l_2)!}{\Gamma(l_2^2 + l)!} [\frac{\Gamma(l_2)!}{\Gamma(l_2^2 - l_2)!}],
\]

so that the solution is

\[
G_{l_1 l_2 l} = \delta_{l_1, l_2} (-1)^{l_1 - l_2} (4\pi \langle \hat{l}_1 \hat{l}_2 \rangle)^\frac{1}{2} (-1)^l \frac{\Gamma(2l_2)!}{\Gamma(l_2^2 - l_2)!} \frac{\Gamma(l_2)!}{\Gamma(l_2^2 + l)!} \frac{\Gamma(l_2)!}{\Gamma(l_2^2 - l_2)!},
\]

\[
= \delta_{l_1, l_2} (4\pi \langle \hat{l}_1 \hat{l}_2 \rangle)^\frac{1}{2} (\frac{\hat{l}}{2l_2})^\frac{1}{2}.
\]

Therefore equation (F.1) becomes

\[
x^l y^m_l(r) = \sum (4\pi \langle \hat{l}_2 \rangle)^\frac{1}{2} (\hat{r}_2)^\frac{1}{2} x_1^{l-l_2} x_2^{l_2} \sum \gamma_{l_1 l_2 l} (r_1) \gamma_{l_2} (r_2) \gamma^m_l
\]

(F.2)

The bipolar expansion (Au 64) is employed to reexpress a function of the form

\[
f(r_1, r_2) [\gamma_{l_1} (r_1) \gamma_{l_2} (r_2)]^m
\]

as

\[
\sum \gamma_{l_1 l_2 l} (r, r) [\gamma_{l_1} (\hat{r}) \gamma_{l_2} (\hat{r})]^m
\]

when the coordinates are related by a linear transformation.

\[
\begin{align*}
\hat{r}_1 &= S_1 \hat{r} + t_1 \hat{r}, \\
\hat{r}_2 &= S_2 \hat{r} + t_2 \hat{r}
\end{align*}
\]

(F.4)
Equation (F-2) implies that

\[
\begin{align*}
\gamma_{l_1}^{(r_1)} \gamma_{l_2}^{(r_2)} & \in L = \\
& = r_1^{-\ell_1} r_2^{-\ell_2} f(r_1, r_2) \lambda_1 \lambda_2 \frac{4\pi}{\ell_1} \frac{1}{\ell_2} \left( \begin{array}{c}
\ell_1 \\
\ell_2 \\
\end{array} \right) \left( \begin{array}{c}
\lambda_1 \\
\lambda_2 \\
\end{array} \right) \times \\
& \times (S_1 R)^{\ell_1,\lambda_1} (t_1 r) \lambda_1 (S_2 R)^{\ell_2,\lambda_2} (t_2 r) \lambda_2 \\
& \times \left[ \gamma_{l_1-\lambda_1}^{(R)} \gamma_{l_1}^{(r)} \right]_{l_1} \left[ \gamma_{l_2-\lambda_2}^{(R)} \gamma_{l_2}^{(r)} \right]_{l_2} \\
& \text{(F-5)}
\end{align*}
\]

The spherical harmonics are easily recoupled and combined.

\[
\begin{align*}
\left[ \gamma_{l_1-\lambda_1}^{(R)} \gamma_{l_1}^{(r)} \right]_{l_1} \left[ \gamma_{l_2-\lambda_2}^{(R)} \gamma_{l_2}^{(r)} \right]_{l_2} & \in L = \\
& = \sum_{L', \ell'} (\hat{L}_2, \hat{L}_1, \hat{L}, \hat{\ell}, \hat{\ell}', \ell) \times \\
& \times \left[ \gamma_{l_1-\lambda_1}^{(R)} \gamma_{l_1}^{(r)} \right]_{L, \ell} \left[ \gamma_{l_2-\lambda_2}^{(R)} \gamma_{l_2}^{(r)} \right]_{L', \ell'} \\
& \times \left[ \gamma_{l_1-\lambda_1}^{(r)} \gamma_{l_1}^{(R)} \right]_{L, \ell} \left[ \gamma_{l_2-\lambda_2}^{(r)} \gamma_{l_2}^{(R)} \right]_{L', \ell'} \\
& \times \left( \frac{\ell_1}{\ell_1 \lambda_1} \lambda_1 \ell_1 \right) \left( \frac{\ell_2}{\ell_2 \lambda_2} \lambda_2 \ell_2 \right) \times \\
& \times C_{0-\ell_1}^{\ell_1-\lambda_1} C_{0-\ell_2}^{\ell_2-\lambda_2} C_{0-\ell_1}^{\ell_1} C_{0-\ell_2}^{\ell_2} \left[ \gamma_{L, \ell}^{(R)} \gamma_{L, \ell}^{(r)} \right] \\
& \text{(F-6)}
\end{align*}
\]
With the identity
\[
(\hat{\xi}_1 \xi_1^{-(\lambda_1)} \hat{\xi}_2) = \hat{\xi}_1 \frac{2}{\xi_1} \xi_2
\]
equations (F-5) and (F-6) give
\[
f(r_1, r_2) \left[ \gamma \left( \frac{r_1}{\xi_1} \right) \otimes \gamma \left( \frac{r_2}{\xi_2} \right) \right] = 
\]
\[
= r_1^{-\lambda_1} r_2^{-\lambda_2} f(r_1, r_2) \sum_{\lambda_1, \lambda_2} \hat{\xi}_1 \xi_1 \hat{\xi}_2 \frac{2}{\xi_1} \xi_2 \times 
\]
\[
x(\xi_1 S_1 \xi_2) (t_1 r_1) \xi_1 (\xi_2 S_2 \xi_2) (t_2 r_2) \xi_2 \times \left\{ \begin{array}{c}
\lambda_1 \lambda_2
\lambda_1 \lambda_2
\end{array} \right\} \times 
\]
\[
x C \xi_1^{-\lambda_1} \xi_2^{-\lambda_2} \xi_1 \xi_2 \left[ \gamma \left( \frac{\xi_1 \xi_2}{\xi_1} \right) \otimes \gamma \left( \frac{\xi_1 \xi_2}{\xi_2} \right) \right].
\]

The transformation of
\[
\begin{pmatrix}
-\xi_1 & -\xi_2
\end{pmatrix}
\begin{pmatrix}
r_1
r_2
\end{pmatrix}
\]
to a function of \( \hat{\xi} \) and \( \hat{\tau} \) is accomplished by integration.
\[
g_K(R, r) \equiv \int_{-1}^{1} dx \, P_K(X) \begin{pmatrix}
r_1' & r_2'
\end{pmatrix} f(r_1', r_2'),
\]
in which \( r_1' \) and \( r_2' \) are given by
\[
r_1' = S_1^2 R^2 + t_1^2 r^2 + 2 S_1 t_1 R r X,
\]
\[
r_2' = S_2^2 R^2 + t_2^2 r^2 + 2 S_2 t_2 R r X.
\]

\[1\text{The numerical integration methods employed are discussed in Appendix I.}\]
The desired function is given by the infinite series

\[ \sum_{K=0}^{\infty} \hat{g}_K(R/r) p_K(R/r) = \int_{-1}^{1} dx \ p_1^{1-l_1} p_2^{1-l_2} f(r_1', r_2'), \]

\[ = r_1^{-l_1} r_2^{-l_2} f(r_1, r_2). \quad (F-9) \]

When this summation is inserted in (F-7), the only dependence on the directions \( \hat{R} \) and \( \hat{r} \), is given by the product

\[ p_K(\hat{R}, \hat{r}) \gamma_{L'}(\hat{R}) \otimes \gamma_{L'}(\hat{r}) = \]

\[ = \frac{4 \pi}{\hat{R}} \left( \gamma_{K}(\hat{R}) \otimes \gamma_{K}(\hat{r}) \right) \gamma_{L'}(\hat{R}) \otimes \gamma_{L'}(\hat{r}) \]

\[ = \frac{4 \pi}{\hat{R}} (-1)^K \left[ \gamma_{K}(\hat{R}) \otimes \gamma_{K}(\hat{r}) \right] \gamma_{L'}(\hat{R}) \otimes \gamma_{L'}(\hat{r}) \]

\[ = \frac{4 \pi}{\hat{R}} (-1)^K \sum_{\ell=0}^{\infty} C_{\ell}^{\ell'} \gamma_{\ell'}(\hat{r}) \otimes \gamma_{\ell'}(\hat{r}) \]

\[ x \left[ \left[ \gamma_{L'}(\hat{R}) \otimes \gamma_{L'}(\hat{r}) \right] \gamma_{L'}(\hat{R}) \otimes \gamma_{L'}(\hat{r}) \right] \]

There is one nonzero term in the sum, for which the Clebsch-Gordan coefficient is one. As before, the spherical harmonics are recoupled and combined.

\[ p_K(\hat{R}, \hat{r}) \gamma_{L'}(\hat{R}) \otimes \gamma_{L'}(\hat{r}) = \]

\[ = (-1)^{L'+\ell'} K \sum_{\ell} \left\{ \begin{array}{c} L' \\ \ell \\ \ell \end{array} \right\} \left\{ \begin{array}{c} K \\ K \\ 0 \end{array} \right\} x \]
The symmetry properties of the 9j-symbol and the 3j-symbols can now be exploited.

\[
\begin{align*}
\{L', L', L\} & = \{L, L, L\} = (-1)^{L+L'-L-K} (2K)^{-1/2} W(L L L'; \mathcal{A} K), \\
(L' L' L) & = (L L L) = (-1)^{L-L'} (2K)^{-1/2} C_0 L L' K \\
\end{align*}
\]

These results, together with the analogous one for \( \ell' K \ell' \), allow (F-10) to be rewritten as

\[
P_K (\hat{R} \hat{r}) [\gamma_L, (\hat{R}) \otimes \gamma_L, (\hat{r})] \chi =
\]

\[
\sum_{L' L' L} (-1)^{L+L'-L'} (L', L', L')^{1/2} K W(L L L'; \mathcal{A} K) \chi
\]

\[
\times C_0 \ell' K \ell \ell' K \chi C_0 \ell K \ell' \ell' K \chi \frac{\gamma_L (\hat{R}) \otimes \gamma_L (\hat{r})}{\chi}.
\]

The coefficient \( C_0 \ell' K \ell \ell' K \chi C_0 \ell K \ell' \ell' K \chi \) is nonzero only if the sum \( L+L'+K \) is even. Thus, the phase \( (-1)^{L-L'} \) can be written simply \( (-1)^L \).

Collection of the results (F-7), (F-9), and (F-11) gives the proper form of the bipolar expansion.
\[ \mathcal{L}_{l_1 l_2 L \ell}^{(R,r)}(R, r) = \sum_{\lambda_1 \lambda_2 K} G_{l_1 l_2 L \ell}^{(S_1 R)} (l_1 - \lambda_1)^{l_1} (S_2 R) (l_2 - \lambda_2)^{l_2} q_K^{(R, r)} \]

\[ \mathcal{L}_{l_1 l_2 L \ell}^{(R,r)} = \frac{1}{2} \left( -1 \right)^{\lambda_1 + \lambda_2 + K} \sum_{L', \ell'} \Delta_{L L'}^{l_1 l_2} \Delta_{\ell \ell'}^{2 \lambda_1 2 \lambda_2} \left( \begin{array}{c} \ell_1 - l_1 \\ \ell_2 - l_2 \\ L' \ell' \end{array} \right) \left( \begin{array}{c} \ell_1 - 1 \\ \ell_2 - 2 \\ L \ell \end{array} \right) W(L \ell L' \ell'; 2 K) \left( \begin{array}{c} 1 - l_1 \\ 1 - l_2 \\ 1 \end{array} \right) \left( \begin{array}{c} 2 - l_1 \\ 2 - l_2 \\ 2 \end{array} \right) \left( \begin{array}{c} L \ell \end{array} \right) \]  

and the integrals \( q_K^{(R, r)} \) are defined by (F-8).

Certain important parity constraints follow immediately from the fact that the sum of the "j indices" must be even for each of the Clebsch-Gordan coefficients. First, if \( l_1 \) and \( l_2 \) are specified, in the sum over \( L \) and \( \ell \) nonzero terms must satisfy the relation

\[ (-1)^{L + \ell} = (-1)^{l_1 + l_2} \]

In the sum over \( \lambda_1 \), \( \lambda_2 \), and \( K \), the constraint is
\[
\lambda_1 + \lambda_2 + K = (-1)^L.
\]

And finally, in the sum of \( L' \) and \( l' \),

\[
(-1)L' + l' = (-1)^{L_1 + L_2}.
\]
APPENDIX G

SOURCE TERM FORMULAS

G.1 Two-Particle Transfer

G.1.1 General Case

Equations (4-31), (4-38), (4-36), and (4-25) can be collected for the general transfer of two protons or two neutrons.¹

\[ \rho_{\gamma}(r_{\gamma}) = \]

\[ = \sum_{\zeta} \left( \sum_{J_{AB}} \int_{T_{PT}}^{J} \int_{T_{P}}^{J} \lambda_{r} \lambda_{r} \right) \]

\[ x(K_{A} B_{B} J_{T} J_{P} J_{T}) \]

\[ x^{J_{T} J_{P} J_{T}}(K_{B} J_{AB} I_{A} ; K_{A} J_{B}) \]

\[ \frac{1}{J} \left( \sum_{J_{T} J_{P} J_{T}} \right) \int_{L_{T} L_{P}}^{J} \lambda_{r} \lambda_{r} \lambda_{L_{T} L_{P}}(r_{T}, r_{P}). \]

The form factor is given by the bipolar expansion (denoted by the arrow) of the center of mass function.

\[ \frac{J}{J_{T} J_{P} J_{T}}(r_{T}, r_{P}) + \frac{J}{J_{T} J_{P} J_{T}}(r, R) = \]

\[ = \sum_{\zeta} (-1)^{\rho} \lambda_{r} \lambda_{r} \lambda_{L_{T} L_{P}}(r_{T}, r_{P}) \]

\[ x^{J_{T} J_{P} J_{T}}(J_{T} J_{P} J_{T} ; S) \]

\[ x^{J_{T} J_{P} J_{T}}(J_{T} J_{P} J_{T} ; S) \]

\[ \frac{L_{T} L_{P}}{L_{T} L_{P}}(r_{T}, r_{P}) \]

¹In these formulas sums over \( \zeta \) refer specifically to sums over \( L_{\zeta} \) and \( K_{\zeta} \) because the sums over internal quantum numbers are incorporated in the sums over \( t, t', b, \) and \( b' \).
\[ F_{\lambda, \rho}^T(r, \rho) = \]
\[ = \sum_{t \neq t'} \beta_{t, t'}^{JP} I_{[P, T]} \left[ 2 \left[ 1 - (-1)^{J_{PT}} \delta_{t, t'} \right] \right] \frac{1}{2} (\hat{j}_{t}, \hat{\lambda}_{t}) \cdot \hat{L} \]
\[ \times \left\{ \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array} \right\} \frac{1}{2} [1 + (-1)^{\rho + S}] F_{t, t'}^{\lambda, \rho} (r, \rho). \]

\[ F_{R, \rho}^{t, \lambda} (R, \rho) = \]
\[ = \sum_{b \neq b'} \beta_{b, b'}^{ab} [A, B] \left[ 2 \left[ 1 - (-1)^{J_{AB}} \delta_{b, b'} \right] \right] \frac{1}{2} (\hat{j}_{b}, \hat{\lambda}_{b}) \cdot \hat{L} \]
\[ \times \left\{ \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array} \right\} [1 + (-1)^{\rho + S}] F_{b, b'}^{R, \lambda} (R, \rho). \]

If the T-system interaction is used,
\[ F_{t, t'}^{\lambda, \rho} (r, \rho) + \left\{ v_{t}^{(N)}(r_{1}) + v_{t}^{(N)}(r_{2}) \right\} \hat{\rho}_{t} (r_{1}) \hat{\rho}_{t} (r_{2}), \]
and
\[ F_{b, b'}^{R, \lambda} (R, \rho) + \hat{R}_{b} (R_{1}) \hat{R}_{b} (R_{2}). \]
G.1.2 $O^+$ Projectile System

Many heavy-ion reactions involving even-even projectiles can be analyzed within the constraint $J_p = J_p T = J_T = 0$. The source term formulas are simplified considerably.

\[ K_A = L_T, \]
\[ K_B = L_p, \]
\[ J = J_{AB}, \]
\[ \ell = S, \]
\[ j_t, = j_t. \]

\[ \rho \gamma (r)/r = \]
\[ = \sum L^2 \rho \biggl[ \frac{W(\lambda_{AB} \lambda_{AB}; L_p L_p) F_{AB} L_T L_T (r_T, r_p)}{L_T L_T} \biggr] \frac{J_{AB}}{L_T L_T} (r, r). \]

\[ \int_{(r, r)} f_{AB} L_T L_T \]
\[ = \sum L^2 \rho \biggl[ \frac{W(\lambda_{AB} \lambda_{AB}; \lambda_{AB} j_t,)}{L_T L_T} \biggr] \int_{\rho} d \rho \int_{\rho} \lambda_{AB} j_t, (r, r). \]

Other required quantities have formulas as given in Section G.1.1.
If the additional assumption of equivalent orbits for the T-system \((t'=t)\) is used, then \(lt'=lt\). Moreover, since \(|\lambda_r-\lambda_p| \leq l\), and \((-1)^{\lambda_r+\lambda_p}(-1)^{l+\lambda_{t'}} = 1\), \(\lambda_p\) must equal \(\lambda_r\).

G.1.3 \(0^+\) Target System

If the target and residual nucleus systems happen to remain in \(0^+\) states, similar relationships hold.

\[ J_A = J_B = 0 \quad \text{implies} \]
\[ J_{AB} = 0, \]
\[ K_A = K_B = I, \]
\[ J = J_{PT}, L = S, \]
\[ j_b' = j_b. \]

\[ \rho_{\gamma^I}(\zeta)/\zeta = \]
\[ = \sum \int d\zeta \zeta^U_{\gamma^I}(\zeta) J_{PT}^A R^A (\zeta) \left( -1 \right)^{l_T} \hat{J}_T^h \]
\[ x W(L_P J_{PT} I_T; L_T J_P) \hat{J}_{\lambda_T^R \lambda_T^P} (r_T, r_P). \]

\[ J_{PT} \hat{J}_{\lambda_T^R L_T^P} (r_T, r_P) = \hat{J}_{\lambda_T^R} (r, R) = \]
\[ = \sum (-1)^{\gamma^P} \left( \hat{J}_{PT}^L \right)^{\gamma^P} W(\lambda_T^R S^R; \lambda_{PT}^R) \]
\[ x \int d\rho \rho^2 \hat{J}_{\lambda_T^R \rho} (r, \rho) \hat{J}_{\lambda_T^R \rho} (r, \rho). \]
\[ \mathcal{F}^{(R,\rho)}_{\lambda} = \int_{R}^{R} \lambda \rho \]

\[ = \sum_{b} \, \mathcal{F}_{bb'} \, [A, B] \, [l+\delta_{bb'}] \, \prod_{j} \, j_{b}^{j_{b}} \, (-1)^{l_{b}+\frac{1}{2}-j_{b}-S} \]

\[ \times (j_{b} S)^{\frac{1}{2}} \, W(l_{b}^{l_{b}}, l_{b}, j_{b}; S) \, [l+(-1)^{S}] \, \mathcal{F}_{bb'} \lambda \rho \lambda \rho \, (R, \rho). \]

Other required quantities have formulas as given in Section G.1.1.

With the assumption of equivalent orbits (b' ≡ b), \( \lambda \rho \) must equal \( \lambda \rho \).

G.2 One-Particle Transfer

G.2.1 One-Nucleon or Cluster-With-Spin Transfer

The source term formulas for one-particle transfer can be derived directly, as in Sections 4.5 and 4.6, with the wavefunctions (4-17) and (4-18) used (with the appropriate interaction) instead of equation (4-26). However, the same results can be obtained with less tedium by taking the appropriate limit from the general two-particle transfer formulas. (\( \lambda \rho = 0 \), etc.)

The following formulas are applicable to the transfer of \(^3H\) and \(^3He\) clusters as well as nucleons when \( S = \frac{1}{2} \). Deuteron cluster transfer can be described when \( S = 1 \).
\[ \rho_{\gamma} (r_{\gamma})/r_{\gamma} = \]
\[ = \sum J_{p+j} \left( (-1)^{J_{p}+J} J_{A}-J_{B}+K_{A}-K_{B} (-1) J_{p}+j \right) \]
\[ \times \left( \frac{K_{A} K_{B}}{J_{T}} J_{p} J_{t} \right) \]
\[ \times \left( K_{B} J_{J} (r_{T}, r_{p}) \right) \]
\[ = (-1)^{J_{t}+J} \left( \sum J_{t} J_{b} \right) W(j_{t} l_{b} l_{b};SJ) \times \]
\[ \times \sum \beta_{t}[P,T] \beta_{b}[A,B] v^{(N)}_{t} (r) \rho_{t}(r) \rho_{b}(r). \]

The summation over \( t \) and \( b \) may include contributions from different model states with the same spin and parity.

\[ \text{G.2.2 Spinless Cluster Transfer} \]

For the case in which a spin-zero cluster is transferred, e.g. two neutrons, two protons, or an alpha particle, the form factor becomes particularly simple.

\[ J_{t} J_{b} L_{T} L_{p} (r_{T}, r_{p}) + J_{t} J_{b} (r, r) = \]
\[ = (-1)^{J_{b}} \sum \beta_{t}[P,T] \beta_{b}[A,B] v^{(N)}_{t} (r) \rho_{t}(r) \rho_{b}(r). \]
The source term formula is the same as that given in Section G.2.1 with the constraint \(l_t=j_t\) and \(l_b=j_b\).

If in addition the projectile or target systems remain in spin zero states, the source terms are given respectively by

\[
\rho_{\gamma \zeta}^I (r_\gamma)/r_\gamma =
\]

\[
= \sum_{L_\zeta} u^3 \int d\zeta r_\zeta u^I (r_\zeta) (-1)^{-l_b} (-1)^{J_A-J_B+L_T} \frac{L_T^\zeta \frac{1}{2}}{j_B^\zeta}
\]

\[
\times W(L_p j_t I T; L_T J_T) \int_0 j_D^L T_T^L \rho I (r_T, r_P)
\]

or

\[
\rho_{\gamma \zeta}^I (r_\gamma)/r_\gamma =
\]

\[
= \sum_{L_\zeta} u^3 \int d\zeta r_\zeta u^I (r_\zeta) (-1)^{L_T} \frac{L_T^\zeta \frac{1}{2}}{j_T^\zeta}
\]

\[
\times W(L_p j_t I T; L_T J_T) \int_0 j_t^O L_T^L \rho I (r_T, r_P)
\]
H.1 One-Nucleon Transfer

H.1.1 One-Particle States

The one-nucleon transfer spectroscopic amplitudes are introduced by equations (4-17) and (4-18). The appropriate overlap of (4-17) yields

\[ \beta_t [P, T] = (T) \frac{1}{2} J_p^\dagger \psi_{P} J_p \tau_p (\rho) \otimes \psi_{\tau_j \tau_t} (\chi) J_T \psi_{\tau_T} (\rho) \psi_{\tau_T} (\chi) \]

\[ = (T) \frac{1}{2} \sum_{M_p m} C_{P T m} \int dT \psi_{\tau_T} (\tau_T) \psi_{P P_T T} (\rho) \psi_{\tau_T} (\chi) \]

With the expansion\(^1\)

\[ \psi_{\tau_T} (\tau_T) = (T) \frac{1}{2} \sum_{t, m, m'} \psi_{t, m, m'} (\chi) \psi_{t, \tau_j, \tau_t} (\chi) \psi_{t, m, m'} \]

and the relations

\[ \psi_{P P_T T} (\rho) = \langle P | P_T M_P | P \rangle \]

\[ \psi_{\tau_T} (\chi) = \langle \chi | t, j_t, m_T, \tau_t, m \rangle \]

\(^1\)See footnote 6 of Chapter 4.
it follows that
\[
\beta_{t}[p,T] = \sum_{M_p m M_T} C_{M_p m M_T} \langle \tau_{\frac{1}{2}} \gamma_{\frac{1}{2}} \tau_{\frac{1}{2}} \gamma_{\frac{1}{2}} | p M_p \tau M_T | p M_p \tau M_T \rangle.
\] (H-1)

The matrix element in (H-1) can be reduced in both ordinary space and isospin space. Application of the Wigner-Eckart theorem twice yields
\[
\beta_{t}[p,T] = (-1)^{2j_t} (-1)^{2\tau_t} \langle J_{T \frac{1}{2}} \gamma_{\frac{1}{2}} | d_{t \frac{1}{2}} \rangle \langle \gamma_{\frac{1}{2}} | p M_p | p \rangle.
\]

The present reduced matrix element can be related to that of French, et al. (equation 4.1 of Fr 69) by a simple phase.
\[
\langle T | d_{t \frac{1}{2}} \rangle_{\text{French, et al.}} = (-1)^{2j_t} (-1)^{2\tau_t} \langle T | d_{t \frac{1}{2}} \rangle_{\text{P}}.
\]

This difference arises because of their choice not to use Racah's phase (Ra 42) for the reduced matrix element, but the difference has no significance because
\[
(-1)^{2j_t} (-1)^{2\tau_t} = +1.
\]

It is also possible to relate the reduced matrix elements to coefficients of fractional parentage (CFP), as given in (Co 67) for example.
\[
\langle J_{T \frac{1}{2}} \gamma_{\frac{1}{2}} | \tau_{\frac{1}{2}} \gamma_{\frac{1}{2}} p_{\frac{1}{2}} \rangle = (-1)^{2j_t} (-1)^{2\tau_t} \langle T \gamma_{\frac{1}{2}} \gamma_{\frac{1}{2}} | d_{t \frac{1}{2}} \rangle_{\text{P}}.
\]

These connections are presented for those cases in which it is desired to use spectroscopic amplitudes from shell
model calculations.

In some cases it is convenient to depart from the isospin formalism to represent neutrons and protons individually. For example if the P- and T-systems can be described respectively by a particle-vacuum state and a single-particle state i.e.

\[ |PJ_p M_p \rangle = |POO \rangle = |\omega \rangle, \]

and

\[ |TJ_T M_T \rangle = d_T \delta_T^+ |\omega \rangle, \]

then the spectroscopic amplitude is simply unity by (H-l):

\[ \beta_t [P,T] = \delta_T \delta_T^+ . \]

Similar relationships hold for \( \beta_b [A,B] \) of course.

H.1.2. Particle-Phonon States

For those reactions in which transfers between phonon states and particle-phonon states are of interest,

\[ |AJ_A^M \rangle = v_A^M |\omega \rangle, \]

and

\[ |BJ_B^M \rangle = [d_b^+, \delta_b^+] |\omega \rangle. \]

The matrix element of (H-l) is
Thus, by (H-1),

\[ \beta_{b}[A,B] = (-1)^{j_b+J_A-J_B} \delta_\lambda J_A \delta_{b',b} \delta_{j_b,j_b} \]

In Chapter 6 the model for the excited states of the B-system employs a coherent mixture of particle-phonon and single-particle states.

\[ |BJ_B^{M_B}> = (C_{P,P} \cdot [d_{b',j_b}, d_B^{M_B}] + C_{S,P} \cdot d_{BJ_B}^{M_B} ) |\omega> \]

The coefficients \( C_{P,P} \) and \( C_{S,P} \) are normalized to one, so only one additional parameter is introduced, their ratio \( C_{S,P}/C_{P,P} \). Within this model the spectroscopic amplitude is either

\[ \beta_{b=\omega,B} = C_{S,P} \]

for the A-system in the vacuum state, or

\[ \beta_{b=\lambda,B} = (-1)^{j_b+J_A-J_B} C_{P,P} \]
for the $A$-system in a one-phonon state.

H.2. Two-Nucleon Transfer

H.2.1 Simultaneous Transfer

The two-nucleon transfer spectroscopic amplitudes are introduced by equations (4-19) and (4-20). The appropriate overlap of (4-19) yields

$$
\beta^{\beta \pi \pi \pi \pi \pi}_{[P,T]} = (\mathcal{T})^{1/2} \left[ \tilde{\psi} \hat{P}(P) \otimes \psi_{\pi \pi \pi \pi \pi}^{\pi \pi \pi \pi \pi} \tilde{\psi}_{\pi \pi \pi \pi \pi}^{\pi \pi \pi \pi \pi} \right]_{\pi \pi \pi \pi \pi}^{\pi \pi \pi \pi \pi} \left[ \psi_{\pi \pi \pi \pi \pi}^{\pi \pi \pi \pi \pi} \tilde{\psi}_{\pi \pi \pi \pi \pi}^{\pi \pi \pi \pi \pi} \right]_{\pi \pi \pi \pi \pi}^{\pi \pi \pi \pi \pi}.
$$

The transferred particles' wavefunction is antisymmetrized as in (4-21).

$$
\psi_{\pi \pi \pi \pi \pi}^{\pi \pi \pi \pi \pi} = \sum_{m} \tilde{\psi}_{\pi \pi \pi \pi \pi}^{\pi \pi \pi \pi \pi} \tilde{\psi}_{\pi \pi \pi \pi \pi}^{\pi \pi \pi \pi \pi} \left[ \psi_{\pi \pi \pi \pi \pi}^{\pi \pi \pi \pi \pi} \tilde{\psi}_{\pi \pi \pi \pi \pi}^{\pi \pi \pi \pi \pi} \right]_{\pi \pi \pi \pi \pi}^{\pi \pi \pi \pi \pi}.
$$

in which
With the expansion

\[ \phi_{\tau_{1}, \tau_{2}} = \frac{1}{2} \sum_{m} \left( \phi_{m}^{+} \phi_{m} \right) \]

and the previous relation

\[ \psi_{\tau_{1}, \tau_{2}} = \psi_{\tau_{1}, \tau_{2}} \]

the spectroscopic amplitude can be written

\[ \beta_{\tau_{1}, \tau_{2}} = \frac{1}{2} \sum_{m} \left( \phi_{m}^{+} \phi_{m} \right) \]

The particle creation operators can be recoupled after they are commuted, and then used to define a two-particle creation operator.
The overall minus sign in the definition

\[ \beta_{P_t \alpha_{P_t}}^{(P_t \alpha_{P_t})} = \sum_{M_p M_{P_t} M_{P_t}} \frac{J_{P_t \alpha_{P_t}} (P_t \alpha_{P_t})}{C_{M_p M_{P_t} M_{P_t}}^{M_p M_{P_t} M_{P_t}}} [\chi^{(P_t \alpha_{P_t})}]^{M_p M_{P_t} M_{P_t}} \delta_{P_t \alpha_{P_t}}^{M_p M_{P_t} M_{P_t}} \times \langle T J_{M_p M_{P_t} M_{P_t}} | \chi^{(P_t \alpha_{P_t})} \rangle_{J_{P_t \alpha_{P_t}}^{M_p M_{P_t} M_{P_t}}}^{M_p M_{P_t} M_{P_t}}. \] 

Equation (H-2) can be reduced in the same manner that (H-1) is.

Once again the reduced matrix element is related to that of (Fr 69, equation 4.2) by the phase
\[ (-1)^{2J_{PT}} (-1)^{2\tau_{PT}} = +1 \]

However, the two-nucleon CFP (Co 70) is given by

\[ \langle J_T^T T^T \frac{1}{2} J_P^T P^T, J_{PT}^T P_{PT}^T \rangle = \]

\[ = -(-1)^{2J_{PT}} (-1)^{2\tau_{PT}} (\frac{1}{2}, -\frac{1}{2}) \langle J_T^T T^T \rangle - \frac{1}{2} \langle J_T^T T^T \rangle - \frac{1}{2} \langle T \mid \tau_{PT}^T \mid J_T^T T^T \rangle \langle J_{PT}^T P_{PT}^T \rangle. \]

Consequently the CFP has a sign opposite to that of the reduced matrix element (Ku 77). A good illustration of this connection is the \( ^{16}\text{O}, ^{14}\text{C} \)B reaction. If only the \( 1p \) nucleons are considered active, the required CFP's are

\[ \langle 00 \mid 16\text{O} \frac{1}{2} 01^{14}\text{C}, 01 \mid 1p_{1/2} \rangle^2 = -0.1949 \]

and

\[ \langle 00 \mid 16\text{O} \frac{1}{2} 01^{14}\text{C}, 01 \mid 1p_{3/2} \rangle^2 = -0.0864 \]

from (Co 70). Since

\[ (T) = \frac{12.11}{2} = 66, \]

and

\[ C_{1}^{1} \frac{1}{1} 0 = 3^{-\frac{1}{2}}, \]

the spectroscopic amplitudes are
H.2.2. Sequential Transfer

The spectroscopic amplitudes for sequential transfer are just the amplitudes for the constituent one-particle transfers, which are given in Section H.1.1. In this section, the sequential transfer reaction \( A(^{16}_0,^{15}_N)B=A'(^{15}_N,^{14}_C)B' \) is presented as an example.

The spectroscopic amplitudes for the first step can be found in a number of ways. If all the \( l_p \) nucleons are again considered active, the required CFP's are

\[
<00^{16}_0|J_p^{15/2}N, j_t=J_p> = (j_t/6)^{1/2}.
\]

Consequently, the spectroscopic amplitudes are

\[
\beta_c^{15}/_N,^{16}/_0=\left(\frac{12}{1}\right)^{1/2}(j_t/6)^{1/2}C_{1/2}^{1/2}0 = j_t^{1/2}.
\]

These amplitudes are perhaps more simply found by considering the \( l_p^{1/2} \) and \( l_p^{3/2} \) nucleons to be members of different shells.\(^2\) The spectroscopic amplitudes can then be

\(^2\)See Example 4. on p. 618 of (Ma 60).
deduced from the unit CFP's for the single-hole $^{15}\text{N}$ states ($1P_\frac{1}{2}$ and $1P_\frac{3}{2}$). Thus, the amplitudes are

$$\beta_{\frac{1}{2}}[^{15}\text{N},^{16}\text{O}]=\left(4\right)^{\frac{1}{2}} c_{\frac{1}{2}}^+ c_{0}^+ 0 = 2^+$$

and

$$\beta_{3/2}[^{15}\text{N},^{16}\text{O}]=\left(8\right)^{\frac{1}{2}} c_{\frac{3}{2}}^+ c_{-\frac{1}{2}}^- 0 = 4^+$$

Here $T = \hat{j}_t \hat{t}_t$ = number of nucleons in complete shell.

Finally the same results can be obtained without the isospin formalism by noting that with

$$T = \hat{j}_t = \text{number of protons in complete shell},$$

$$\beta_{\frac{1}{2}}[^{15}\text{N},^{16}\text{O}]=\left(2\right)^{\frac{1}{2}},$$

and

$$\beta_{3/2}[^{15}\text{N},^{16}\text{O}]=\left(4\right)^{\frac{1}{2}}.$$

The spectroscopic amplitudes for the second step, $^\Lambda'(^{15}\text{N},^{14}\text{C})B'$, can be found from the CFP table of (Co 67).

(All $1p$ nucleons, with the isospin formalism, are considered to be active here.)

$$<\frac{1}{2},^{15}\text{N} || 01^{14}\text{C},\frac{1}{2} >= 0.3376,$$

and

$$<\frac{3}{2},^{15}\text{N} || 01^{14}\text{C},\frac{3}{2} >= 0.1059.$$
Now with $T=11$, and

$$C^{1 \frac{1}{2} \frac{1}{2}}_{-\frac{1}{2} \frac{1}{2}} = \frac{2}{3} \frac{1}{2},$$

the amplitudes are

$$\beta_{\frac{1}{2}}^{14\text{C},15\text{N}} = 0.9142,$$

and

$$\beta_{\frac{3}{2}}^{14\text{C},15\text{N}} = 0.2868.$$

Incidentally, the two-nucleon CPP's for simultaneous transfer are simply related to those for the associated sequential transfers (Ku 77). For these cases ($t=t'$)

$$<00^{16}\text{O}\mid 01^{14}\text{C},01t^2> =$$

$$= -<00^{16}\text{O}\mid J_p^{j_{15}\text{N}},j_t=J_p><J_p^{j_{15}\text{N}}\mid 01^{14}\text{C},j_t=J_p>. $$
NUMERICAL TECHNIQUES

I.1 Flow Charts

A heavy-ion induced transfer reaction calculation typically begins with the generation of the bound-state wavefunctions and potentials for the transferred particles. The function $\mathcal{R}_t(r)$ and the interactions $V_t^{(N)}(r)$ and $V_t^{(C)}(r)$ are computed in the program FORM for one-nucleon or cluster transfer. Similar quantities for the B-system are produced by a second execution of FORM. (Section 4.3 specifies the conventions employed.) The necessary wavefunctions and potentials are then used by COMBE to compute the form factors $J_{t_{1_{b_{L_{T_{L_{P}}}}}}}^{(L_{T_{P}})}(r_T,r_p)$ given in Section G.2. These form factors are stored on magnetic tape for later computation of source terms. This process is diagramed in figure (I-1).

For two-nucleon transfer FORM is used to generate $\mathcal{R}_t(r_1), \mathcal{R}_t(r_2)$ and their corresponding interaction potentials. The analogous quantities for the B-system are produced by a second execution of FORM. The transformations from single-particle coordinates to center of mass and relative coordinates are done in PROBE and VIBE for the T- and B-systems respectively. The resulting functions, $J_{\lambda_T \lambda_P}^{S}(r,\rho)$ from PROBE and
Figure (I-1). Form factor generation flow chart. Bound-state wavefunctions are generated in FORM. T-system and B-system bipolar expansion transformations are done in PROBE and VIBE. The bipolar expansion transformation from \( \mathbf{r} \) and \( \mathbf{R} \) to \( \mathbf{r}_T \) and \( \mathbf{r}_p \) is done in COMBE.
FORM FACTOR GENERATION

FORM

COMBE

FORM

ONE-NUCLEON or CLUSTER TRANSFER

FORM

PROBE

FORM

VIBE

COMBE

TWO-NUCLEON TRANSFER
from VIBE, are integrated over $\rho$ in COMBE. (Section G.1.2 contains the relevant formulas.) The form factors

$$\mathcal{J}_{AB}^{L} \mathcal{L}^{L} \mathcal{W}(S_{R}^{L} L_{R}^{L}; \lambda_{R} \rho, \lambda_{P} \rho)$$

are computed and summed over $\lambda_{R}$ and $\lambda_{P}$ in COMBE. Finally the summed form factors are stored on magnetic tape. Figure (1-1) contains a flow diagram for the generation of two-nucleon transfer form factors.

Once the form factors are computed, they can be used in the solution of the coupled complex differential equations. A typical one-nucleon transfer reaction with inelastic excitation is diagramed in Figure (I-2). The programs LM2 and LD2 are used to set up and solve the coupled differential equations. Their first execution generates the relative motion wavefunctions, or distorted waves, for the entrance channels. The transfer source terms are computed in LT2 with the form factors from COMBE. The exit channel solutions are generated in LD2, and the S-matrix elements are passed to LC, in which the differential cross section is computed.

Usually this procedure is made more tractable by computing the transfer routes individually and summing their S-matrix elements coherently. If, for example, the excitation of the P- and T-systems need not be considered, and the A- and B-systems can each be described
Figure (I-2). Differential equation solution flow chart for one-nucleon transfer with inelastic excitation. LM2 and LD2 are used to set up and solve the coupled complex differential equations. Transfer source terms are computed in LT2, and LC generates differential cross sections.
DIFFERENTIAL EQUATION SOLUTION

 ONE-NUCLEON TRANSFER with INELASTIC EXCITATION
by the ground state and one excited state, then there are four possible transfer routes connecting the target and residual nuclear systems. These routes are numbered "one" through "four" in figure (I-3). The population of a residual nuclear state is described by the coherent sum of S-matrix elements resulting from the four transfer routes, each of which is preceded and followed by inelastic scattering. That is, the procedure described by figure (I-2) is repeated four times, once each for

\[ 0 + 1 + 5, \]
\[ 0 + 2 + 5, \]
\[ 0 + 3 + 5, \]
\[ 0 + 4 + 5. \]

For the present case in which the excited states are weakly coupled to the ground states, the process is further simplified. Although three excited states in \(^{49}\text{V}\) are considered, they can be individually treated in conjunction with the ground state. Thus each excited state can be thought of as a separate problem of the type diagramed in figure (I-3). Since the inelastic couplings are weak, only three routes must be calculated for each of the three excited states. They are the two indirect routes, \(0 + 4\) and \(1 + 5\), which populate the particle-phonon components of the excited states, and the direct route, \(2\), which contributes to the small single-particle component. The
Figure (I-3). Enumeration of excitation and transfer routes for the case in which one excited state is considered in each of the target and residual nuclear systems.
EXCITATION and TRANSFER ROUTES

Target Nucleus

Residual Nucleus

0

g.s.

4

3

2

1

5

g.s.
S-matrix elements resulting from each of the second executions of LD2 for a given excited state are stored on magnetic tape. After the three routes are computed, the program SSUM adds the S-matrices for use with LC.

For second-order sequential transfer reactions (without inelastic excitation) the flow diagram is extended as shown in figure (I-4). REAC is used to set up and solve the coupled complex differential equations. Its first execution generates the relative motion wavefunctions for the entrance channels. The transfer source terms for the first rearrangement of particles are computed by the first execution of SORT. The intermediate channel relative motion wavefunctions are generated by the second execution of REAC, from which differential cross sections for the population of the intermediate states can be extracted if desired. The transfer source terms for the population of the exit channels are computed by the second execution of SORT; and finally, the exit channel relative motion wavefunctions are generated by the third execution of REAC. The resulting S-matrix elements are used to obtain differential cross sections from CSA or PDC when $J^{\pi}_\gamma = 0^+$ or $J^{\pi}_\gamma \neq 0^+$ respectively.
Figure (I-4). Differential equation solution flow chart for sequential transfer. REAC is used to set up and solve the coupled complex differential equations. Transfer source terms are computed in SORT. CSA and PDC generate differential cross sections when $J_\gamma^m = 0^+$ and $J_\gamma^m \neq 0^+$ respectively.
DIFFERENTIAL EQUATION SOLUTION

COMBE

REAC

SORT

REAC

CSA or PDC

COMBE

SORT

REAC

CSA or PDC

SEQUENTIAL TRANSFER
I.2 Relative Angular Momentum Truncation

The computation of full-recoil microscopic two-nucleon transfer form factors involves an infinite sum over $\lambda_p$, the multipolarity of the relative coordinate $\vec{r}$. (The appropriate formulas are given in Section G.1.2.) The convergence and truncation of this series are studied by Takemasa (Ta 74b, Ta 75, Ta 75a) for the population of $0^+$ states of the residual nucleus; i.e. for the case in which

$$J_p = J_{PT} = J_T = J = J_A = J_{AB} = J_B = 0.$$  

In that work it is found that the first four terms in the sum (through $\lambda_p=3$) are sufficient to ensure an accuracy of better than 1-2% in the differential cross section. Recently, calculations for an analysis (Tu 77) associated with this thesis have indicated that this truncation after four terms is sufficient for the more general case in which $2^+$ states may be populated:

$$J_p = J_{PT} = J_T = 0,$$

$$J = J_{AB} = 0 \text{ or } 2.$$  

This rapid convergence can be understood, and applied to even more general cases, by the following qualitative explanation.
When a given pair of bound-state wavefunctions is transformed from single-particle coordinates to relative and center of mass coordinates, e.g.

\[ \mathcal{F}_{bb',\lambda_R}^{\lambda_p}(R, \rho) + \mathcal{Q}_b(R_1) \mathcal{Q}_{b'}(R_2), \]

the transformed functions generally decrease in magnitude as \( \lambda_p \) is increased. This convergence is rather slow, however, and terms with \( \lambda_p > 3 \) may be expected to contribute nontrivially, at least as far as this first bipolar expansion transformation is concerned. Ultimately the rapid convergence in \( \lambda_p \) occurs because of the subsequent transformation to channel radii coordinates, after the appropriate sums over bound-state quantum numbers and integration over \( \rho \). When this transformation takes place, the functions \( \mathcal{J}_{AB}^{\lambda_R}(r, R) \) appear in the integrand of equation (F-8), weighted with a Legendre polynomial and the inverse radial powers \( r^{-\lambda_R} r^{-\lambda_R} \). These inverse powers effectively diminish the

---

1The transformed wavefunctions decay relatively slowly as functions of \( \rho \), especially for large \( \lambda_b \) and \( \lambda_{b'} \). Consequently transformed bound-state functions are generated out to \( \rho \sim 16 \) fm; a large mesh size (\( \sim 0.3 \) fm) is sufficient.
contributions from all but the smallest multipolarities of $\lambda \rho$. (Note that in these cases $t=t' \neq \lambda_r = \lambda \rho$.) Consequently the rapid convergence in $\lambda \rho$ can be largely attributed to the sensitivity of the form factors to the various multipole terms of $\lambda_r$ and $\lambda_R$.

In the example $^{90}_{\text{Zr}}(^{16}_{\text{O}}, ^{14}_{\text{C}}) ^{92}_{\text{Mo}}(0^+, 2^+)$ the summed form factor

$$J_{AB} \sum_{\lambda_R \lambda_r} \int_{\lambda_R} L_{\lambda r} L_{\lambda r} (\rho_1, \rho_2)$$

is found to vary typically by 10-15% when the $\lambda \rho=3$ terms are omitted (e.g. $L_1=L_2=30, J_{AB}=0$ or 2). The agreement between sums up to $\lambda \rho=3$ and $\lambda \rho=4$ is accurate to about three or four decimal places near the form factor diagonal, but discrepancies increase in regions where the form factor decays.

Finally it should be noted that computation involving higher multipole terms is eventually limited by round-off errors. With the 60-bit words of the CDC6600 used for this calculation (~14 decimal places) round-off errors are apparently negligible for $J_{AB}=0$ with $\lambda_r=\lambda_R \leq 4$, as well as for $J_{AB}=2$ with $\lambda_r \leq 3, \lambda_R \leq 5$; but for $J_{AB}=2$ with $\lambda_r \leq 4, \lambda_R \leq 6$ deviations in the form factors become apparent for large values of the channel radii.
I.3 Angular Integration.

Angular integrals given by equation (F-8) must be computed for each bipolar expansion transformation required for the generation of form factors. For microscopic two-nucleon transfer reactions, each of the transformations of the bound-state wavefunctions require integration over the angle between the center of mass and relative coordinates. Standard Gauss-Legendre integration formulas (Ab 64) are applied to the entire integration range for these expansions. Twenty-four point formulas are found to be more than adequate for all cases considered. Computer coding for these expansions and the subsequent integration over \( \rho \) has been extensively checked by comparison with the analytic transformation of harmonic oscillator wavefunctions.

The angular integration necessary for all full-recoil transfer calculations corresponds to the transformation from center of mass coordinates to channel radii. If this integration over the angle between the channel radii is performed over the full range, \(-1 \leq x \leq 1\), a high order of Gauss-Legendre integration is required (Po 74). However, for heavy-ion induced transfer reactions, the integrand is negligibly small except where the channel coordinates are nearly aligned \((x \approx 1)\). Consequently, a relatively low order of integration (twenty-four roots) suffices when the integral is computed over this much restricted range.

The following rationale illustrates the angular behavior
connected with the asymptotic limits of the reaction, which are described in Section 4.8. The coordinate transformation (E-12) yields

\[\begin{align*}
t^2 &= s^2 r_T^2 + u^2 r_p^2 - 2 s u r_T r_p X, \\
R^2 &= u^2 r_T^2 + t^2 r_p^2 - 2 u t r_T r_p X,
\end{align*}\]

in which \(X = \frac{r_T^+ r_p^+}{(r_T r_p)}\). Near the T-channel limit, the magnitudes of the channel radii are parameterized by

\[r_p = \alpha (S/u) r_T,\]

with \(\alpha \ll 1\). The magnitude of the B-system center of mass coordinate is approximated by

\[R \approx r_T.\]

These relations substituted into equation (I-2) yield an approximation to the cosine

\[X \approx \frac{u^2 + t^2 \alpha^2 (s/u)^2 - 1}{2 u t \alpha (s/u)}\]

\[= \frac{(u+1)/\alpha + (u-1)\alpha}{2u}.\]

(Here the relation \(u^2 - st = u\) is employed for simplification.) With \(X\) treated as a function of \(\alpha\), a "minimum" occurs for

\[\alpha = \left(\frac{u+1}{u-1}\right)^{\frac{1}{2}},\]

and

\[X = (1-u^{-2})^{\frac{1}{2}}\]
Near the P-channel limit the relations corresponding to (I-3) and (I-4) are

\[ r_p = \alpha(u/t)r_T, \quad (I-7) \]

and

\[ r \approx r_p. \quad (I-8) \]

Relations (I-7) and (I-8) substituted into (I-1) yield the cosine approximation

\[ x \approx s^2 + \frac{(u^2-1)\alpha^2(u/t)^2}{2su\alpha(u/t)} \quad (I-9) \]

\[ = \frac{(u-1)/\alpha + (u+1)\alpha}{2u}. \quad (I-10) \]

Differentiation with respect to \( \alpha \) yields a "minimum" for

\[ \alpha = \left(\frac{u-1}{(u+1)}\right)^{1/2}, \]

and

\[ x = (1-u^{-2})^{1/2} \] again.

Hence it is convenient to compute the \( g_K \) integral over the range \((1-u^{-2})^{1/2} \leq x \leq 1\), which generally corresponds to an angular range of only a few degrees for heavy-ion induced transfer reactions. The approximation argument above is not to be construed as a "derivation of a lower limit." In principle, transfers can occur at any angle. These ideas are perhaps best interpreted as the motivation for a very useful default parameter, which
is expected to become irrelevant when the transferred mass is an appreciable fraction of the core masses, or when the binding energy of the T- or B-system is very small.

1.4 Form Factor Generation and Storage

As mentioned in Section 4.8, the transfer form factors are treated as functions of $r_T$ and $r_p - r_T$. This parameterization is useful not only for computer storage purposes but also for the generation of the form factors. Usually the form factors change rather rapidly as functions of $r_p - r_T$ (going across the diagonal), but they fall off relatively slowly as functions of $r_T$ (going along the diagonal). See, for example, figure (4-4). Consequently, a great deal of computation time is saved by using a larger distance between mesh points along the diagonal than across it, and then by interpolating along the diagonal to obtain an array of mesh points equally spaced in either direction. (Such a form factor is then equally applicable to integration for pick-up or stripping reactions.)

Typically the integrations are done with points spaced by $\sim 0.08$ fm. This distance is also the spacing of the mesh points across the diagonal. Points along the diagonal are generally spaced by three times this amount initially. The seven-point Lagrange interpolation formula is then applied along the diagonal to fill in the remaining mesh points. This method has yielded agreement with a direct
form factor computation to four decimal places; the inter-
polation procedure increases the speed of COMBE by a
factor of $\xi 3$.

The two-dimensional form factor arrays must be placed
on the mass storage disk facility for accessibility during
the source term computation. For efficient use of the storage
space, triangularity and parity constraints among $L_T, L_p,$
and $J$ (which specify the summed form factors) are exploited
by the indexing procedure. Each of the arrays is specified
by an integer given by one of the following expressions.

Two-nucleon (or spinless cluster) transfer:

$$\left(\frac{J_{\text{max}}}{2}+1\right)^2L_T + \frac{(L_p - L_T)}{2} + \frac{J}{2}(J/2+1)+1. \quad (I-11)$$

One-nucleon transfer:

$$\left\{\left[\{\frac{(J_{\text{max}}+1)}{2}+\text{JPTY}\}/2\right]-\left[\{\frac{J_{\text{min}}}{2}+\text{JPTY}\}/2\right]\right\}^{2}_{L_T} +$$

$$+ \left[\frac{(L_p - L_T)}{2}\right]+J(J+1)/2 - \frac{J_{\text{min}}(J_{\text{min}}+1)}{2}+1, \quad (I-12)$$

$$\text{JPTY} = \left\{(\frac{L_p + L_T + 1}{2}\right\}/2 = \left\{(-1)^{\ell_i+\ell_j}+1\right\}/2.$$}

The index (I-11) is appropriate to the formulas of Section
G.1.2; the index (I-12) is for use with the formulas of
Section G.2.1. In these index expressions $J_{\text{min}}$ and $J_{\text{max}}$
denote respectively the lower and upper values taken on
by the angular momentum transfer $J$. For two-nucleon transfer,
\( J_{\text{max}} \) equals the largest angular momentum \( J_{AB} \) to be considered for the transferred particles in the B-system.

\[
|J_B - J_A| \leq J_{AB} \leq J_B + J_A.
\]

For one-nucleon transfer, the limits are set by the constraints.

\[
|\ell_b - \ell_t| \leq J \leq \ell_b + \ell_t,
\]

and

\[
|j_b - j_t| \leq J \leq j_b + j_t.
\]

In (I-12) the square brackets denote the greatest integer function.
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