MEASUREMENT OF TRANSITION
QUADRUPOLE AND HEXADECAPOLE MOMENTS
IN EVEN-EVEN RARE EARTH NUCLEI

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ABSTRACT

The deformations of the charge distributions of the eight even-even rare earth nuclei $^{152}$Sm, $^{154}$Sm, $^{158}$Gd, $^{160}$Gd, $^{164}$Dy, $^{166}$Er, $^{168}$Er, and $^{174}$Yb have been investigated in a Coulomb excitation experiment in which elastically and inelastically scattered $^4$He projectiles were detected at back angles in an annular Si surface barrier detector. Up to ten determinations of the $2^+$ and $4^+$ state excitation probabilities for each nucleus, with individual accuracies for each measurement of $1\%-2\%$ and $2\%-4\%$ respectively, were analyzed both with the semi-classical Coulomb excitation calculation corrected for quantal effects using second order perturbation theory, and a full quantal calculation, to yield the reduced transition electric multipole matrix elements (transition moments) $M(E2;0^+\rightarrow 2^+)$ and $M(E4;0^+\rightarrow 4^+)$. These moments are directly related to the intrinsic electric quadrupole and hexadecapole moments, respectively, within the context of the collective model. Large hexadecapole moments and accompanying pronounced E4 contributions to the excitation of the $4^+$ state were observed in $^{152}$Sm, $^{154}$Sm, and $^{158}$Gd; these were found to be smaller for the other nuclei studied. Differences between the perturbation theory quantal corrections and the full quantal calculation were explored. Investigations of the interference between Coulomb and nuclear potential scattering amplitudes were
conducted in order to determine the incident projectile energy range within which the interaction can be considered pure Coulomb. A deformed Fermi charge distribution was assumed in order to calculate the Coulomb deformation parameters $\beta_2^C$ and $\beta_4^C$, from intrinsic moments. These deformation parameters were examined as a parameterization of the charge distribution, and the degree to which they could be related to measurements of the deformation of the nuclear potential was explored.
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CHAPTER I
INTRODUCTION

The discovery of the effect of the nuclear magnetic dipole and electric quadrupole moments on hyperfine atomic spectra precipitated interest in the measurement of these moments, so that by 1950 a large body of data on nuclear ground state moments had been accumulated by methods such as atomic spectroscopy and molecular beams. The inability of the single particle nuclear model to account for the large static quadrupole moments found in the rare earth region was part of the impetus behind the development of the collective model of nuclear structure. While the Bohr and Mottelson model made definite predictions as to the consequences of a statically deformed nucleus—predictions whose experimental verification would soon prove the validity of the model in certain regions—it provided neither a method of predicting ground state deformations, nor a solid connection with nuclear shell theory. These shortcomings were bridged by the model proposed by Nilsson and formulated about two years later. This model provided a framework within which single particle orbits could be calculated from a deformed potential, and ground state deformations could be predicted from these single particle levels.

The approximately simultaneous development of the collective description of nuclear structure and the method
of Coulomb excitation made possible rapid progress in the measurement of quadrupole moments in deformed nuclei, especially in the rare earths. By the middle 1960's more quadrupole moments had been measured in the rare earths than had been known throughout the periodic table ten years earlier. Methods such as Coulomb excitation, nuclear electron scattering, and mesic atoms made it possible to measure quadrupole deformation in even-even nuclei whose zero ground state spin makes these moments unobservable by conventional atomic spectroscopy.

Several recent studies of static shape deformations in nuclei have focused attention on the effects of higher moments of both the charge and mass distributions on single particle dynamics and on collective structure of nuclei, especially in the lanthanide and actinide regions. In particular hexadecapole deformation, the next highest order after quadrupole, has been studied by a variety of techniques. The existence of a significant hexadecapole component in the shapes of rare earth nuclei is indicated both by the successful inclusion of a \( \beta_4 \) term (that is a term proportional to the spherical harmonic \( Y_{40} \)) in the potentials employed in recent nuclear structure calculations in this region, and by the measurement of hexadecapole contributions to deformed optical potentials used to describe nuclear scattering from these nuclei. Hexadecapole deformations can have a significant effect on the magnitude and sign of the dominant quadrupole shapes of
deformed nuclei\textsuperscript{12}, and the extrapolation of these influences into the region of postulated stable superheavy nuclei may have an important effect on the determination of islands of stability.

The effects of higher deformations on nuclear scattering were demonstrated by studies conducted by Hendrie, et al.\textsuperscript{9} and Aponick, et al.\textsuperscript{10} of elastic and inelastic scattering of $^4$He projectiles to members of the ground state rotational bands of rare earth nuclei. Both investigations were performed with $^4$He incident energies significantly above the Coulomb barrier, where nuclear potential scattering dominates. The analysis of the Hendrie, et al. experiments, which were performed at an incident energy of 50 MeV, involved the use of the same spherical optical potential parameters throughout the region bordered by $^{148}$Sm and $^{178}$Hf. The dominant dependence of the rapidly varying differential cross-sections on the deformation parameters demonstrated the considerable sensitivity of the $(\alpha,\alpha')$ probe to the shape of the nuclear potential surface.

The quadrupole and hexadecapole deformation parameters, $\beta_2$ and $\beta_4$, determined from the analysis of both the experiment conducted by Hendrie et al., and the experiment performed by Aponick, et al. are consistent with present theoretical predictions\textsuperscript{12,19}. The latter experiment extended these measurements to several lower incident energies to test the calculational procedures used to extract the deformation parameters and the interpretation of these
parameters as a physical representation of the shape of the nuclear potential. An attempt was also made to observe the effects reflecting Coulomb excitation which would yield information on the charge deformation. This second objective was not realized, due in part to the fact that the lowest incident energy employed was more than twice as great as the Coulomb barrier for scattering at back angles, so that the Coulomb contribution was obscured by nuclear scattering effects.

Although the description of the nuclear deformation in terms of the parameters of a deformed surface $\beta_\lambda$ fits conveniently into theoretical calculations of nuclear deformation and the formulation of deformed optical potentials, the nuclear moments can only be calculated from measured deformation parameters through the use of a specific model of the nuclear charge distribution, and several important assumptions about the relation of charge, mass and potential deformations must be made in order to do so. Coulomb excitation experiments yield nuclear moments more directly, owing to the nature of the analysis of the experiment and the fact that the interaction of the incident projectile is with the electromagnetic field of the nucleus. Much of the information accumulated on quadrupole moments has come from the measurements of static and transition $E2$ matrix elements. In rotational nuclei, such as those studied in this experiment, the static matrix elements are rather difficult to measure, and the transition $E2$ matrix elements are directly
related to the quadrupole moment through the rotational model. The measurement of $E4$ transition matrix elements can yield a similar determination of the hexadecapole moment. In the context of the rotational model which, it will be seen in a later section, provides a good description of low-lying states in even-even rare earth nuclei, the $0^+ - 2^+$ E2 matrix element is equal to the intrinsic quadrupole moment of the nucleus and the $0^+ - 4^+ E4$ matrix element is equal to the intrinsic hexadecapole moment of the nucleus. These moments can be related to a description of the deformation in terms of $\beta^*$ deformation parameters through the use of a model of the charge distribution as mentioned above; the nature of that relationship will be discussed in a later section.

The first Coulomb excitation measurement of hexadecapole deformations in the rare earths, by Stephens et al. \cite{40} employed alpha particles as incident projectiles. By detecting the de-excitation gamma-rays following Coulomb excitation of $^{152}\text{Sm}$ and $^{154}\text{Sm}$ they were able to deduce from measured matrix elements $\beta_2^C$ and $\beta_4^C$ (where the superscript refers to Coulomb deformation) for a particular specific charge distribution. The values of $\beta_4^C$ are larger than the nuclear potential deformation parameters $\beta_4^N$ determined by the higher energy scattering experiments discussed above by roughly a factor of two. An electron scattering measurement of $\beta_4^C$ of $^{152}\text{Sm}$ \cite{13} determined a value which, while not as large as that determined by Stephens, et al. \cite{40} is still
significantly larger than the measured $\beta_4^N$.

The present experiment was conducted to make accurate measurements of the transition quadrupole and hexadecapole moments of these samarium isotopes by a Coulomb excitation technique which avoids the uncertainties involved in detecting the de-excitation gamma-rays, and to extend these measurements through the first half of the rare earths where the hexadecapole deformation is expected to fall from the large values at the samarium isotopes to approximately zero. In addition, an exploration of the dependence of the extracted Coulomb deformation parameters on the form and parameterization of the charge distribution was undertaken in an effort to understand in more detail the disagreement with the nuclear potential deformation parameters observed by Stephens, et al. Concurrent measurements have been reported by Brückner, et al.\textsuperscript{29)}, and Ebert, et al.\textsuperscript{43)} for the samarium isotopes, and by Erb, et al.\textsuperscript{42)} for several nuclei through the region. Similar measurements have been conducted in the actinide region by Bemis, et al.\textsuperscript{44)}.

The various theoretical formulations and experimental methods mentioned in the preceding discussion are all attempts to describe and measure the same thing: the deformation of the nucleus. While connections among them are not always perfectly clear, complementary information from these many sources can be drawn together to produce a coherent picture of nuclear deformation. The next several sections will deal with models of nuclear deformation, and
methods of probing the nuclear potential and charge distributions. First the theoretical basis for the treatment of nuclear deformations will be outlined, with special emphasis on the features of the rotational model that are fundamental to the analysis of this experiment. This is followed by a discussion of different descriptions of nuclear deformation and the information that can be obtained through various experimental probes of the nuclear potential surface and the charge distribution.

A. MODELS OF NUCLEAR DEFORMATION

While the approach toward nuclear deformations taken in this dissertation is fundamentally an experimental one, the study of such deformations has a firm theoretical basis. The analysis of this experiment is dependent upon well established and experimentally verified features of the collective model of nuclear structure, and the optical model potentials used in the analysis of the nuclear potential scattering experiments mentioned in the previous section are intimately related to the potentials used in formulations of nuclear deformation based upon single particle dynamics. These theoretical approaches are summarized here with particular emphasis upon the important features of the collective model pertaining to this experiment. The brief descriptions of these models presented here also indicate the importance of measured deformations to the understanding of nuclear structure through such models.
l. The Collective Model

The large quadrupole moments found in the rare earth region prompted J. Rainwater to speculate that a spheroidal nuclear shape might give a better description of nuclei in that region than spherical shell models. Bohr and Mottelson calculated the behavior of a more general deformed body by beginning with the quantum mechanical analog of the hydrodynamic model. The underlying assumption of their model is that the nuclear Hamiltonian is separable into two parts:

\[ H = H_{\text{collective}} + H_{\text{intrinsic}} \]

with the addition of a coupling term, if necessary. The collective frequencies are assumed to be much smaller than the intrinsic particle frequencies, so that the wave function may be written in product form:

\[ \Psi = \phi_{\text{collective}} \chi_{\text{intrinsic}} \]

The collective behavior is treated as an analogue of an incompressible liquid drop whose surface is given by:

\[ R(\theta, \phi) = R_0 \left[ 1 + \sum_{\lambda, \mu} \alpha_{\lambda \mu} Y_\lambda^\mu(\theta, \phi) \right] \]

where the \( \alpha_{\lambda \mu} \) are treated as generalized co-ordinates. The expansion is truncated at \( \lambda = 2 \), and transformed into the body-fixed co-ordinate system:

\[ a_{2\nu} = \sum_{\mu} D_{\mu \nu}^2 a_{2\mu} \]

With a further change of co-ordinates
\[ a_{20} = \beta \cos \gamma \]
\[ a_{2\pm 2} = \frac{1}{\sqrt{2}} \beta \sin \gamma \]

kinetic energy becomes:

\[ T = \left( \frac{1}{2} \beta \dot{\gamma}^2 + C \dot{\beta}^2 \gamma^2 \right) + \frac{1}{2} \sum_{k=1}^{3} I_k \omega_k^2. \]

The kinetic energy consists of a vibrational part (the first term), and a rotational part. If this separation is to be valid, the rotational frequencies must be smaller than the vibrational frequencies. The spectrum of this idealized collective nucleus therefore consists of a rotational band built on the ground state, with quadrupole vibrational states (with rotational bands built upon them) at higher energies, and intrinsic particle excitations at higher energies yet. This is analogous to the spectra of strongly deformed molecules. If this complete separation is valid, the rotational part of the wavefunction is given by the rotational function \( D^J_{MK} \) and the rest of the nuclear properties are contained in the wavefunction \( \chi_K \). The complete normalized and symmetrized wavefunction is given by:

\[ \psi(JMK, \alpha) = \left[ \frac{2J + 1}{16\pi^2 (1 + \delta_{K0})} \right]^{1/2} \left( D^J_{MK} \chi_K^\alpha + (-)^J D^J_{MK, -K} \chi_K^\alpha \right) \]

where \( J \) is the nuclear spin, \( M \) the projection of the space-fixed z-axis, and \( K \) the projection on the body-fixed
symmetry axis. Only these three quantum numbers are necessary to describe the rotational state; $\alpha$ represents all other quantum numbers needed to describe the total nuclear state.

The total angular momentum $J$ is the vector sum of the rotational contribution and the intrinsic spin of the particle configuration. In the ground state the rotation contribution is zero, so that $J$ is the intrinsic spin. The lowest energy occurs for the intrinsic spin projection equal to its maximum value\(^5\), so that in the ground state $K = J = \text{the intrinsic spin}$. In the case of even-even nuclei, $K = J = 0$.

Since there can be no rotation about a symmetry axis, in an axially symmetric nucleus the projection of the rotational angular momentum on the symmetry axis is zero, and $K$ has the same value for the entire rotational band.

Since $K = 0$ for the ground state band of an even-even nucleus, the wavefunction vanishes unless $J$ is even:

$$\psi(JM, K=0, \alpha) = \left[ \frac{2J+1}{16\pi^2(2)} \right]^{1/2} \left( D_{M0}^J x_0^\alpha + (-)^J D_{M0}^J x_0^\alpha \right)$$

so that the spin sequence of the ground state band of an even-even nucleus is $0^+, 2^+, 4^+, \ldots$. For such a rotational band the rotational energy is given by:

$$E(J) = \frac{\hbar^2}{2I_0} J(J+1)$$

and the reduced electric multipole transition matrix
elements are:

\[ \langle J_f | | M(\lambda \alpha) | | J_i \rangle = Q^\lambda \langle J_i \lambda_00 | J_f 0 \rangle \sqrt{2J_i + 1} \]

where \( Q^\lambda \) is the intrinsic electric moment of multipolarity \( \lambda \). \( \langle J_i \lambda_00 | J_f 0 \rangle \) is the Clebsch-Gordan coefficient for the angular momentum coupling \( J_i + \lambda = J_f \) with all projections equal to zero. This relationship is demonstrated in Appendix II, as is the relationship of the intrinsic moments to the spectroscopic moments:

\[ Q^\lambda = \langle J\lambda_00 | JJ \rangle \langle J\lambda K0 | JK \rangle Q^\lambda \]

For a direct transition from the ground state the matrix element is given by:

\[ \langle \lambda | | M(\lambda \alpha) | | 0 \rangle = Q^\lambda \]

While the spin sequence of the low-lying states is the most easily identifiable "signature" of a rotational nucleus, quantitative data on the degree to which the properties of those states conform to rotational predictions is best obtained (and most usually obtained) from the energies of these states and the reduced E2 transition rates. The latter are defined by:

\[ B(E2; J_i \rightarrow J_f) = (2J_i + 1)^{-1} |\langle f | | M(E2) | | i \rangle|^2 \]
and predicted by the rotational model to be:

\[(2J_1 + 1)^{-1} (Q_2^2)^2 |\langle J_1 \lambda 00 | J_\pi 0 \rangle|^2\]

For the first two excited states in an even-even rotor nucleus the following relationships therefore hold:

\[\frac{E_{4+}}{E_{2+}} = \frac{4(5)}{2(3)} = 3.333\]

\[\frac{B(E2; 4^+ \rightarrow 2^+)}{B(E2; 2^+ \rightarrow 0^+)} = 1.43\]

A reasonably large body of data now exists on energies of and quadrupole transition rates between low-lying states in even-even rare earth nuclei. Parts of this data have been compiled in several places.\(^{15,16,17}\) It is pointed out by the authors of these compilations that the data come from many different types of experiments conducted over a period of roughly fifteen years during which the understanding of the mechanisms involved in those experiments has increased. The data are therefore sometimes conflicting and subject to different interpretations. In general, however, the data indicate that from \(^{154}\)Sm to \(^{180}\)Hf the experimental determinations of \(^2^+\) and \(^4^+\) state excitation energies and \(^0^+\rightarrow^2^+\) and \(^2^+\rightarrow^4^+\) transition rates agree with the rotational model predictions. The energy ratios are rotational to within 1%-2%, and the ratios of the transition rates agree with
the model to within one standard deviation, typically 5%. Bar, et al.\textsuperscript{17}) calculate an average value of the ratio $B(E2; 4^+ \rightarrow 2^+)/B(E2; 2^+ \rightarrow 0^+)$ for the nuclei from $^{152}\text{Sm}$ to $^{188}\text{Os}$. They obtain a value of $1.43 \pm 0.07$, the implication of which is that the properties of the $2^+$ and $4^+$ states within this region are rotational to about 5%. Such an analysis, of course, glosses over the differences between the individual nuclei, but also does the same for experimental differences, and provides an overall picture of the region. In addition to this quantitative agreement with the rotational model, there is also agreement in the general structure: the ground state rotational bands have been observed up to $8^+$ and $10^+$ states, with the sequence agreeing with rotational predictions, and there are one phonon vibrational states roughly one order of magnitude higher in energy than the first $2^+$ state, with rotational bands built upon those states. It should be noted that agreement with rotational behavior decreases as the spin of states within a band increases, and for states of very high spin (i.e. 14 to 22)\textsuperscript{45}) large significant deviations appear.

This agreement with rotational model behavior has several practical consequences. The collective model description of the low excited states of these nuclei being due to rotations of a rigid body having a fixed shape is valid within the limits of the above measurements, and in the body-fixed co-ordinate system the shape of the nucleus in its lowest states can be considered the same as the
intrinsic shape in the ground state. More rigorously, the separation of the wavefunction into a rotational part and a part containing all other nuclear properties is consistent with existing data, and the intrinsic wavefunction is the same in the first $2^+$ and $4^+$ states as it is in the ground state. This implies that the deformation and the intrinsic electric moments are the same in the low-lying rotational states as they are in the ground state. Furthermore, the use of rotational values for properties that have not been directly measured—for example, the diagonal E2 matrix elements—or those that have not been measured to high accuracy, is reasonable.

2. Motion of Nucleons in Deformed Potentials

Because it is based on the assumption of the separability of collective and particle degrees of freedom, the formulation of the collective model avoids the issue of the origin of the nuclear shape. In this sense the approach is phenomenological: given a body whose shape is given by the hydrodynamic form and which can oscillate about this form, what are the consequences? The equilibrium shape is calculated from experimental data on nuclear deformation; for example, measured $\beta_2$'s—rather than from more fundamental principals which are then used along with the model to predict the deformation. It provides neither a method to calculate shapes from first principles nor a connection with earlier nuclear theory based on the shell model.
It would be clearly advantageous to supply the "missing link" so that the shell model and collective model could be used to provide a uniform model of the nucleus. A model provides a more satisfying description of nuclear structure if it can be used to calculate both parts of the wavefunction.

The first attempt to calculate the orbits of single particles in a deformed potential was by Nilsson\(^6\). His calculations were done for an axially symmetric shape of purely quadrupole deformation, which is essentially the same shape used by Bohr and Mottelson\(^15\) when truncating the expansion of the nuclear surface at \(\lambda=2\).

Nilsson's approach was to solve the single particle wave equation for an oscillator potential with spin orbit coupling.

\[
H = H_0 + \frac{\hbar}{2} \vec{s} \cdot \vec{l} + D\ell^2
\]

\[
H_0 = -\frac{\hbar^2}{2m} \vec{v}^2 + \frac{m}{2} \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right)
\]

The deformation was included by deforming the oscillator wells in the following manner:

\[
\omega_x^2 = \omega_0^2 \left[1 + \left(\frac{2}{3}\right) \delta \right] = \omega_y^2
\]

\[
\omega_z^2 = \omega_0^2 \left[1 - \left(\frac{4}{3}\right) \delta \right]
\]

\[1\]
with volume conservation leading to the requirement:

$$\omega_x \omega_y \omega_z = \text{constant}$$

$$\omega_0(\delta) = \omega_0 \left[1 - \left(\frac{4}{3}\right)\delta^2 - \left(\frac{16}{27}\right)\delta^3 \right]^{-\frac{1}{6}}$$

$$\omega_0 = \omega_0(\delta=0)$$

Nilsson's deformation parameter is related to the Bohr and Mottelson quadrupole expansion by:

$$\delta \approx \left(\frac{3}{2}\right) \sqrt{\frac{5}{4\pi}} \beta \approx 0.958$$

Ground state deformations were found by minimizing the total energy as a function of $\delta$:

$$\left(\frac{\partial E(\delta)}{\partial \delta}\right)_{\delta\text{equil.}} = 0$$

for the odd nucleon in deformed regions. It should be noted that Nilsson later chose a second parameterization of the nuclear deformation which was used in the subsequent papers by Mottelson and Nilsson. 18)

The calculation of the Nilsson levels for a single nucleon in a deformed potential allows the determination of the ground state configuration and the lower-lying intrinsic states in odd mass nuclei, as well as the equilibrium
deformation, through measurements of intrinsic state excitation energies and spins. Most important, it demonstrates the connection between the collective model and the well known shell structure—the unified model. In order to extend the calculation to even-even nuclei, it is necessary to include the pairing of both the neutrons and the protons. More recent calculations include the pairing force, the effect of the Coulomb energy, and the $\beta_4$ deformation.19,20,12

It is thus possible to start from first principles, calculate single particle orbits in a deformed potential, add the contributions of these orbits to predict a ground state shape, and calculate the consequences of collective motion of that shape—i.e. the energies of rotational states, the transition matrix elements, and the intrinsic moments. The measurement of these quantities then provides evidence against which the theoretical calculation can be checked. Properly obtained experimental data on these nuclear properties form a body of independent data on the nuclei being studied. It is against these data that the validity of the theoretical calculation is gauged.

The various nuclear models discussed above have all used similar parametrizations of the nuclear shape. The collective model formulation begins with the general form taken from the hydrodynamic model for the nuclear surface:

$$R(\theta, \phi) = R_0 \left[ 1 + \sum_{\lambda=1}^{\infty} \sum_{\mu=-\lambda}^{\lambda} \alpha_\lambda^\mu \, Y_\lambda^\mu (\theta, \phi) \right]$$
which should be completely general, since it is an expansion over a complete set of orthonormal functions, the spherical harmonics. The transformation into the body fixed coordinates as performed by Bohr and Mottelson\(^5\) is accompanied, however, by a truncation of the series at \(\lambda = 2\), so that the highest order deformation is quadrupole. The original Nilsson\(^16\) formulation keeps the truncation, but changes the form, allowing only shapes that are axially symmetric spheroids. An approximate equivalence of Nilsson's deformation parameter and that of Bohr and Mottelson for axially symmetric shapes (\(\gamma = 0\)) is calculated. More recent models include the \(\lambda = 4\) term, but are not necessarily consistent with each other, either in definition of deformation parameters or in the assumed shape: for example, some treat only axially symmetric shapes while others treat a more general shape. Reference 12) treats non-axially symmetric \(\beta_2\) distortions, but confines \(\beta_4\) to be the coefficient of \(Y_4^0\) only. In Reference 20), \(\beta_2\) and \(\beta_4\) multiply only Legendre polynomials of the appropriate order. The potentials used to analyze some recent experiments have had a surface shape given by:\(^{21,9,10}\)

\[
R(\theta) = R_0 \left[ 1 + \beta_2 Y_2^0(\theta) + \beta_4 Y_4^0(\theta) + \ldots \right]
\]

which, like those used in the models, is restricted in its generality, although not severely so, as evidence strongly indicates that the ground state shapes of most even-even rare earths are axially symmetric.\(^{12}\)
B. MEASUREMENTS OF DEFORMATION OF NUCLEAR POTENTIAL SURFACE

The deformation of the nuclear potential has been studied by the analysis of elastic and inelastic scattering of alpha particles\textsuperscript{19,10,29} and protons.\textsuperscript{21,30} In these experiments the scattering is conducted at energies far above the Coulomb barrier, so that the nuclear force dominates by large factors, and the Coulomb interaction need not be specified very accurately. These experiments are, therefore, very sensitive to the shape of the nuclear potential and very insensitive to the moments of the nuclear-Coulomb interference.\textsuperscript{29}

The methods followed in all the analyses were in general the same. The rotational transitions were treated explicitly in a coupled-channels calculation, and the effects of all other states (which presumably had a much smaller probability of entering into the interaction) were approximated by the optical model. The optical model is used to reduce an infinite number of channels to a small number, possibly one.\textsuperscript{31} The aim is to parameterize the effects of all possible interactions without treating them explicitly or knowing their specific form. The basic optical potential consists of a real well plus terms that account for absorption, spin orbit interaction, etc. (see Preston chap. 18).\textsuperscript{32} The deformation is added by expanding about the spherical potential:\textsuperscript{33}

\[
R(\theta) = R_0 \left[ 1 + \sum_{\lambda} \beta_\lambda Y^0_\lambda (\theta) \right] = R_0 (1 + \delta R)
\]
The real part of the potential has a Saxon-Woods form, which is identical to a spherical Fermi distribution used for the charge discussed in the next section, so that in this sense the potential is assumed to have the same shape as a deformed Fermi distribution whose surface is given by the above expression for \( R(\theta) \).

To obtain spherical optical parameters for the rare earth region from \(^{178}\text{Hf}\), Hendrie, et al\(^9\)) and Aponick, et al\(^10\)) studied scattering under identical conditions from \(^{148}\text{Sm}\), which is a spherical nucleus and applied the parameters thus obtained throughout the entire region, varying only the radius by \( A^{1/3} \). These experiments
demonstrated the sensitivity of the cross sections to the surface shape and the validity of fixing the spherical optical parameters by fitting the data throughout the region by varying the deformation parameters only. Aponick et al. measured the same deformation parameters at several incident energies. The quantities that actually parameterize the deformed optical potential are $\beta \cdot R_0^8$ (29). This can be seen by examining the above expression for $N_L$. For $n=1$, $N_L$ is proportional to $R_0^8 \beta_L$, for $n=2$, it is proportional to $R_0^8 \beta_L + R_0^2 \beta_L^2 \beta_L^0$, and so on. In addition, the $r_0$ obtained from the optical model includes the radius of the projectile as well as the target nucleus, and this must first be corrected for (the projectile is assumed to be spherical). Extracted $\beta_\lambda$'s must then be scaled according to the $r_0$ used if comparisons between different determinations are to be made. The results of the experiments discussed above are listed in Table I-1.

C. DETERMINATIONS OF THE SHAPE OF THE NUCLEAR CHARGE DISTRIBUTION

The charge distribution of deformed nuclei can be specified in the same manner as the nuclear potential, by postulating a model which has a deformed surface. However, the shape of the charge distribution can also be specified in terms of the multipole moments of the charge distribution $q_{lm}$, which is an equivalent description which can be easily related to any model parameterization, and is more compatible with the analysis of a Coulomb excitation experiment.
Deformation parameters measured by various nuclear scattering experiments

<table>
<thead>
<tr>
<th>NUCLEUS</th>
<th>$\beta_2$</th>
<th>$\beta_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{152}_{\text{Sm}}$</td>
<td>$0.246^a)</td>
<td>$0.048^a)</td>
</tr>
<tr>
<td></td>
<td>$0.25^{+0.02}_c)</td>
<td>$0.050^{+0.015}_c)</td>
</tr>
<tr>
<td></td>
<td>$0.279^{+0.009}_e)</td>
<td>$0.050^{+0.009}_e)</td>
</tr>
<tr>
<td>$^{154}_{\text{Sm}}$</td>
<td>$0.270^a)</td>
<td>$0.054^a)</td>
</tr>
<tr>
<td></td>
<td>$0.225^{+0.005}_b) ($0.280$</td>
<td>$0.050^{+0.005}_b) ($0.062$</td>
</tr>
<tr>
<td></td>
<td>$0.250^d)</td>
<td>$0.050^d)</td>
</tr>
<tr>
<td>$^{158}_{\text{Gd}}$</td>
<td>$0.282^a)</td>
<td>$0.036^a)</td>
</tr>
<tr>
<td>$^{166}_{\text{Er}}$</td>
<td>$0.276^a)</td>
<td>$0.00^a)</td>
</tr>
<tr>
<td></td>
<td>$0.230^{+0.005}_b) ($0.286$</td>
<td>$0.00^{+0.005}_b) (0.0)</td>
</tr>
<tr>
<td>$^{174}_{\text{Yb}}$</td>
<td>$0.276^a)</td>
<td>$-0.048^a)</td>
</tr>
<tr>
<td>$^{176}_{\text{Yb}}$</td>
<td>$0.276^a)</td>
<td>$-0.054^a)</td>
</tr>
<tr>
<td></td>
<td>$0.230^{+0.005}_b) ($0.286$</td>
<td>$-0.035^{+0.01}_b) (-0.044)</td>
</tr>
</tbody>
</table>

a) reference 9
b) reference 10; $r_0 = 1.49$. The numbers in parenthesis represent $\beta_\lambda$ scaled so that $r_0 = 1.2$.
c) reference 30
d) reference 21
e) reference 29
The electric potential outside any region containing charges can be written as:

\[ \phi(\vec{x}) = 4\pi \sum_{\ell,m} \frac{1}{2\ell+1} q_{\ell m} r^{-\ell-1} Y_{\ell}^{m}(\theta,\phi). \]

This is a general expression from classical electrostatics which holds for any point outside a region containing charges. This can also be written as:

\[ \phi(\vec{x}) = \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \, d\tau', \]

where \( \rho(\vec{x}') \) is the charge density. The primed co-ordinates refer to the region containing the charges, the un-primed to the region outside. This is expanded to give:

\[ \frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{\ell,m} \frac{1}{2\ell+1} \frac{r_<'}{r_>^{\ell}} Y_{\ell}^{m*}(\theta',\phi') Y_{\ell}^{m}(\theta,\phi) \]

\[ \phi(\vec{x}) = \int \left\{ 4\pi \sum_{\ell,m} \frac{1}{2\ell+1} \frac{r_<'}{r_>^{\ell}} Y_{\ell}^{m*}(\theta',\phi') Y_{\ell}^{m}(\theta,\phi) \right\} \rho(\vec{x}') \, d\tau', \]

and

\[ \phi(\vec{x}) = 4\pi \sum_{\ell,m} \frac{1}{2\ell+1} Y_{\ell}^{m}(\theta,\phi) \int r_<' Y_{\ell}^{m*}(\theta',\phi') \rho(\vec{x}') \, d\tau' \]

so that the multipole moments can be obtained from:

\[ q_{\ell m} = \int r_<' Y_{\ell}^{m*}(\theta',\phi') \rho(\vec{x}') \, d\tau'. \]
The multipole moments of a nucleus are defined by:
\[
Q_\lambda = \int \psi^*_J,M=J Q_{\lambda J} \psi_J,M=J \, d\tau'
\]
where the expectation value is taken in the state of maximum angular momentum projection. The charge distribution can be exactly specified if all the multipole moments can be measured, so that describing the deformation this way has the same generality as specifying the complete set of deformation parameters.

It is important to note that \(Q_{\lambda m}\) are the same operators used to obtain the \(E\lambda\) matrix elements. The rotational model relationships obtained in Appendix II and discussed in an earlier section which give the \(E\lambda\) matrix elements as proportional to the intrinsic moments of the same multipolarity seem almost intuitively obvious, because of the basic assumption of the rotational model that the charge distribution in the excited state is the same as the charge distribution in the ground state.

The multipole moments of a nucleus whose surface shape is defined by a set of deformation parameters \(\beta_\lambda\) can be calculated if several assumptions are made:

1) The nuclear charge and mass are distributed in the same manner.

2) The nuclear shape is the same as the shape of the nuclear potential—since the deformation parameters are calculated either from the motion of nucleons in a deformed potential or from the analysis of a scattering experiment using a deformed potential.
3) The shape is well determined by the $m=0$ spherical harmonics, since the deformation parameters are only defined for this set.

4) The size of the contributions decrease with $\lambda$, and the series---both the $B^\lambda$ and the multipole expansion---can be terminated after a finite, hopefully small, number of terms.

5) The form of the nuclear charge distribution, $\rho(\tilde{x})$, is known.

The nuclear moments can always be calculated from a postulated (or measured) charge distribution. In this manner measurements of nuclear moments and information from other probes of the charge distribution can be combined to produce a coherent picture of the shape of the charge distribution while avoiding the first two assumptions, which are far from obvious.

Although both the parameterization of the deformed shape in terms of $B^\lambda$ deformation parameters and the description in terms of multipole moments are expansions in spherical harmonics, the coefficients of the same spherical harmonic in both expansions will not, in general, be the same, and in general each multipole moment will contain contributions from all the $B^\lambda$'s. For example, one commonly used charge distribution is the uniform charge distribution with a sharp cut-off defined by:
\[ p(r) = \begin{cases} 
0 & r < R(\theta) = R_0 \\
1 + \beta_2 Y_2(\theta) + \beta_4 Y_4(\theta) + \beta_6 Y_6(\theta) & r > R(\theta) 
\end{cases} \]

\[ \rho(r) = \begin{cases} 
0 & r < R(\theta) = R_0 \\
\rho_0 & r > R(\theta) 
\end{cases} \]

\[ Q_2' \text{ and } Q_4' \text{ are given by:} \]

\[ Q_2' = \int d\Omega \int_0^{R(\theta)} dr \ r^2 \ \rho_0 \ r^2 \ Y_2^0 \]

\[ Q_4' = \int d\Omega \int_0^{R(\theta)} dr \ r^2 \ \rho_0 \ r^4 \ Y_4^0 \]

Performing the \( dr \) integral:

\[ Q_2' = \rho_0 \frac{R_0^5}{5} \int d\Omega \left( 1 + \beta_2 Y_2(\Omega) + \beta_4 Y_4(\Omega) + \beta_6 Y_6(\Omega) \right)^5 Y_2(\Omega) \]

\[ Q_4' = \rho_0 \frac{R_0^7}{7} \int d\Omega \left( 1 + \beta_2 Y_2(\Omega) + \beta_4 Y_4(\Omega) + \beta_6 Y_6(\Omega) \right)^7 Y_4(\Omega) \]

and doing the angular integration:

\[ Q_2' = ze \left( \frac{1}{\Omega^3} \right)^2 \left[ .2387 \beta_2 + .0861 \beta_2^2 + .2309 \beta_2 \beta_4 
+ .0782 \beta_4^2 + .0766 \beta_6^2 + .2278 \beta_4 \beta_6 \right] \]
\[ Q_4 = Ze \left( \frac{1}{r_0 A^{\frac{1}{3}}} \right)^4 \left[ 0.2387 \beta_4 + 0.1732 \beta_2^2 + 0.2346 \beta_2 \beta_4 \\ + 0.3417 \beta_2 \beta_6 + 0.0908 \beta_6^2 + 0.2038 \beta_4 \beta_6 \\ + 0.0981 \beta_4^2 \right] \]

which yields each moment as a function of all the deformation parameters. The integrations are shown in detail in Appendix I.

An important consequence of this is that in the region of mass 165 to 175 where \( \beta_4 \) is predicted to be close to zero, the hexadecapole moment should be non-zero and therefore measureable. However, in the region where \( \beta_4 \) becomes negative the hexadecapole moment may vanish, and may be difficult to measure even for large negative values of \( \beta_4 \).

This calculation was performed for the simplest deformed charge distribution. Experimental evidence from sources such as electron scattering indicate that the charge distribution actually has a more complicated shape, and the moments are therefore not such "simple" functions of the deformation parameters.

A large body of experimental information on the nuclear charge distribution has been accumulated from Coulomb excitation and lifetime measurements, from electron scattering, and from studies of x-ray transitions in mu-mesic atoms. By measuring E\( \lambda \) matrix elements Coulomb excitation
and lifetime experiments provide a determination of the intrinsic nuclear moments which describe the deformation in a model independent manner, as demonstrated in the preceding discussion. These other experiments probe the interior of the nucleus, to which Coulomb excitation is insensitive, and provide a test of the models of the charge distribution. This is important if the information contained in the moments is to be described in terms of deformation parameters. The nuclear charge distribution has also been studied by atomic spectroscopy, molecular beams techniques, and isotope shifts in mesic atoms and normal atoms.

1. Electron Scattering

Electron scattering has been a valuable tool for determining which postulated charge distributions are most realistic. Low energy electron scattering is capable of determining only one parameter (the mean nuclear radius), but higher energy elastic scattering can determine two or more parameters of the radial distribution, and inelastic scattering can be used to study the deformation. Since the middle 1950's good fits have been obtained to electron scattering data in the rare earth region using a spherical Fermi distribution:

\[ \rho(r) = \rho_0 \left[ 1 + e^{\left(\frac{r-c}{t}\right)} \right]^{-1} \]
although several others of the same general form have been tried. Reference 25) contains a sample of fits to data for scattering from neodymium isotopes using various forms of the charge distribution. Only those distributions which are similar in form to the Fermi distribution produce reasonable fits; the uniform distribution is worse by a factor of two, and shapes which do not have a flat central region are much worse. In general, as the nuclear mass decreases, the flat region becomes narrower and narrower (consistent with a reduction in the nuclear radius) until it no longer exists. For light nuclei a distribution closer to an exponential provides the best fits to the data.24)

The cross-sections for the scattering of relativistic electrons from nuclei are obtained by solving the Dirac equation in the Born approximation:34)

\[
\frac{d\sigma(\theta)}{d\Omega} \bigg|_{\text{B.A.}} = \left( \frac{Ze^2}{2E_1} \right) \frac{\cos^2(\theta/2)}{\sin^4(\theta/2)} \left| F \right|^2 \left( 1 + \frac{2E_1}{M_Tc^2} \sin^2 \theta \right)
\]

where

\[ q = |\vec{p}_1 - \vec{p}_f| = \text{momentum transfer} \]

\[ M_T = \text{nuclear mass} \]

\[ \theta = \text{scattering angle} \]
All the nuclear structure information is contained in the form factor, $|F|^2$, which for elastic scattering is given by:

$$|F_{el}|^2 = |1 - \frac{q^2}{6} <r^2> + \frac{1}{120} q^4 <r^4> + \ldots|^2$$

and for inelastic scattering:

$$|F|^2 = \sum_{\lambda} |F_{c\lambda}|^2 + \sum_{\lambda} |F_{e\lambda}|^2 + \sum_{\lambda} |F_{m\lambda}|^2$$

where

$c\lambda = $ coulomb multiples

e$\lambda = $ transverse electric multiples

$m\lambda = $ transverse magnetic multiples

The transverse magnetic form factor is small enough to be neglected, and the transverse electric form factor can be written in terms of, and as a correction to, the Coulomb form factor, but is also usually considered small enough to be neglected\(^34\). The resulting form factor for a one-step inelastic excitation takes the form:
\[ |F_{\text{inel}}|^2 = q^{2L} \frac{4\pi}{Z^2} \frac{B(EL^+)}{[(2L+1)!!]^2} |1 - \frac{q^2}{2(2L+3)} \frac{I_{L+2}}{I_L} |^2 \]

\[ + \frac{q^4}{8} \frac{1}{(2L+5)(2L+3)} \frac{I_{L+4}}{I_L} + \ldots |^2 \]

where

\[ I_k = \int_0^\infty r^{k+2} \rho_L(r)dr \]

\[ <LM|\rho|00> = \sum_{L,M} \rho_L(r)Y_L^M*(\Omega) \]

This is the same as expanding the charge distribution in a series:

\[ \rho(r,\theta) = \rho_0(r)Y_0^0(\theta) + \rho_2(r)Y_2^0(\theta) + \rho_4(r)Y_4^0(\theta) + \ldots \]

\[ |F|^2 \] can usually be truncated:

\[ |F_{\text{el}}|^2 = |1 - \frac{q^2}{6} <r^2>|^2 \]

\[ |F_{\text{inel}}|^2 = q^{2L} \frac{4\pi}{Z^2} \frac{B(EL^+)}{[(2L+1)!!]^2} |1 - \frac{q^2}{2(2L+3)} \frac{I_{L+2}}{I_L} |^2 \]

The elastic scattering data can be used to extract the mean square radius, but since measurements can be made at
several different values of the momentum transfer, it
is also possible to obtain several parameters of a charge
distribution by measuring several data points. As might be expected, the elastic cross-section is most sensitive to the radial part of the charge distribution; it is often calculated in terms of \( \rho_0 \). As discussed above elastic electron scattering data has been used to successfully discriminate between different postulated charge distributions, and is used to define a region in \( c, t \) space for a Fermi distribution. It is most useful in this application rather than in the measurement of the mean square radius which can be obtained more accurately from mu mesic atoms, and isotope shift studies. Determinations of \( c, t \) can in principle be made with more certainty from electron scattering data than by studies of mu mesic atoms since measurements at many values of \( q \) can be used to determine the two parameters.

Using the parameters of the radial charge distribution established from the elastic scattering, the inelastic form factors can be used to extract the deformation parameters: \( \beta_2 \) from the \( 2^+ \) state, \( \beta_4 \) from the \( 4^+ \) state, and so on. In practice, experimental resolution makes it difficult to resolve the \( 2^+ \) states of highly deformed nuclei in the rare earths, but the \( 4^+ \) can be more cleanly separated. In Reference 13), the elastic cross-section is used in conjunction with the radius obtained from mu mesic atoms and a \( B(E2;0^+\rightarrow2^+) \) from Coulomb excitation work to obtain
c, t, and $\beta_2$; $\beta_4$ is then obtained from the $4^+$ cross-section. It should be noted that while it is, in principle, possible to obtain B(EL) from the form factors by extrapolating to $q = 0$, typical experimental resolution makes that determination an order of magnitude worse than can be obtained from Coulomb excitation or lifetime measurements. The accuracy of measurements is also limited by the fact that as experimental accuracy increases, more terms must be fit in the infinite series that define the form factors.

2. Mu Mesic Atoms

If an atom is prepared in which some of the atomic electrons have been replaced by muons, the muon orbits will have mean radii much smaller than those of the corresponding electrons, owing to the greater mass of the muon. The $1s_{1/2}$ muon in a muonic atom of a heavy element has a probability of close to 50% of being within the nuclear charge distribution. In addition, although the magnetic dipole hyperfine splitting is increased by a factor of $4 \times 10^4$ over that observed in normal atoms, the electric quadrupole splitting is enhanced by a factor of about $8 \times 10^6$, and therefore dominates the hyperfine splitting, making this splitting sensitive to the nuclear quadrupole moment. In the analysis of experimental data, both energy levels and splittings are calculated simultaneously from a postulated charge distribution, so the radial parameters and the deformation are determined in a correlated manner.
The sensitivity to the details of the nuclear charge distribution decreases with increasing \( n \) (the principal quantum number of the muon orbit), and in practice only the \( 1s_{1/2} \), \( 2S_{1/2} \), \( 2p_{1/2} \), and \( 2p_{3/2} \) levels are affected enough by the deviation of the nuclear charge distribution from a point charge to have any detectable effect on the spectrum in the rare earth region. The problem of extracting information about the charge distribution from the experiment is basically one of how many quantities can be measured and how many parameters must be determined. In the case of atoms with \( Z \) less than about 50, only K x-ray transitions (to the \( 1s_{1/2} \) state) can be measured with reasonable accuracy, so that only one parameter can be extracted from the analysis: the mean square radius \( \langle r^2 \rangle \). For heavier nuclei the L transitions can also be measured (although with somewhat reduced accuracy compared to K x-ray measurements) so that two parameters of the charge distribution can be measured. Since a Fermi distribution is usually employed in the analysis, this means that both \( c \) and \( t \) can be determined for:

\[
\rho(r) = \rho_0 \left[ 1 + \exp \left( \frac{r-c}{t} \right) \right]^{-1}
\]

The method usually employed is to plot the region of \( c \), \( t \) corresponding to one standard deviation in the K energies, on a plot of \( c \) vs. \( t \). This takes the form of two parallel lines running from the \( c \) axis to the \( t \) axis. The \( c \), \( t \)
plot for L transitions is then plotted over this, and the region bounded by these four lines defines the region in which c and t lie. \( <r^2> \) can always be determined much more accurately than c and t due to the fact that the K measurements are more accurate than the L measurements (the latter contribute most of the error to the c, t plot); \( <r^2>^{1/2} \) can usually be determined to .01 fm out of 5 fm. These plots are generated by numerically calculating the energies from the potential parameters. If the hyperfine splitting is measured (for example in even-even rare earths), the potential can be calculated with the inclusion of a deformation, and a three parameters fit done to the data. A good example of this is the experiment by Hitlin, et al. A deformed Fermi distribution with \( \beta_2 \) deformation only:

\[
\rho(r, \theta) = \rho_0 \left[ 1 + \exp \left\{ \left[ r - c \left( 1 + \beta_2 Y_2^0(\theta) \right) \right] / t \right\} \right]^{-1}
\]

was used to fit K energies and relative intensities, L energies and relative intensities, the hyperfine splitting of the K and L lines, and \( B(E2; 0^+ \rightarrow 2^+) \) obtained from Coulomb excitation experiments. Determination of hexadecapole deformations from such experiments would seem rather unlikely due to the small size and complicated form of the hexadecapole interaction on the atomic orbit. The former would make it difficult to observe experimentally and the latter would complicate already difficult computations of the energies
of the atomic muon states and the hyperfine splittings. Calculations show that the hexadecapole effect can always be simulated by adjusting the quadrupole parameters, and is therefore not observable. \(^{41}\)

3. Coulomb Excitation and Lifetime Measurements

These two methods are treated together here because they are both methods used to measure electric multipole matrix elements, and because Coulomb excitation is often the mechanism used to excite the states the lifetimes of which are to be measured. The Coulomb excitation process will be discussed in the following chapter, and this section will therefore be devoted to a discussion of the information on the nuclear charge distribution obtained from Coulomb excitation experiments.

As was discussed in a previous section, to the extent that low-lying states of even-even rare earth nuclei are rotational, the reduced transition rates are proportional to the squares of the intrinsic electric multipole moments, and the reduced matrix elements are given by:

\[
< J_f | M(\ell \lambda) | J_i > = Q_\lambda^\ell < J_i \lambda 00 | J_f 0 > \sqrt{2J_i +1}
\]

Therefore measuring the transition rates (or the matrix elements) between rotational levels constitutes a measurement of the intrinsic moments. This relationship holds, in part, because the shape of the nucleus and the charge distribution are assumed to be constant in all states that are
rotationally related. Therefore the first two excited states in an even-even nucleus, the $2^+$ and $4^+$, have the same intrinsic moments as the $0^+$ ground state. This model dependency is only necessary in the case of the off-diagonal matrix elements (the transition moments), since the diagonal matrix elements of the charge multipole operators are the moments of the states in any case. Even if the states are not rigorously rotational, the transition moments still yield valuable information in that they measure the overlap of the wavefunctions of the initial and final states, and therefore the extent to which the charge distributions in those two states are the same.

The $E\lambda$ matrix elements are determined in the analysis of a Coulomb excitation experiment from the measured excitation probabilities, the relative differential cross-sections. The excitation probabilities are completely determined by the reduced matrix elements if the spins and energies of all states are known and if the interaction is known to be pure Coulomb. The matrix elements thus determined from a measurement of the excitation probabilities are independent of any model of nuclear structure, although the interpretation in terms of intrinsic moments is bound up with the rotational model.

The effect of the diagonal matrix elements (the static moments) on the excitation probability of a given state (the re-orientation effect) is always smaller than, and of a higher order than the effect of the off-diagonal matrix elements (the transition moments). The static moments cannot therefore be determined to the same degree of accuracy as
can the transition moments. For example, the $0^+ \rightarrow 2^+$ E2 matrix element can be determined from the excitation probability of the $2^+$ state to better than 1% in some cases while the $2^+ \rightarrow 2^+$ E2 matrix element, which only accounts for a few percent of the excitation of the state and contributes only in second order has only been measured with accuracies of 10%-50% in the lanthanide region. While the re-orientation effect measurements are most useful in regions where the rotational model does not correctly predict nuclear properties, (for example, in regions where low-lying levels demonstrate pronounced vibrational characteristics) uncertainties involved in using the rotational model in the region of the rare earths studied in this experiment are much smaller than the experimental uncertainties in measuring static moments through the re-orientation effect.

Much available data on quadrupole moments (especially of even-even rare earth nuclei) comes from Coulomb excitation measurements of transition quadrupole matrix elements. Quadrupole moments measured up to 1965 are tabulated in Reference 7). In addition, much of the verification of rotational behavior and analysis of deviations from rotational behavior is due to measurements of these matrix elements.

The matrix elements for transitions between excited states are often difficult to determine accurately because they contribute to multiple excitation processes which
depend upon several simultaneous transitions (for example, see Figure 11-4). These matrix elements can also be determined by measuring the lifetimes of the excited states,$^{14,15,16,35,36,37,38}$ since the de-excitation processes proceed by a gamma-ray cascade where each individual transition is observable and depends upon only one matrix element. Such measurements are of particular importance to the analysis of this experiment because they are a source of information on the matrix element $M(E2; 2^- \rightarrow 4^-)$ which is not determined from this experiment and enters into the extraction of $M(E4; 0^+ \rightarrow 4^+)$ in a fundamental manner.

In summary, the following points should be noted. The information obtained in the most straightforward manner from Coulomb excitation and lifetime measurements is a measurement of reduced transition matrix elements, which can be related to the intrinsic moments of the nuclear charge distribution through the relationships determined by the collective model. These moments can be interpreted in terms of Coulomb deformation parameters only with additional information on the shape of the charge distribution. This information can be obtained from mu-mesic atom experiments and electron scattering experiments, which are also sensitive to charge deformations. While electron scattering is sensitive to $\beta_2^C$ and $\beta_4^C$, mu-mesic atom experiments are capable of determining $\beta_2^C$ deformation only. It should be emphasized that although these experiments are capable of determining which model of the charge distribution best fits
available data, these models are subject to change as more
detailed information becomes available, and data presented
in terms of such models are very dependent upon the valid-
ity of the models. It should also be emphasized that
unique determinations of the parameterizations of these
models is not always possible.

The nuclear potential scattering experiments measure
the parameters of the deformed optical potentials, $\beta_\lambda R_0$,
the relationship of which to the shape of the charge
distribution is not well established.
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Chapter II
COULOMB EXCITATION

The Coulomb excitation process is well understood; the general problem has been treated in great detail and the important features of the theory have been well summarized (see for example Reference 1 and Reference 7). It has been an established experimental tool for well over a decade. This chapter will briefly outline the basic features of the standard semi-classical approach to Coulomb excitation, and the more exact quantum mechanical Coulomb scattering problem; the differences between these two calculations are significant in the scattering of \(^4\)He projectiles from rare earth nuclei. Two small corrections to the Coulomb scattering interaction which enter into the analysis of this experiment are also discussed. The influence of electric hexadecapole (E4) excitation is discussed in more detail, with specific emphasis on the situation encountered in this experiment.

When a positively charged nuclear projectile interacts with a target nucleus, the Coulomb component of the interaction is always present and can contribute to the excitation of states in the target. If the incident energy is sufficiently high so that the projectile comes close enough to the target to interact through the strong nuclear force,
the Coulomb excitation is a small perturbation on the larger contribution of the nuclear interaction. If, however, the incident projectile energy is low enough so that the distance of closest approach of the projectile to the target due to the long range Coulomb force keeps the two nuclei beyond the range of the nuclear force, the entire interaction is Coulomb, and is in principle exactly known and calculable. It should be emphasized that although the nuclear force is of finite range, it is not possible to a priori determine the incident energy below which the interaction is pure Coulomb and above which it is predominantly nuclear, rather there is a range of incident energies within which the Coulomb-nuclear interference effects become pronounced. This will be discussed in Section V.D.

A. THE SEMI-CLASSICAL FORMULATION OF COULOMB EXCITATION

In most cases of Coulomb excitation due to incident heavy ions, the interaction has been found to be described well by a semi-classical treatment.¹,²) In this approach the nucleus is treated as a finite body of charge distribution ρ(r), the states of which are described quantum mechanically, and the projectile is treated as a point charge whose orbit is determined by Rutherford scattering. The nuclear states are solutions of the static Schrödinger equation of the nuclear Hamiltonian, and the transitions are calculated from time-dependent perturbation theory and
integrated along the classical path of the projectile. The validity of this approach depends upon two primary conditions being satisfied:

1) \( n \gg 1 \)

2) \( \Delta E/E \ll 1 \)

where \( n \), the Sommerfeld parameter, is the ratio of one half the distance of closest approach to the wavelength of the projectile, and \( \Delta E/E \) is the ratio of the transition energy to the energy of the incident projectile.

The first condition implies that the spatial extent of the projectile is small in the context of the scattering, and therefore the projectile is validly treated as a point charge. It also implies that the projectile and target never get close enough to interact via the strong nuclear force, and it is therefore valid to assume the interaction to be pure Coulomb. This is equivalent to the classical statement that the interaction takes place below the Coulomb barrier. The second condition implies that the energy lost in an inelastic scattering (an excitation of a nuclear state) is small compared to the energy which defines the orbit. The orbit is therefore not perturbed significantly by it, and may still be treated as a well-defined classical orbit.

In the context of the semi-classical formulation the nuclear state is treated as a superposition of the stationary states which are eigenstates of the nuclear
Hamiltonian, so that at some time during the interaction the nuclear state is given by:

\[ |n(t)> = b_1(t)|1> + b_2(t)|2> + \ldots + b_i(t)|i> + \cdots \]

Before the interaction occurs the nucleus is assumed to be totally in the ground state:

\[ |n(-\infty)> = b_1(-\infty)|1> = |\text{g.s.}> \]

For simplicity, consider a two state system. The amplitude for the excitation of the excited state \(|f>\) is given, in perturbation theory, by:

\[ b_f = \frac{1}{i\hbar} \int_{-\infty}^{+\infty} <f|H_e(t)|i> e^{i\omega t} dt \]

where the time integral is carried out along the path of the projectile orbit. The excitation probability of the final state is defined by:

\[ P(f) = \sum_{M_iM_f} |b_f|^2 \left[ 2I_i + 1 \right]^{-1} \]

where the sum is over the magnetic substates of the initial and final states.

The total Hamiltonian for the Coulomb interaction is given by:

\[ H = \int \rho_n(r)\phi(r,t)dt \]

\[ = H_0 + H_e(t) \]
where \( \rho_n(\vec{r}) \) is the charge distribution of the nucleus, and \( \phi(\vec{r},t) \) is the potential due to the projectile. \( H_0 \) is the monopole term which causes the Rutherford scattering.

Expanding the potential in multipoles yields:

\[
\phi(\vec{r},t) = 4\pi Z e \sum_{\ell, m} \frac{1}{2\ell + 1} \sum_{p} \frac{r^\ell_p}{r^\ell + 1} Y_\ell^m(\theta_p, \phi_p) Y_{\ell}^{m*}(\theta, \phi)
\]

\[
H = 4\pi Z e \sum_{\ell, m} \sum_{p} \frac{r^\ell_p(t) - \ell - 1}{2\ell + 1} Y_\ell^m(\theta_p(t), \phi_p(t)) \int \rho_n(\vec{r}) Y_{\ell}^{m*}(\theta, \phi) r^\ell dt
\]

which gives the perturbation as:

\[
H_e(t) = 4\pi Z e \sum_{\ell} \sum_{m=\ell}^{\ell+1} Y_\ell^m(\theta_p(t), \phi_p(t)) \frac{r^\ell_p(t) - \ell - 1}{2\ell + 1} q^m_{\ell}
\]

\( H_0 \) is just the monopole term of the expansion, and \( q^m_{\ell} \) are the multipole moments of the interaction. These are the operators defined by:

\[
q^m_{\ell} = M(E\lambda, \mu) = \int \rho_n(\vec{r}) Y_{\ell}^m(\theta, \phi) r^\lambda dt
\]

The expression for the transition amplitudes therefore becomes a sum of products of a part which depends on nuclear structure and a part which is orbit-dependent only:

\[
b_f = \frac{4\pi Z e}{\hbar} \sum_{\ell, \mu} I_1^f \langle I_1 M_1 | M(E\lambda, \mu) | I_1 M_f > S_{E\lambda, \mu}
\]

\[
S_{E\lambda, \mu} = \int_{-\infty}^{\infty} e^{i\omega t} Y_{\ell}^m(\theta_p, \phi_p) r^\lambda dt
\]
The expression can be further simplified by defining the reduced matrix elements,

\[ <I_{M_1}M_f|M(E\lambda,\mu)|I_{M_1}M_f> = (-)^{I_{M_1}M_f}M_{M_1}(I_{M_1}M_f) <I_{M_1}||M(E\lambda)||I_f> \]

reducing the sum to a sum over \( \lambda \) only. In general, similar terms enter due to the excitation of the state \(|f>\) from states other than the ground state, so that the total excitation is a sum of contributions from direct excitation and "multiple excitation" through various other states. The coupled equations of the Coulomb excitation of a many state system are solved to all orders by a computer code, the original version of which is due to Winther and de-Boer.\(^2\) This code includes a "symmetrization" of the orbits to account for the inelasticity of the interaction; i.e., \( \Delta E \neq 0 \).

The behavior of the excitation probabilities with various parameters of the interaction such as incident energy, multipolarity, scattering, etc. are discussed at some length in Reference 1, and will be quickly summarized to the extent that they bear on the excitation of rotational states in even-even deformed nuclei.

Since the angular momentum change between adjacent states in a \( K = 0 \) ground state band is two units, the lowest contributing multipole is \( \lambda = 2 \), that is \( E2 \), and
because all states in a rotational band have the same parity, and the parity change associated with the transfer of an $E\lambda$ photon is $(-1)^\lambda$, only even multipoles contribute. In general, the excitation amplitudes fall off rapidly with increasing $\lambda$, so that $E2$ excitation dominates, followed by $E4$, with the contribution of higher multipoles being negligible in most cases. In addition, there is excitation due to $M\lambda$ matrix elements. Following the above arguments, the lowest multipole magnetic transition between states in $M3$, since only odd $M\lambda$ multipoles can contribute because of parity considerations. The cross sections for magnetic excitations of multipolarity $\lambda$ are about $(v/c)^2 (v/c)^{-1}$ of the excitation due to $E\lambda$ excitations of the same multipolarity. For the conditions encountered in this experiment, this ratio is about $0.01$, so that magnetic excitations can also be ignored. It should be noted that due to angular momentum considerations there can be no direct $M\lambda$ excitation of states in the ground state band of an even-even nucleus.

The Rutherford cross section decreases rapidly with increasing scattering angle, but the excitation probability increases with scattering angle, making it advantageous to use a backscatter geometry. The excitation of states of higher spin increases rapidly with the charge of the projectile. Thus protons excite only the $2^+$ state significantly, while alpha particles excite the $2^+$, with the $4^+$ being excited to a much lesser but still detectable
extent. Using $^{16}$O projectiles the $6^+$ and $8^+$ states can be readily excited, and so on. In general, for incident energies below the Coulomb barrier the amplitudes rise with bombarding energy. However, increasing excitation of higher states by multiple excitation can deplete lower lying states and cause the excitation probabilities of those states to decrease as the bombarding energy is increased still further.

B. THE QUANTUM MECHANICAL FORMULATION OF COULOMB EXCITATION

If alpha particles of 12MeV incident energy are scattered from rare earth targets the interaction has $n=10^{-13}$, which may not satisfy the condition for the validity of the semi-classical Coulomb excitation calculation, $n \gg 1$. Alder, et al. $^3$ have calculated that the difference between the semi-classical calculation of the excitation probability of the $4^+$ state of $^{152}$Sm by competing direct $E^4$ and double $E^2$ (via the $2^+$ state) and the quantum mechanical calculation would be on the order of 6%, the quantum mechanical calculation yielding a smaller excitation probability. Neglecting this correction would result in a serious error in the determination of the $E^4$ contribution to the $4^+$ excitation probability.

The exact quantum mechanical calculation follows the coupled channels partial wave expansion formalism of nuclear scattering (e.g. Reference 4) which has been applied specifically to the Coulomb excitation process.
in several treatments\(^5,6,7\) and is briefly outlined here.

The radial equations of the scattering interaction are given by:

\[
\left( \frac{d^2}{dr^2} + \frac{k_I^2 - 2n_I k_I - \ell(\ell+1)}{r^2} \right) g_{\ell I J}(r) = \sum_{\ell', I'} V_{I \ell I' \ell'}^J(r) g_{\ell' I' J}(r)
\]

where \( \ell \) is the orbital angular momentum, \( I \) is the nuclear spin, and \( \hat{J} = \hat{I} + \hat{\ell} \). \( V_{I \ell I' \ell'}^J(r) \) is the matrix element given by:

\[
V_{I \ell I' \ell'}^J(r) = \frac{\sqrt{16 \pi m_2 e^2}}{\hbar^2} \sum_{\lambda} \sqrt{(2\ell+1)(2\ell'+1)} \frac{\langle I' | | M(E\lambda) || I \rangle}{2\lambda + 1} (-)^{J+\lambda+I'} \binom{\ell \ell' \lambda}{000} \binom{J I' J'}{\lambda I \ell} \frac{1}{r^{\lambda+1}}
\]

where the sum is over all contributing multipoles, and \( \langle I' | | M(E\lambda) || I \rangle \) is the same reduced matrix element used in the semi-classical calculation. The solution requires the following boundary conditions:

1) \( g_{\ell I J}(r) \) are finite at the origin

2) \( g_{I \ell I J}(r) \frac{\delta_{II_0 \delta_{\ell \ell_0} - i\phi_{I \ell}}(r)}{\sqrt{V_I}} - \frac{r_{I \ell I_0 \ell_0} \phi_{I \ell}}{\sqrt{V_I}} \)
which is a combination of incoming and outgoing waves for $I = I_0$, $\ell = \ell_0$, and an outgoing wave otherwise.

Listing the results from Reference 7:

\[
\frac{d\sigma}{d\Omega} I_{o} \rightarrow I = \frac{1}{2I_0 + 1} \frac{V_I}{V_0} \sum_{\text{MM}} |f_{I_0 M_0} \rightarrow I M(\theta, \phi)|^2
\]

\[
f_{I_0 M_0} \rightarrow I M(\theta, \phi) = \frac{1\sqrt{\pi}}{\sqrt{k_0 k_I}} \sum_{J_{\ell} \text{Im}} \frac{\lambda_{\ell}-\ell}{2\ell + 1} \langle I_0 O I_0 O | J M >
\]

\[
\langle \text{lmIM}|J M_O > (\delta_{II_O} \delta_{\ell \ell_0} - e^{i(\sigma_{\ell_0}(\eta_0) - \sigma_{\ell}(\eta))})
\]

\[
\frac{r_{I I_O \ell_0}}{r_{J I_O \ell_0}} Y_{\ell}^{m}(\theta, \phi)
\]

\[
\phi_{I\ell}(r) = k_I r - \eta_I \log(2k_I r) - \frac{\pi\ell \sigma_{\ell}(\eta)}{2} \text{ for } kr >> 1
\]

\[
\sigma_{\ell}(\eta) = \arg \Gamma (\ell + 1 + i\eta) \text{ The Coulomb phase shift}
\]

In practice, the calculation involves the simultaneous solution of the equations for all contributing partial waves, which may number as many as several hundred for alpha particles scattered from rare earths. Alder, et al. \textit{7}) have calculated solutions in second order perturbation theory and have tabulated the results as a correction to the semi-classical results. The results of interest in this experiment are the correction to the two step excitation of the $2^+$ via the reorientation effect (static quadrupole moment of the $2^+$ state), and the correction to
the excitation of the $4^+$ by double E2 excitation in competition with direct E4. For He excitation with $E \approx 10-12$ MeV the former amounts to a lowering of the calculated $2^+$ cross-section by about 2%, and the latter to a decrease in the calculated $4^+$ excitation probability of about 6%. In the analysis of this experiment use was made both of the perturbation solution and of a full solution of the quantum mechanical problem. The differences in the results of these two calculations will be discussed in Chapter IV.

C. SMALL CORRECTIONS: ATOMIC SCREENING AND VACUUM POLARIZATION

The treatments considered thus far have been based on the assumption of a projectile scattered from a charged nucleus the electric potential of which at large distances (or in the point charge limit) is given by the monopole Coulomb potential:

$$\phi(r) = \frac{Z \epsilon}{r}$$

In the semi-classical calculation the monopole term accounts for elastic scattering and defines the orbit of the projectile. To be more correct, this must be modified for two effects: the screening due to the atomic electrons of the nuclear potential seen by the projectile and vacuum polarization at distances close to the nucleus. The former tends to decrease the electrostatic potential energy between projectile and target, and the latter tends to increase it. The two effects are generally assumed to result
in small equal and opposite effects on the Coulomb scattering of projectiles such as alpha particles from heavier nuclei. Calculations by D. Cline of both effects as a function of the charge of projectiles scattered from $^{114}$Cd targets indicate that the vacuum polarization correction to the calculated excitation probability is about twice as large as the screening correction. Since these calculations were carried out mainly for heavier projectiles and a lighter target, it is important to consider both these effects in somewhat greater detail for the case of scattering of $^4$He projectiles from nuclei of mass 150-180.

The screening correction to the potential energy of a projectile of charge $Z_1$ near a nucleus of charge $Z_2$ has been calculated to be: $^{9)}$

$$\Delta V = Z_1(32.65Z_2^{7/5} - 40Z_2^{2/5}) \text{ eV}.$$ 

The authors $^{9)}$ state that this has the effect of changing the distance of closest approach, and can be accounted for by modifying the center of mass bombarding energy used in the Coulomb excitation calculation to $E' = E + \Delta V$. This is equivalent to modifying the bombarding energy in the laboratory system by $\Delta V$, since $m_1/m_2 = 4/152$ in the case of alpha particles scattered from $^{152}$Sm.

The static potential energy of two charged particles at rest corrected for vacuum polarization effects has been calculated by Foldy and Eriksen: $^{10)}$
They estimate that at separations of the order of 1 fermi, the correction to the Coulomb energy is of the order of 1/2%. A computer program was written which numerically calculated the integral. The correction calculated from the program was 0.7% for a 1 fm separation.

Since both atomic screening and vacuum polarization change the potential energy of the projectile in the region of the target, both should be accounted for (at least in lowest order) by altering the distance of closest approach used in the deBoer-Winther program. This is not, however, identical to modifying the bombarding energy used in the program. The explicit expression used for the equations solved in the program are:

\[
\frac{d \alpha_r(\omega)}{d \omega} = \frac{4\pi Z_1 e^2}{\hbar v} \sum_{\lambda \mu s} \frac{Y^\mu_\lambda(\theta, \phi) \exp[i \xi_{rs}(\epsilon \sinh \omega + \omega)]}{\lambda \mu s \alpha \lambda \epsilon \sinh + 1} (2\lambda + 1)
\]

\[x <s|M(\lambda, \mu)|r>a_s(\omega)\]

where \(a = Z_1 Z_2 e^2 / M_0 v^2\) is one half the distance of closest approach, and

\[\xi_{rs} = \frac{a}{\hbar v} (E_r - E_s).\]
The charge of the target $Z_2$, enters only through $a$, but $Z_1$ and $v$ which is proportional to $\sqrt{E}$ enter explicitly. When the equations are set up in the program, $(Z_1/v^\lambda)$ becomes $a (v/a^\lambda Z_2)$, so that there is now a factor multiplying the entire expression which is proportional to $v/Z_2$. Therefore if $Z_1$ is changed to correct $a$ only $a$ is altered, but if either $Z_2$ or $E$ is changed still another change is made in the expression for the amplitudes. In addition, when the expressions are symmetrized, the bombarding energy enters into the expression for the parameter $a$ in a more complicated manner:

$$a + a_{MN} = \frac{Z_1 Z_2 e^2}{M_0 v N V_M}$$

$$v_N = \sqrt{2[E - (1 + \frac{A_1}{A_2})(E_M - E_1)]} / M_P$$

so that for heavy projectiles the change in $a$ will not necessarily be proportional to the change in $E$. In addition, $E$ enters into the lab to center of mass transformation.

In sum, the corrections are most simply accounted for by altering the charge of the projectile used in the calculation. If the charge of the target is altered, the effect of the explicit factor of $1/Z_2$ can be accounted for in the final excitation probabilities. The effect of changing $E$ is not so simple.

As would be expected, changing the incident energy used in the calculation does not yield the same correction as changing the charge of either the target or the
projectile, since changing $E$ alters more than just $a$. Altering the energy results in corrections that are consistent with those of Cline $^8$; the ratio of vacuum polarization to screening effects for $^{114}$Cd varies slowly with incident projectile charge; both corrections decrease as $Z_1$ increases. Altering $Z_1$ produces effects that are constant with $Z_1$; this is consistent with calculation of the vacuum polarization effect by Berant, et al. $^{11}$

While both treatments produce different absolute effects in the case of alpha particle scattering from rare earths, the screening and vacuum polarization effects calculated in both treatments are small and opposite in sign. The magnitude of each correction to the excitation probability of the $2^+$ state is less than 0.5% in either treatment, and the difference between the two treatments is less than $1/4\%$. While it appears that it is more correct to alter the distance of closest approach used in the calculation by artificially changing $Z_1$ rather than the bombarding energy, most analyses have utilized the latter approach, and the difference between the net corrections calculated in the two cases is negligible.

D. THE CONTRIBUTION OF HEXADECAPOLE EXCITATION

Although an accurate calculation of the $E4$ contribution to any Coulomb excitation process can only be approached by solving the complete set of coupled equations for that process, some insight can be gained, and a fairly accurate estimate can be made of the effect, by examining
the simplified case of a target nucleus with only $0^+$, $2^+$, and $4^+$ states excited by incident $^4$He projectiles. Only first and second order processes are considered. This is a reasonable approximation to the situation encountered in this experiment, since the excitation of all other states is almost negligible.

Using the formulation of Alder et al. the probability of exciting the $4^+$ state of the ground state rotational band by direct $E4$ and double $E2$ excitations can be written, in second order perturbation theory,

$$P(4) = P(2,2)_{0 \rightarrow 2 \rightarrow 4} \frac{X^{(4)}_{0 \rightarrow 4}}{(2)} \frac{d(\xi_1, \xi_2, n, \theta)}{X_{0 \rightarrow 2} X_{2 \rightarrow 4}} + \frac{X^{(4)}_{0 \rightarrow 4}}{(2)} \frac{2a(\xi_1, \xi_2, n, \theta)}{X_{0 \rightarrow 2} X_{2 \rightarrow 4}}$$

where

$$P(2,2)_{0 \rightarrow 2 \rightarrow 4} = (X_{0 \rightarrow 2})^2 (X_{2 \rightarrow 4})^2 \pi_4(\xi_1, \xi_2, n, \theta)$$

is the excitation probability due to $E2$-$E2$ excitation and where $X^\lambda_{I_0 \rightarrow I}$ and $a$ are given by

$$X^\lambda_{I_0 \rightarrow I} = \frac{\sqrt{16 \pi}(\lambda-1)!}{(2\lambda+1)!!} \frac{Z_1 e}{\hbar \sqrt{V_{I_0} V_I}} \frac{<I_0 || M(E^\lambda) || I>}{a^{\lambda 2 I_0 + 1}}$$

$$a = \frac{Z_1 Z_2 e^2}{m v_{I_0} v_I}$$
The notation is from Alder et al. where the functions $d(\xi_1, \xi_2, n, \theta)$, $a(\xi_1, \xi_2, n, \theta)$, and $P_4(\xi_1, \xi_2, n, \theta)$ are tabulated. The excitation probability $P(4)$ can thus be written as

$$P(4) = P^{(2,2)}_{0+2\to4}[1+C(\xi_1, \xi_2, n, \theta)\frac{Q'_4}{Q'_2} + C'(\xi_1, \xi_2, n, \theta)\frac{Q'_4}{Q'_2}^2]$$

where the functions $C$ and $C'$ depend upon the kinematics and contain the functions $d(\xi_1, \xi_2, n, \theta)$ and $a(\xi_1, \xi_2, n, \theta)$, respectively, as multiplicative factors. The quantities $Q'_4$ and $Q'_2$ are the intrinsic hexadecapole and quadrupole moments calculated from the reduced matrix elements by the rotational model prescription

$$M(E\lambda; I_1\to I_\tau) = \langle I_1 | | M(E\lambda) | | I_\tau \rangle = \sqrt{(2I_1+1)}<I_10\lambda0|I_\tau0>Q'_\lambda$$

The behavior of the functions $C$ and $C'$ with $n$ are shown in Figure 1, as is the variation of the total $E4^+$ contribution to the excitation probability of the $4^+$ state for a typical case. In the range of useful bombarding energies below the Coulomb barrier (8 to 12 MeV) the variation of the $E4^+$ contribution is slight. The behavior with scattering angle is also not excessive, as is illustrated in Figure 2. In that part of the lanthanide rare earths where the rotational model is known to give an accurate description of the low-lying states of even-even nuclei, $Q'_2$ varies little with mass number, and the magnitude of
the $E^4$ contribution to $P(4)$ is dictated solely by $Q'_4$.

Although these estimates are reasonably valid in the situation usually encountered with light projectiles, it is not generally applicable in the case of heavy projectiles, since multiple processes proceeding through many other states can effect the excitation of the $4^+$ state as much as the direct $E^4$ excitation. The general problem is thus best examined in the terms of a full coupled channels Coulomb excitation calculation for various projectiles exciting a given target under conditions of specified beam energy and scattering angle.

The variation with incident projectile charge of the $E^4$ contribution to the excitation of the states in the ground state band of $^{154}$Sm is shown in Figure 3. For a given state, the magnitude of the contribution decreases with increasing $Z_p$; for a given projectile the magnitude of the contribution increases with the spin of the state being studied. Since the excitation probability, in general, decreases with increasing state spin for a given projectile, it is not always possible, in practice, to take advantage of the increase in the magnitude of the $E^4$ contribution with state spin by observing states higher up in the band. A contribution of 20% to 30% is the largest observable in a nucleus such as $^{154}$Sm under realistic experimental conditions; measuring the larger ratios $R_1$ predicted for the higher spin states would necessitate prohibitively long data accumulation times.
Figure II-1. Variation with $n$ of the contributions to $E_4$ excitation

$$P(4) = P_{o}^{(2,2)}[1+C(\xi_1,\xi_2,n,\theta)\frac{Q_4'}{(Q_2')^2} + C'(\xi_1,\xi_2,n,\theta)\left(\frac{Q_4'}{(Q_2')^2}\right)^2]$$

of a typical rotational rare earth nucleus, as given by:

- $E_2^+ = 0.080$ MeV \(\xi_1 = 2.62 \times 10^{-5}n^3\) at \(\theta = 180^\circ\)
- $E_4^+ = 0.260$ MeV \(\xi_2 = 2.25\xi_1\)

a) The function $C(\xi_1,\xi_2,n,\theta)$
b) The function $C'(\xi_1,\xi_2,n,\theta)$
c) The total $E_4$ contribution

$$\left[\frac{C}{(Q_2')^2} + C'(\frac{Q_4'}{(Q_2')^2})^2\right]$$

for $Q_4'/(Q_2')^2 = 0.1$

This is equal to $R_4$ shown in Figure 3.

---

Figure II-2. The variation with laboratory scattering angle of the $E_4$ contribution, normalized to

the value at $\theta = 180^\circ$. $R_4(\theta)/R_4(\theta = 180^\circ)$

---

Figure II-3. Variation of the magnitude of the $E_4$ contribution to the excitation process with projectile charge and final state spin.
Figure II-1
Figure II-2
Figure II-3

$\text{Sm}^{154}$ (semiclassical calc.)

\[ R_I = \frac{\bar{P}_I(E4+E2) - \bar{P}_I(E2)}{\bar{P}_I(E2)} \]

$M(E2; 0 \rightarrow 2) = -2.066 \text{ eb}$

$M(E4; 0 \rightarrow 4) = +0.653 \text{ eb}^2$

$\theta_{sc} = 174.5^\circ$

$12 \text{ MeV} \quad \text{He}^4$

$41 \text{ MeV} \quad \text{C}^{12}$

$54 \text{ MeV} \quad \text{O}^{16}$

$110 \text{ MeV} \quad \text{S}^{32}$
Figure 3 represents the results of calculations performed for back angle scattering with incident energies close to the Coulomb barrier. Calculations carried out as a function of incident energy show trends very similar to those displayed in Figure 1. The variation with scattering angle of the E4 contribution to the excitation of the 4+ state using 4He projectiles shown in Figure 2 was obtained with a coupled channel calculation. The E4 excitation increases slowly as the scattering angle decreases from about 180° to roughly 90°. Forward of 90° the increase is more rapid, but offers no experimental advantage due to the sharp decline in the excitation probability at forward angles.

In principle, by taking advantage of the variation of the E4 contribution with bombarding species and state spin; and the slow variation of that contribution and rapid change in multiple excitation with incident energy, it should be possible to judiciously select experimental situations in which each contributing matrix element is uniquely determined from a measured excitation probability. Thus in principle it is possible to construct the entire transition matrix for the ground state band. For example, if low energy 4He projectiles are employed, the excitation of the 2+ state depends only on M(E2; 0+→2+), since the E4 contribution is very small and the multiple excitation of the 4+ state from the 2+ is negligible at low enough bombarding energies. Using low energy 16O beams to excite the 4+
state, \( P(4) \) depends predominantly on the product
\[
M(E2; 0^+ \rightarrow 2^+) \cdot M(E2; 2^+ \rightarrow 4^+)
\]
with \( M(E2; 0^+ \rightarrow 2^+) \) already determined to high accuracy from the previous measurement, the \( M(E2; 2^+ \rightarrow 4^+) \) matrix element is uniquely determined from \( P(4) \). The use of \(^4\text{He}\) projectiles or Li projectiles at incident energies close to the Coulomb barrier to excite the \( 4^+ \) state by double \( E2 \) and competing direct \( E4 \), leads to the extraction of \( M(E4; 0^+ \rightarrow 4^+) \) as the only additional unknown. Since multiple \( E2 \) and multiple \( E4 \) excitation becomes increasingly important to the excitation of states with spin higher than \( 4 \), the situation would become more complicated for these states, necessitating some assumptions and independent measurements of inter-band transitions which would begin to make significant contributions\(^2,3,4\).

Figure 4 illustrates the typical transitions which occur in the excitation of an even-even rare earth nucleus by incident heavy ions.

Although proceeding in the manner outlined above would eventually yield a set of reduced matrix elements which constitutes a body of primary, model-independent information about the nucleus, much of the most significant information can be obtained by only considering \(^4\text{He}\) excitation. As the above discussion indicates, the measurement of \( P(2) \) and \( P(4) \) for \(^4\text{He}\) excitation yields \( M(E2; 0^+ \rightarrow 2^+) \) and \( M(E4; 0^+ \rightarrow 4^+) \) which are directly related to the intrinsic quadrupole and hexadecapole moments, respectively, within
Figure II-4. Transitions in the Coulomb excitation of a typical even-even rare earth nucleus. The downward transition corresponding to each upward transition is also allowed. Heavy arrows indicate direct excitation.
the rotational model formulation outlined in Appendix II. Determining the moments from these two matrix elements: (which are of the form $M(E\lambda;0\rightarrow\lambda)$) makes it only necessary to assume that the rotational model provides a valid description of the $2^+$ and $4^+$ states, whereas the determination of the intrinsic moments from transition matrix elements between higher lying states would involve the assumption of rotational behavior in these higher states. In general, as the spin of the state within the band increases, the validity of the assumption of rotational behavior and the experimental precision to which such behavior has been tested decrease.

In addition, a Coulomb excitation experiment using $^4\text{He}$ projectiles has the following advantages:

1) The E4 contribution to the excitation of the $4^+$ state is as large as the E4 contribution expected in any reasonable situation.

2) The only matrix element which contributes significantly and is not determined in the experiment is $M(E2;2^+\rightarrow4^+)$. Because the re-orientation contributions to the excitation probabilities are small, the static moments need not be determined to a high degree of accuracy, whereas all contributing transition moments must be determined to at least the degree of accuracy to which the matrix element of interest is to be determined. For example, if $M(E2;4^+\rightarrow6^+)$ is to be determined from the excitation
of the $6^+$ state by incident heavy ions, $M(E2;0^{+}\rightarrow2^{+})$ and $M(E2;2^{+}\rightarrow4^{+})$ must be known to the accuracy to which $M(E2;4^{+}\rightarrow6^{+})$ is to be determined.

3) The excitation of the $2^+$ and $4^+$ states dominates, so that multiple processes which depopulate the $4^+$ state need not be accurately determined, and the final spectrum has few prominent spectral lines. The excitation probability of the $6^+$ state, the next most highly excited state in the ground state band, is roughly two orders of magnitude less than that of the $4^+$ state, and the presence of that state has therefore little effect on the excitation of the $4^+$. The excitation of the $2^+$ members of the beta and gamma vibrational bands (the states most highly populated after the first $2^+$ and $4^+$ states) changes the excitation probabilities of the states of interest by small amounts and introduces negligible uncertainties.

4) The 13 to 18 keV resolution obtainable for the detection of the scattered alpha particles makes it possible to separate the inelastically scattered particles from those elastically scattered, so that the excitation probabilities can be extracted either directly from the particle spectra or from the detection of de-excitation gamma rays. The resolution obtainable for the detection
of heavier projectiles, such as $^{12}\text{C}$ would be on the order of 50 to 60 keV, making it necessary to detect the de-excitation gamma rays if the excitation probabilities of the low-lying states are to be measured.

In summary, Coulomb excitation of rotational rare earth nuclei by incident $^4\text{He}$ projectiles can be used to determine $M(E2;0^+\rightarrow2^+)$ and $M(E;0^+\rightarrow4^+)$ from the direct $E2$ excitation of the $2^+$ state and the large $E4$ component of the excitation of the $4^+$ state. The only matrix element which must be determined accurately from other sources is $M(E2;2^+\rightarrow4^+)$. Because of the rotational nature of these states, the measured matrix elements yield the intrinsic quadrupole and hexadecapole moments of the nuclei. The implementation of this method is the subject of the next Chapter.
REFERENCES

CHAPTER II

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CHAPTER III

THE EXPERIMENT

The experiment was performed by directly detecting the \(^4\text{He}\) projectiles scattered at backward angles into an annular surface barrier detector. The direct detection of the scattered alpha particles avoids the uncertainties involved in detecting the de-excitation gamma rays, and facilitates the measurement of \(M(E2;0^+ \rightarrow 2^+)\) to the high degree of accuracy necessary to obtain a meaningful measurement of \(M(E4;0^+ \rightarrow 4^+)\). The detection of the scattered particles provides a straightforward normalization and avoids the necessity for an accurate calibration of absolute and relative counting efficiencies that would be inherent in a gamma-ray measurement. In addition, the calculation of the excitation probabilities from gamma-ray spectra requires an accurate determination of internal conversion coefficients, especially for the low energy \(2^+ \rightarrow 0^+\) transition.\(^8,9\) The detection of the incident particles scattered at back angles into an annular detector centered on the incident beam axis offers several advantages, besides the obvious one of symmetry. a) The excitation probabilities are maximized near 180°, while the Rutherford cross section is smallest, and varies most slowly with angle there. b) The kinematic spread in the peaks is minimized at back angles, allowing a large
solid angle without undue loss in energy resolution. The large solid angle is dictated by the high statistical accuracy required for these measurements. (The detectors employed spanned a polar angle of approximately three degrees, which corresponds to roughly a change of 1/2\% in $P(2)$ and $P(4)$ across the face of the detector, or $\pm 1/4\%$ about the central angle, the variation being roughly linear with angle.) c) The kinematic separation of the contaminant lines is maximized at back angles, so that the elastic lines from possible light element contaminants not present in the isotope sample--such as zirconium from the target preparation process and copper from the electrodes of the evaporation equipment--would, if present, be well separated from the lines of interest.

The contributions from contaminants in the samples from which the targets are fabricated are a major problem with a scattering experiment of this type. High purity samples are required. Another problem is resolution loss due to the projectiles losing energy in the target, which dictates the use of very thin targets. These thin targets must be supported on carbon backings, introducing pile-up problems due to the large count rate originating from the $^{12}\text{C}(\alpha,\alpha')^{12}\text{C}$ and $^{16}\text{O}(\alpha,\alpha')^{16}\text{O}$ reactions. These problems are dealt with in more detail below.

The magnitude of the uncertainties in determining $M(E4;0^+\rightarrow4^+)$ from such measurements can be estimated by noting that in the most favourable cases it is expected...
that the E4 contribution to P(4) is approximately 25%-30%, whereas the smallest contributions are expected to be 10% and less. The uncertainty in determining the contribution from E4 excitation varies from four times to possibly ten times the percentage uncertainty in the excitation probability of the 4+ state. The extraction of the hexadecapole contribution to the excitation of the 4+ state depends directly upon the accuracy with which the multiple E2/E2 contribution can be correctly calculated. An uncertainty in M(E2;0^+→2^+) of 1/2% results in an uncertainty in the calculated P(4) of about 1-1/2% and an uncertainty in the E4 contribution of 5% to 15%, assuming the uncertainty in M(E2;2^+→4^+) to be the same as that in M(E2;0^+→2^+). The best measurements of M(E2;2^+→4^+) claim an accuracy of 1-1/2% to 2%\(^{13}\), and these measurements exist only for the samarium isotopes. If the rotational model is used to calculate that matrix element from M(E2;0^+→2^+), its uncertainty must reflect not only the uncertainty in M(E2;0^+→2^+), but also the degree to which the model is applied to correctly predict it; this will be treated at greater length in a subsequent section. The uncertainty in the calculation of P(4) is therefore expected to be roughly three per cent, assuming the uncertainty in M(E2;0^+→2^+) to be small compared to the uncertainty in M(E2;2^+→4^+). It was therefore decided to attempt to measure P(4) to 2%-3% accuracy and P(2) to 1% accuracy (which yields M(E2;0^+→2^+) to about 1/2% accuracy).
The high accuracy with which the small inelastic lines, close in energy to the much larger elastic line, must be determined dictates careful attention to experimental resolution. For example, if the full width at half maximum (FWHM) of the spectral lines is 25 keV, the separation of the 80 keV $2^+$ peak is only three times FWHM; if however, the resolution is improved to 13 keV that separation increases to over five times the FWHM. This is especially significant because the line shapes fall rapidly to zero as the distance from the center of the peak increases. The specific shapes of the spectral lines and the influence on resolution will be discussed in detail in Chapter IV, but the general sharp increase in peak separations—measured as the ratio of the inelastic peak to the valley between the elastic and inelastic peaks—with comparatively small gains in FWHM requires that every possible contribution to resolution be carefully examined. For example, if the FWHM is 15 keV, the $2^+$ peak to valley ratio in the spectrum of a nucleus such as $^{154}\text{Sm}$ where the excitation energy is about five times the FWHM (80 keV) is about 20:1, while the peak to valley ratio for $^{152}\text{Sm}$ where the inelastic line is separated by eight times the FWHM (120 keV) is about 70:1 to 80:1.

The experimental apparatus was therefore designed with careful attention to spectral resolution, and the effects of contributions from the beam, the detector and pulse height analysis electronics, and the target were examined.
He beams of 8 to 17 MeV were produced by the Yale University MP1 tandem Van de Graaff accelerator. The energy is regulated by requiring the beam to pass through a 90° analyzing magnet and remain well collimated. The short term (1/2 hr.) stability of the magnet power supply maintains the beam energy within about 1/4 keV, while the long term stability is better than one keV. The beam energy was monitored during long runs, and was readjusted if it wandered more than + 1/2 keV. The absolute energy calibration was determined by comparing elastic scattering to the lines from a $^{212}$Pb source described below. It was found to be within 5-10 keV of its nominal value.

After leaving the analyzing magnet, the beam is focused in a quadrupole triplet lens, steered into the proper beam line, and focused onto the target by a quadrupole doublet lens. Steering magnets were employed to maintain the position of the beam on the optical axes of the quadrupole lenses to insure that the lenses were non-steering. A final set of steering magnets steered the beam into the collimation system shown in Figure 1, which was assembled in an ORTEC 2800 series scattering chamber.

The spacing and sizes of the first two apertures were calculated from the divergence of the beam when focused to a one mm. spot (the smallest spot obtainable without delimiting the beam with apertures that remove sections of the beam). The edges of the tantalum apertures were polished
Figure III-1. Arrangement of equipment inside the scattering chamber. (Not to scale) Slits are round apertures; the inner edges are rounded and highly polished.
Figure III-1
in order to reduce slit scattering, and the slit at the back of the detector mount acted as an anti-scatter slit. The presence of a halo beam slightly degraded in energy by slit scattering would broaden the spectral lines due to the spread in incident energies. The beam current collected at the slits was monitored, and the beam optics were adjusted so that more beam was lost on the first slit than on the second, and almost none was collected at the third. This implies that the beam was not being focused somewhere between the slits and diverging to the target, and was not wandering across the face of the target. It also indicates that very few particles were being scattered out of the beam by the first two slits to be collected at the third. The polished lucite detector shields cover the region where the detector is attached to its mount and the region around the central hole in order to prevent scattered projectiles from being detected in these regions where charge collection is expected to be poor.

Detector noise was reduced by cooling the detector until the leakage current, typically 0.25 to 0.5 microamps at room temperature, was reduced to below 0.05 microamps. In order to prevent contaminants in the vacuum system from building up on the target or the cooled detector, a copper cryogenic trap was constructed to surround the target, and to provide a colder surface close to the detector face. The trap was kept cold by a continuous flow of liquid nitrogen. Pressure inside the chamber was typically $10^{-6}$ Torr.
The response of a surface barrier particle detector to incident particles typically results in a gaussian peak with a long low energy tail. The shape and relative height of the tail were studied as a function of detector bias; the bias was progressively increased until the tail ceased to show improvement. The detectors were therefore operated at large overbias.

Under normal experimental conditions, count rates for alpha particles scattered from the rare earth target material averaged from 5/sec to 50/sec. However, the total count rate on occasion was as large as $10^4$/sec, originating principally from elastic and inelastic scattering from the carbon backing material. The pulse height distribution of the latter pulses was sufficiently separated from the primary data as to constitute a pulse pile-up problem only. The use of a pile-up rejection system made it possible to use incident beam currents of up to 350 nanoamperes without observing the characteristic high energy peak broadening due to pulse pile-up. Target degeneration problems, however, dictated a limit in beam intensity of 200-250 nanoamps.

The pile-up rejection system, consisting of the timing filter amplifier, constant fraction timing discriminator, and pile-up inspector as shown in Figure 2, determines whether two consecutive pulses from the preamplifier fall within a specified resolving time of each other, in this case sixteen microseconds. The constant fraction timing discriminator output occurs at a fixed time after the
Figure III-2. Electronics. All units are ORTEC standard NIM modules except the ADC and the multi-channel analyzer.
Figure III-2
pulse from the preamplifier has begun to rise. All pulses from this unit go to the pile-up inspector, which delivers a pulse to the coincidence circuit if two consecutive pulses are within the resolving time; this pulse is used in anti-coincidence in the universal coincidence unit. The adjustment of the lower level discriminator of the constant fraction unit—the voltage level below which a pulse would be ignored—was found to be very important; if adjusted at too high a level the very low energy pulses would be missed by the pile-up rejector, if adjusted too low the unit would trigger on random noise and all pulses would be gated out. The second input to the universal coincidence from the pile-up rejection system simply counts all pulses counted by the constant fraction unit.

The lower line of Figure 2 comprises a DC coupled linear pulse height analysis system. The single channel analyzer which receives a pulse from the main amplifier and sends a pulse to the universal coincidence is used to set a lower limit on the heights of the pulses that will be analyzed in the multichannel analyzer. The coincidence of the output of this unit and all pulses, if not eliminated by an anti-coincidence pulse from the pile-up rejection system, is used to gate the biased amplifier. The spectrum of pulses at the output of the biased amplifier therefore consists of all pulses above a set lower discriminator level which have not occurred within sixteen microseconds of another pulse. Thus a large number of pulses which are of
no interest to the experiment are prevented from reaching the ADC, which has a comparatively large dead time.

Careful attention was paid to the settings of DC output levels, pulse shaping in the main amplifier, and the setting of the baseline restorers, both the separate unit before the ADC and the unit internal to the amplifier. Resolution tests were run for various combinations of these settings. The resolution of the entire system was tested using a thin $^{212}$Pb radioactive source. $^{212}$Pb decays by $\beta$ emission to $^{212}$Bi which emits alpha particles at 6.0897 MeV and 6.0498 MeV. $^{212}$Bi also $\beta$ decays to $^{212}$Po, which in turn decays emitting an alpha particle at 8.78 MeV. The FWHM of the single line was typically 13-14 keV, and the lower doublet was well resolved, with a peak to valley ratio between 12:1 and 15:1. When sources with strong activity were used, the anti-pile-up system increased the peak to valley ratio of the doublet and visibly improved the shape of the leading edge of the upper peak.

Since 12 MeV alpha particles will lose approximately 0.15 keV for each microgram/cm$^2$ of rare earth material they pass through, and the problem is compounded at backward angles due to the fact that some of the scattered particles will pass through the entire target twice, targets with areal densities of only a few micrograms/cm$^2$ could be employed if the contribution from the targets was not to significantly degrade the system resolution. Such thin targets cannot be made self-supporting. The targets were also required to be reasonably uniform and as free from contaminants close in
mass to the target material as possible. For example, the elastic scatter line from an impurity with a 1% concentration will be ten times as intense as the \(4^+\) line.

The metallic rare earth targets were prepared from isotopically enriched oxide samples obtained from the stable isotope separation facility at Oak Ridge National Laboratory. These samples are listed in Table 1. Most of the samples are better than 98.5\% enriched, and several have purities in excess of 99.9\%. All targets, with the exception of the Gadolinium isotopes, were prepared by reduction and simultaneous evaporation onto commercially prepared 10 microgram/cm\(^2\) carbon foils. The gadolinium targets were fabricated by evaporating previously reduced metallic samples. Target fabrication is discussed in Appendix VII.

Spectra obtained for elastic scattering from the thinnest targets, which typically produced count rates of 5-10 per second, resulted in a broadening of the system resolution by less than one keV. \(2^+\) peak to valley ratios were as good as 80:1 for \(^{152}\)Sm, and 25:1 for the more rotational nuclei. Elastic peak to background ratios were 20,000:1 to 30,000:1. These thinnest targets were used to measure \(P(2)\); spectra with 12,000 to 15,000 counts in the \(2^+\) peak were accumulated in two to five hours. Such count rates are impractical for the measurement of \(P(4)\), which is typically two orders of magnitude smaller than \(P(2)\). Raising the beam current from the 150-200 nA used in the \(2^+\) measurements to above 300 nA caused noticeable resolution degradation, probably due to
Table III-1

Enrichment of Isotopic Samples used in target fabrication. These are the purities listed in the assay supplied with the rare earth oxide samples obtained from Oak Ridge National Laboratory.

<table>
<thead>
<tr>
<th>ISOTOPE</th>
<th>SAMPLE NUMBER</th>
<th>PURITY (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{152}_{\text{Sm}}$</td>
<td>114701</td>
<td>99.06</td>
</tr>
<tr>
<td>$^{154}_{\text{Sm}}$</td>
<td>167703</td>
<td>98.69</td>
</tr>
<tr>
<td>$^{158}_{\text{Gd}}$</td>
<td>109690</td>
<td>97.58</td>
</tr>
<tr>
<td>$^{160}_{\text{Gd}}$</td>
<td>164401</td>
<td>99.9932</td>
</tr>
<tr>
<td>$^{164}_{\text{Dy}}$</td>
<td>171701</td>
<td>98.43</td>
</tr>
<tr>
<td>$^{166}_{\text{Er}}$</td>
<td>140801</td>
<td>99.97</td>
</tr>
<tr>
<td>$^{168}_{\text{Er}}$</td>
<td>140901</td>
<td>99.987</td>
</tr>
<tr>
<td>$^{174}_{\text{Yb}}$</td>
<td>124701</td>
<td>98.97</td>
</tr>
</tbody>
</table>
target deterioration; for example, the peak to valley ratio would decrease from 25:1 to 15:1 over the course of several hours. In order to avoid this problem and obtain reasonable counting statistics in the $4^+$ peak it was necessary to employ thicker targets, which broadened the resolution by about three keV and produced peak to valley ratios of 17:1-20:1 for the $2^+$ states. Peak to valley ratios for $^{152}$Sm were considerably better. The accompanying decrease in resolution was acceptable due to the greater separation of the $4^+$ peak. Spectra with 2500 counts in the $4^+$ peak were accumulated in 10 to 20 hours.

In order to determine the "safe" bombarding energy, below which the interaction can be considered pure Coulomb to within the accuracy of this experiment, the behavior of elastic and inelastic scattering was studied as a function of bombarding energy. If the interaction is entirely Coulomb, the total scattering cross-section should be equal to the Rutherford cross-section. To avoid measurements of the absolute cross-section, composite $^{208}$Pb-$^{152}$Sm and $^{208}$Pb-$^{154}$Sm targets were prepared, so that the scattering from samarium could be measured relative to scattering from lead, which should conform to the Rutherford cross-section to several MeV higher bombarding energy than does scattering from samarium. To further reduce target deterioration and wandering of the beam across the surface of the target—both of which could change the ratio of samarium and lead cross-sections—a fourth collimator was placed fifteen centimeters
upstream of the others shown in Figure 1 and the beam intensity was reduced to 30nA-60nA. Composite targets of other isotopes with lead were also prepared. The lead peaks in the spectra obtained for scattering from the composite targets served as a convenient monitor of the peak shape during the experiment, and the $^{208}\text{Pb}$ targets provided a good determination of the elastic peak shape.
REFERENCES

CHAPTER III


A. DATA

The primary data for each isotope consisted of thick and thin target spectra for several incident $^4\text{He}$ energies. The thin target spectra, which typically contained 12,000 to 15,000 counts in the $2^+$ peak, were used for the extraction of the $2^+$ excitation probabilities. Higher counting statistics were found to be, in general, superfluous, as the uncertainty in determining the number of counts in the peak was dominated by the uncertainty in unfolding the spectrum as the total number of counts was increased past this level. This observation was supported by attempts to determine $P(2)$ from thick target spectra which contained up to 300,000 counts in the $2^+$ peak. Thick target spectra were accumulated until there were 2200-3000 counts in the $4^+$ peak; several spectra were accumulated with higher statistics. Some spectra were accumulated at a mean laboratory scattering angle of 171.5°, while the rest were accumulated at 174.5°.

Figure 1 shows thick and thin target $^{152}\text{Sm}$ spectra. Spectra of $^{160}\text{Gd}$ shown in Figure 2 are similar to the spectra obtained for all other isotopes, as can be deduced from the similarity in the energy spacing of the levels shown in...
Figure 4. The effect of $^{208}$Pb deposited on the same target as the rare earth target is shown in Figure 3. A list of spectra from which $2^+$ and $4^+$ excitation probabilities were determined is given in Appendix III. Not all spectra listed in the appendix were used in the determination of the final values of the $M(E2;0^+\rightarrow2^+)$ and $M(E4;0^+\rightarrow4^+)$. Many of the $^{152}$Sm spectra listed in Appendix III were used only in the studies of Coulomb-nuclear interference and the accompanying determination of the maximum "safe" bombarding energy. To this same end, spectra for scattering from combination $^{152}$Sm-$^{208}$Pb and $^{154}$Sm-$^{208}$Pb not listed in Appendix III were accumulated and were used to determine the ratio of scattering from samarium to scattering from lead only. This will be discussed in more detail in the section on Coulomb-nuclear interference.

B. DATA REDUCTION

The extraction of the excitation probabilities from the spectra proceeded in the following manner:

1) Linear backgrounds were subtracted;

2) Contributions from known contaminants were subtracted;

3) The intermediate spectra of long runs were separated, shifted, and readded if necessary to account for any drifts that had taken place;

4) Two runs conducted at the same incident energy and scattering angle could be added; this feature was useful in the rare cases when the
Figures IV-1,2,3. Typical Spectra:

Figure 1. $^{152}$Sm thick and thin target spectra

Figure 2. $^{160}$Gd thick and thin target spectra

Figure 3. $^{164}$Dy–$^{208}$Pb target

Horizontal scale displays channel numbers;
vertical scale lists counts/channel.
Figure IV-1
Figure IV-2

(a)

(b)
Figure IV-4. State spins and energies of the nuclei studied. Not all levels are included. Energies are from Reference 1.
Figure IV-4
accelerator had to be turned off for long periods of time during a long run; and

5) The spectral lines were separated and integrated. The first four steps were accomplished using the computer; two methods were used to unfold the spectral lines, one of which employed the computer.

The backgrounds were mostly flat, varying from 1-2 counts per channel for thin target spectra to 8-9 per channel for some of the thick target spectra. The known contaminant lines were subtracted using the concentrations listed in the assay of the oxide sample. An image of the elastic peak was shifted to the calculated position of the contaminant line, normalized to the calculated height of the contaminant peak, and subtracted from the spectrum point by point. This procedure was repeated for all significant elastic and inelastic contaminant lines. An example of this procedure is shown in Figure 5 which shows a thick target $^{154}$Sm spectrum before and after contaminant subtraction. The presence of the contaminant lines has the following effects in this case: the $^{152}$Sm $2^+$ line is an obvious contaminant; the $^{152}$Sm elastic line broadens the $^{154}$Sm elastic line; and elastic scatter lines from the lighter samarium isotopes fill in the valley. Figure 6 shows a $^{152}$Sm spectrum following the same procedure. The presence of the $^{154}$Sm elastic line distorts the leading edge of the $^{152}$Sm elastic peak and the minute contribution of the first excited state of $^{150}$Sm is sufficient to alter the $4^+$ yield by about 2%, which is on the order of one
standard deviation. Coupled with the lineshape fitting routine, the contaminant subtraction method could also be used to subtract $^{208}\text{Pb}$ lines which put an extra background under some thick target elastic peaks. This is demonstrated in Figure 7.

Figure 8 shows the result of using the low energy tail of a $^{208}\text{Pb}$ peak to draw rough approximations to the $0^+$ tail under the $2^+$ peak and the combined tail under the $4^+$. The errors introduced by simply summing the peaks from valley to valley and hence ignoring the tails are listed in the table at the bottom of the figure. It is clear that the errors involved are far greater than the statistical uncertainties. It is also clear that an accurate method is needed to reliably account for these contributions if experimental accuracy is to approach the 1% and 2% counting statistics sought for the $2^+$ and $4^+$ peaks respectively. It should be emphasized that just drawing a tail on the peak, while more accurate than just summing from valley to valley, has a fairly large uncertainty associated with it, even if the shape of the tail is known. A more reasonable method must put other constraints on the tails.

The shape of a single peak produced by scattering at a given angle of a given projectile (in this case alpha particles) from a very thin target should depend only upon the response of the detector and the electronics used in the pulse height analysis. As the target becomes thicker, the peak should change in width, but not alter its basic
Figures IV-5,6,7. Contaminant subtraction process:

Figure 5. $^{154}$Sm thick target spectrum
Figure 6. $^{152}$Sm thick target spectrum
Figure 7. $^{164}$Dy-$^{208}$Pb including subtraction of $^{208}$Pb elastic peak.

Vertical scale in counts/channel
Horizontal scale in channels
Figure IV-5
$^{152}\text{Sm}(\alpha,\alpha')^{152}\text{Sm}$

$E_{\text{inc}} = 11.75\text{ MeV}$

$^{154}\text{Sm}^2^+$ 0.04%

$^{147}\text{Sm}$ 0.19%

$^{148}\text{Sm}$ 0.07%

$^{150}\text{Sm}$ 0.09%

$^{149}\text{Sm}$ 0.11%

$^{154}\text{Sm}^{0^+}$ 0.48%

BEFORE CONTAMINANTS SUBTRACTED

AFTER SUBTRACTION OF CONTAMINANTS AND BACKGROUND

Figure IV-6
Figure IV-7
shape, since that shape is equivalent to the integral of peaks produced by scattering from vanishingly thin targets over a small range of incident energies. Both the elastic and inelastic peaks in one spectrum should have identical shapes, differing only in overall normalization, and the peaks in different spectra are expected to differ only in some simple way related to their width (i.e. to the thickness of the target).

Examination of elastic peaks from different $^{208}$Pb targets, and comparison with rare earth elastic peaks, demonstrated a constant tail shape over the course of several different runs. A typical $^{208}$Pb peak, as shown in Figure 9, is basically a gaussian with a low-energy tail; it is plotted on semi-logarithmic scale in order to display the tail shape in detail. It was observed that as the target thickness changed, the FWHM of the gaussian and the point at which the tail joined it changed, but tails of different peaks could always be successfully superimposed (within statistics) by shifting the semi-logarithmic plots vertically and horizontally, which corresponds to changing the relative normalization of the tails and the points at which they join the gaussian peak. The $^{208}$Pb peak shapes were compared to $^{152}$Sm elastic peaks where the relatively large separation of the $2^+$ peak makes most of the elastic peak accessible. As a further check, using the computer peak fitting routine discussed below, good fits to elastic peak tails were obtained using the tail of a $^{208}$Pb elastic
Figures IV-8,9

Figure 8. Demonstration of error involved in ignoring backgrounds due to tails of adjacent peaks.

Figure 9. Typical $^{208}$Pb peak.
<table>
<thead>
<tr>
<th>PEAK</th>
<th>VALLEY TO VALLEY SUM</th>
<th>TAIL CORRECTION</th>
<th>CHANGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^+$</td>
<td>5,032,873</td>
<td>+15,600</td>
<td>0.3%</td>
</tr>
<tr>
<td>$2^+$</td>
<td>356,492</td>
<td>-15,450</td>
<td>4.3%</td>
</tr>
<tr>
<td>$4^+$</td>
<td>4,808</td>
<td>-147</td>
<td>3.0%</td>
</tr>
</tbody>
</table>

Figure IV-8
$^{208}\text{Pb}(\alpha,\alpha')^{208}\text{Pb}$

$E_{\text{inc}} = 12.00\text{MeV}$

$\theta_{\text{lab}} = 174.5^\circ$
peak and fitting only its position and normalization. Acceptable fits to $^{152}\text{Sm}$ elastic peak tails were obtained in some cases by using the $^{208}\text{Pb}$ peak tail scaled by the ratio of peak heights and shifted by the separation of the lead and samarium peak centroids, without treating the position and normalization as fitting parameters, i.e. by assuming the peak shapes to be identical.

Two related methods were used to unfold the spectra into their constituent spectra lines. In both methods the criteria were the same: the peaks within one spectrum were required to have the same shape, and the tail of each peak was constrained to be a smooth function similar in shape to the tail of a $^{208}\text{Pb}$ elastic peak. The first method, an iterative manual fitting procedure, proceeded in the following manner:

1) A $^{208}\text{Pb}$ elastic peak tail was used to provide a first approximation of the tail of the elastic peak in a manner similar to that shown in Figure 7. Since typical $2^+$ peak to valley ratios were 20:1 and the excitation probability of the $2^+$ state was usually less than 10%, the elastic peaks were unobscured to about 1/200 of the peak height. The tail was required to join smoothly to the region of the peak just above the valley.

2) The elastic peak including this tail was subtracted point by point from the spectrum.

3) The remaining $2^+$ peak was redrawn and compared to the elastic peak, the requirement being that the leading
edges coincide from the peak channel down to one count.

4) If the fit was unsatisfactory, the procedure was repeated with the joining point and normalization of the reference tail altered.

5) Changes which caused a 5%–10% change in the integrated number of counts under the tail produced readily observable changes in the fit of the inelastic peak to the elastic peak. The uncertainty in the number of counts in the tail was taken to be 10%, which adds an uncertainty in the number of counts in the $2^+$ peak approximately equal to the statistical uncertainty. Attempts to unfold thick target spectra using this technique resulted in realistic uncertainties in the tail of the elastic peak of about 10%, so that the determination of the number of counts in the $2^+$ peak was limited by the unfolding of the tail and not by counting statistics. Uncertainties calculated from the computer fits yielded the same results: if a larger number of counts were accumulated in the spectrum, the uncertainty in the $2^+$ peak expressed as a fraction of the number of counts in the elastic peak tail remained approximately constant.

The background of the $4^+$ peak consists of a composite tail due to both the elastic and the $2^+$ peaks. This shape is not necessarily constant, as it depends upon the spacing of the three peaks. The starting shape of this tail was obtained by drawing a first guess at the elastic peak tail and adding it to an image of the elastic peak tail shifted...
by the separation of the $2^+$ and elastic peaks and normalized by the excitation probability of the $2^+$ state. This tail was then subtracted and the remaining $4^+$ peak compared to the elastic peak shape as outlined above. This composite tail was required to fit the regions above and below the $4^+$ peak. In addition to the matching of the peak shapes, the fits were checked by comparing the number of counts under the trailing edge of the $4^+$ peak with the number of counts in the same region of an image of the elastic peak normalized to the $4^+$ peak. In all cases, the first and last channels of the peak--the channels between which the $4^+$ peak was summed--were determined from the points at which the normalized image of the elastic peak fell below 0.5 counts. It was found that a 10% to 15% variation in the tail was necessary to obtain a noticeable deviation in the resultant peak shape, and this was therefore taken as the uncertainty in the unfolding procedure.

A functional fit to the spectra was explored using a spectrum fitting code based on the routine CURFIT by Bevington. This program made use of the interactive CRT display on the computer to continuously monitor the fitting procedure.

Several functions were tried with varying degrees of success until a function was found which produced reliable fits to the data. A function consisting of a gaussian peak with an exponential tail, which is an inverted parabola with a straight line tail when displayed on a semi-logarithmic
scale, reproduced a single peak and the region where it first deviated from Gaussian, but could not follow the continuously turning tail. Adding more exponentials in sequence progressively improved the fit, but an infinite number of such straight lines would be required to fit the curve. The most successful functional fit of this type was obtained with the function suggested by Aponick, which consists of a Gaussian followed in sequence by an exponential, the exponential of a quadratic and another exponential; this was modified so that the Gaussian was asymmetric. Fits of a single peak were obtained with chi squared per degree of freedom of 1.2 to 1.8. The function became troublesome when attempts were made to fit several peaks simultaneously, as the individual components of the tail could not be sufficiently constrained to produce reliable fits.

A functional fit to the tail was achieved by using a hybrid method which made use of the empirical peak shape of the elastic peak of each spectrum and a tail function which was the exponential of a general hyperbola given by:

\[ f(x) = A_1 \exp \left[ \left( \frac{A_2}{A_7} \right)^2 + \left( x - A_3 \right)^2 \right] . \]

The tail was appended to the rest of the peak shape at a point which was not a variable of the fit, so that the function had the form:

\[ g(x) = \begin{cases} 
  f(x) & x \leq x' \\
  \text{Spect}(x) & x > x'
\end{cases} \]
where \( \text{SPECT}(x) \) is the measured number of counts at the channel \( x \). For a single \( ^{208}\text{Pb} \) peak, fits with chi squared per degree of freedom from 0.8-1.6 were obtained for a wide range of fitting regions; i.e. from tails which only covered a small part of the peak to tails which extended through a large region from just below the peak channel to the region where the peak became indistinguishable from the background. Chi squared was calculated only from the region where \( f(x) \) was used for the tail.

The form chosen for the inelastic peaks was

\[
g_N(x) = c_N g(x-x_N)
\]

so that they differed from the elastic peak only in relative normalization, \( c_N \), and relative position, \( x_N \). The normalizations and positions of the inelastic peaks were treated as parameters of the fit, and all three peaks were fitted simultaneously. At the completion of a satisfactory fit the \( 0^+ \) peak was plotted, summed, stored, and subtracted point by point from the spectrum, and the remaining spectrum was displayed. The \( 2^+ \) peak tail was then plotted on the remaining spectrum only where it was obscured by the \( 4^+ \) peak, and subtracted in the same manner as the elastic peak. The \( 4^+ \) peak was summed between the two points beyond which the image of the elastic peak normalized to the \( 4^+ \) peak falls to below 0.5 counts. The final values were checked by calculating the chi squared of the \( 2^+ \) and \( 4^+ \) peaks with respect to the images of the elastic peak. Since the number of
degrees of freedom is approximately equal to the number of channels, the error in the final extraction was taken to be the statistical error multiplied by chi squared per degree of freedom. In those cases in which both hand and computer fits were obtained, this determination agreed with the method of taking the error in the tail added in quadrature to the statistical error. It should be noted that in the case of 2^+ peaks extracted from thick target spectra with high statistics both the methods yield the same result: the uncertainty is primarily due to the determination of the tail, and a large increase in the number of counts in the peak over 12,000 to 15,000 is superfluous.

The routine CURFIT^2) searches for a minimum of the chi squared surface in N-dimensional space, where N is the number of parameters being determined. The possibility of finding only a local minimum, which is not the best fit, makes it sensitive to the starting parameters. It is therefore advantageous to use starting parameters that are close to the final values. Making use of the general constancy of the tail shape, a fit to a lead elastic peak tail—preferably from the same spectrum—was normalized and shifted, and "grafted" onto the elastic peak. This was then used in a fit in which only the positions and heights of the inelastic lines were varied. Next, the position and normalization of the tail were allowed to vary along with the parameters of the inelastic lines. This is the computer analog of the manual fitting procedure. Finally, all the
parameters were allowed to vary freely, starting with the values obtained from the preceding fit. This last process was iterated several times, each time beginning with the parameters obtained from the preceding fit. The free fit and the fit in which the tail shape was kept constant but varied in position and normalization usually yielded consistent results. In those cases in which the tail contained few counts and was poorly defined, the free fit could produce obviously wrong fits. This was especially true of some $^{152}\text{Sm}$ spectra which had been spread over a large number of channels and consequently had poorly defined valleys.

After discarding the unacceptable fits, an average weighted by chi squared per degree of freedom was taken of the rest. Good fits were defined as those that had reasonable values of chi squared and tail shapes that were not visibly wrong. Table 1 shows a comparison of some hand fits and some computer fits of the same spectra. The computer fits were most successful in extracting the $2^+$ peaks. The excitation probability of the $2^+$ state of $^{152}\text{Sm}$ was studied as a function of bombarding energy from 8 MeV to 17 MeV on two separate occasions. One set of spectra was analyzed by hand and the other by computer. The results of both are consistent.

The laboratory excitation probabilities are defined as:

$$P(2) = \frac{d\sigma_{\text{lab}}(2^+)}{d\sigma_{\text{lab}}(0^+) + d\sigma_{\text{lab}}(2^+) + d\sigma_{\text{lab}}(4^+)}$$
Table IV-1
Comparison of Hand and Computer Fits

Chi squared per degree of freedom is calculated for the computer fit; it is defined as the total chi squared for the fitted region (only up to the point where the tail joins the elastic peak, so that it does not include the region where the elastic peak is copied point by point) divided by the number of channels in that region.

<table>
<thead>
<tr>
<th>Spectrum</th>
<th>CHI Squared per Degree of Freedom</th>
<th>$2^+$ Counts</th>
<th>$P(2)$</th>
<th>$4^+$ Counts</th>
<th>$P(4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{152}$Sm thin target</td>
<td>1.2</td>
<td>25611</td>
<td>25512</td>
<td>.064132</td>
<td>.063884</td>
</tr>
<tr>
<td>$^{152}$Sm thick target</td>
<td>1.9</td>
<td>341700</td>
<td>341407</td>
<td>.06411</td>
<td>.06405</td>
</tr>
<tr>
<td>$^{154}$Sm thin target</td>
<td>2.2</td>
<td>13194</td>
<td>13186</td>
<td>.08868</td>
<td>.08860</td>
</tr>
<tr>
<td>$^{154}$Sm thick target</td>
<td>2.5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
P(4) = \frac{d\sigma_{lab}(4^+)}{d\sigma_{lab}(0^+) + d\sigma_{lab}(2^+) + d\sigma_{lab}(4^+)}

and were calculated by dividing the integrated number of counts in the $2^+$ and $4^+$ peaks by the total number of counts in the entire region from the low energy end of the $4^+$ peak (or the low energy end of the $0^+$ or $2^+$ peak tail if it extended past the $4^+$) to the high energy end of the elastic peak. All measured excitation probabilities are listed in Appendix IV.

C. ANALYSIS

1. Calculation of Excitation Probabilities

The calculations of the excitation probabilities from the reduced matrix elements were carried out in part using a version of the deBoer-Winther semi-classical multiple Coulomb excitation code which includes E1, E2, E3, and E4 transitions. It has been pointed out that for the conditions encountered in this experiment, where the Sommerfeld parameter $\eta = 10-15$, quantal effects are not negligible for multiple excitations. The semi-classically calculated excitation probabilities therefore were corrected for quantal effects in one procedure that was followed, using the second order perturbation calculation by Alder, et al. We note that the quantal corrections modify the excitation probability of the $4^+$ state due to double E2 and competing direct E4 by about 6%. Exploratory calculations of the full quantal
formulation of this scattering problem were carried out using the code LISA\textsuperscript{19}). The results of that calculation differ somewhat from the perturbation treatment. This point is discussed in detail in Section E.

Calculations of the excitation probabilities using the deBoer-Winther code included states up to 8\(^+\) in the ground state rotational band, as well as the 2\(^{+}\gamma\) vibrational state, the 0\(^+\) and 2\(^+\) members of the 8 vibrational band, and the 1\(^-\) and 3\(^-\) states, where these states have been identified. The reduced matrix elements used in the calculation are listed in Appendix VI. Those not listed were calculated from M(E\(\lambda\);0\(\rightarrow\lambda\)) using the rotational model. The effect of excluding all states except the 0\(^+\), 2\(^+\) and 4\(^+\) states in a calculation of P(4) are shown in Figure 11a for \(^{154}\)Sm.

The sensitivity of the calculation of P(2) and P(4) to variations in the matrix elements used in the calculation is shown in Table 2. The sensitivity of P(2) to all matrix elements other than M(E2;0\(^+\)\(\rightarrow\)2\(^+\)) is clearly negligible compared to the accuracy with which that excitation probability is experimentally determined.

From the variation of the calculated P(2) with M(E2;0\(^+\)\(\rightarrow\)2\(^+\)) shown in Figure 10, it can be seen that a 1\% to 2\% uncertainty in the measured excitation probability results in a 0.5\% to 1\% uncertainty in the extracted matrix element.

Table 3 shows the final values of M(E2;0\(^+\)\(\rightarrow\)2\(^+\)) for the nuclei studied. These numbers are averages weighted by the uncertainties in the individual determinations.
The listed changes in the matrix elements used in the calculation are not necessarily meant to represent realistic experimental uncertainties, but are only to demonstrate the effect on the calculation. In most cases matrix elements to other bands have been determined with uncertainties closer to 30% than to 100%.

<table>
<thead>
<tr>
<th>Matrix Element(s) Changed</th>
<th>M(E2;0⁺→2⁺)</th>
<th>M(E2;2⁺→4⁺)</th>
<th>All E4 Matrix Elements Within Ground State Band</th>
<th>All E2 Matrix Elements to β Band</th>
<th>All E2 Matrix Elements to γ Band</th>
<th>All Matrix Elements to 1⁻,3⁻ States</th>
</tr>
</thead>
<tbody>
<tr>
<td>Change</td>
<td>1%</td>
<td>2%</td>
<td>20%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>Effect on Calculated P(2)</td>
<td>2%</td>
<td>0.06%</td>
<td>0.002%</td>
<td>0.22%</td>
<td>0.2%</td>
<td>0.1%</td>
</tr>
<tr>
<td>Effect on Calculated P(4)</td>
<td>2%</td>
<td>4%</td>
<td>4%</td>
<td>0.5%</td>
<td>0.3%</td>
<td>0.01%</td>
</tr>
</tbody>
</table>
Table IV-3

Measured Matrix Elements in e-barns^{\lambda/2}

<table>
<thead>
<tr>
<th>NUCLEUS</th>
<th>$M(E2;0^+-2^+)$</th>
<th>$M(E4;0^+-4^+)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Set I</td>
<td>Set II</td>
</tr>
<tr>
<td>$^{152}$Sm</td>
<td>-1.865±.012</td>
<td>-1.856±.012</td>
</tr>
<tr>
<td>$^{154}$Sm</td>
<td>-2.093±.015</td>
<td>-2.088±.015</td>
</tr>
<tr>
<td>$^{158}$Gd</td>
<td>-2.242±.011</td>
<td>-2.233±.011</td>
</tr>
<tr>
<td>$^{160}$Gd</td>
<td>-2.292±.014</td>
<td>-2.283±.014</td>
</tr>
<tr>
<td>$^{164}$Dy</td>
<td>-2.370±.020</td>
<td>-2.360±.020</td>
</tr>
<tr>
<td>$^{166}$Er</td>
<td>-2.414±.011</td>
<td>-2.404±.011</td>
</tr>
<tr>
<td>$^{168}$Er</td>
<td>-2.454±.021</td>
<td>-2.444±.021</td>
</tr>
<tr>
<td>$^{174}$Yb</td>
<td>-2.448±.012</td>
<td>-2.438±.012</td>
</tr>
</tbody>
</table>

Set I refers to the matrix elements obtained using the semi-classical calculation corrected for quantal effects according to the second order perturbation treatment. Set II refers to the matrix elements obtained with the calculation corrected according to the results of the full quantal calculation.
uncertainty in the average value is given by:

$$\sigma^2 = \sigma_1^2 + \sigma_2^2$$

where:

$$\sigma_1^2 = \frac{1}{\sum_{1}^{N}\left(\frac{1}{\sigma_1}\right)^2}$$

reflects the uncertainties in the individual points, and

$$\sigma_2^2 = \frac{1}{N}\sum_{1}^{N}(x_i - \bar{x})^2$$

accounts for the spread of the individual determinations about the average.

The uncertainties in the calculation of $P(4)$ are more significant than the uncertainties in the calculation of $P(2)$. The variation of $P(4)$ with $M(E4;0^+\rightarrow4^+)$, with all $E2$ matrix elements fixed and all other $E4$ matrix elements varied in rotational relationship to $M(E4;0^+\rightarrow4^+)$ is shown in Figure 11. Figure 11a illustrates the error introduced by including only the $0^+$, $2^+$, and $4^+$ states in the calculation. As would be expected from Table 2, these errors are small compared to the effects of varying $M(E2;0^+\rightarrow2^+)$ and $M(E2;2^+\rightarrow4^+)$ one standard deviation. These latter variations account for the major uncertainty in the calculation of $P(4)$; the effects of variations in $M(E2;2^+\rightarrow4^+)$ are shown in Figure 11b. The uncertainties in $M(E4;0^+\rightarrow4^+)$ indicated in Table 3 were determined by adding the contributions of uncertainties in the calculation of $P(4)$, which are systematic uncertainties,
Figures IV-10,11

Figure 10. Variation of $P(2)$ with $M(E2;0^+\rightarrow 2^+)$, $M(E2;2^+\rightarrow 4^+)$ fixed for Sm isotopes, varied using rotational model for the rest.

Figure 11. Variation of $P(4)$ with $M(E4;0^+\rightarrow 4^+)$. All other $E4$ matrix elements in rotational relation to $M(E4;0^+\rightarrow 4^+)$. All other matrix elements fixed.

Figure 11a illustrates the error introduced into the calculation by including only three levels: $0^+$, $2^+$, and $4^+$.

Figure 11b illustrates the uncertainties in the calculation of $P(4)$ due to varying the value of $M(E2;2^+\rightarrow 4^+)$ used in the calculation by one standard deviation.
$^{154}\text{Sm}$
$E_a=11.25\text{ MeV}$
$\theta_{\text{lab}}=174.6^\circ$

$\bar{P}(2^+)=\frac{d\sigma(2^+)}{\Sigma d\sigma(N)} (1+\text{QUANTUM CORRECTIONS})$

Figure IV-10
\[ P(4^+)/P[4^+; M(E4)=0] \]

\[ M(E2) = -2.03 \text{ e} \cdot \text{b} \]

\[ M(E\lambda) = (2I_i + 1)^{1/2} \left( \frac{2\lambda + 1}{16\pi} \right)^{1/2} Q_{\lambda 0} \langle I_i \lambda K_0 | I_f K \rangle \]

\[ \theta_{sc} = 174.5^\circ \]

\[ E_\alpha = 12.0 \text{ MeV} \]

**LEVEL CALC.**

**3 LEVEL CALC.**

\((0^+; 2^+; 4^+)\)

**(QUANTAL CORRECTIONS INCLUDED)**

**Figure IV-11a**
$E_\alpha = 12 \text{ MeV}, \theta_{sc} = 174.5^\circ$

$P(4^+) \text{ for } M(E2; 2 \rightarrow 4) = (-3.267 \text{ eb}) \pm 1.6\%$
to the uncertainties in the weighted averages calculated in the same manner as the uncertainties in \( M(E2; 0^+ \rightarrow 2^+) \) discussed above.

\( M(E2; 2^+ \rightarrow 4^+) \) has been determined from lifetime measurements for the samarium isotopes\(^6,7\) with a standard deviation of about 1.8%. No such precision measurements exist for the other isotopes studied, and the rotational model was therefore employed to calculate \( M(E2; 2^+ \rightarrow 4^+) \) for those nuclei. The uncertainties assigned to these calculated values must reflect both the experimental uncertainty in the values of \( M(E2; 0^+ \rightarrow 2^+) \) from which they were calculated, and the degree to which \( M(E2; 2^+ \rightarrow 4^+) \) can be expected to conform to the rotational model prediction.

The uncertainties introduced by using the rotational model can be estimated by examining a single model which deals with deviations from rotational model behavior.

Although the energy sequence of the low-lying states in the ground state bands of these nuclei conforms reasonably well to the rotational prediction:

\[
E = AI(I+1)
\]

the fit is improved with the addition of a higher order term:

\[
E = AI(I+1) - BI^2(I+1)^2
\]

the effect of which becomes more evident in the energies of higher spin states. For the nuclei \(^{158}\text{Nd}, \, ^{160}\text{Gd}, \, ^{164}\text{Dy}, \, ^{166}\text{Er}, \, ^{168}\text{Er}, \text{ and } ^{174}\text{Yb} \) the ratio \( B/A \) as calculated from
the energies of the $2^+$ and $4^+$ states is between 0.0004 and 0.0010, indicating that the rotational predictions for the energies of these states is indeed quite good. It should be noted, for comparison, that in the case of $^{152}\text{Sm}$, B/A = 0.0067.

This correction to the energy sequence can be calculated from several theoretical treatments, which also yield related deviations from the rotational prediction for the relationship between all matrix elements of the same multipolarity. If the extreme point of view is taken that all the deviation is attributable to centrifugal stretching, the form of the E2 matrix elements becomes:

$$<I_f|\langle M(E2) |I_i \rangle = (2I_i + 1)^{1/2} <I_i 200|I_i 2I_f 0> <2|\langle M(E2) |0>$$

$$\times \left\{ \left[ 1 + (B/2A) \right] \left[ I_i (I_i + 1) + I_f (I_f + 1) \right] / \left[ 1 + 3B/A \right] \right\}$$

A calculation employing the rotational-vibrational interaction in a procedure which was used to successfully reproduce the prominent deviations from rotational behavior in E2 matrix elements observed in $^{152}\text{Sm}$ has shown that the centrifugal stretching interpretation overestimates these deviations. The uncertainty in the matrix element $M(E2;2^+\rightarrow 4^+)$ due to assuming the rotational model is 0.5% to 0.9%, so that the uncertainty in the calculated P(4) including the uncertainty in $M(E2;2^+\rightarrow 4^+)$ is typically 2.5%. This includes both the uncertainty involved in using the rotational model and the uncertainty in the value of
M(E2;0^+\rightarrow 2^+) from which M(E2;2^+\rightarrow 4^+) is calculated. It should be emphasized that this represents a "worst case" analysis of the errors involved in using the rotational model, since more sophisticated analyses of the deviations from rotational behavior indicate that the centrifugal stretching analysis overestimates these deviations. 20)

It is clear from Figure 11 that there are two possible values of M(E4;0^+\rightarrow 4^+), a positive value and a negative value, which will correctly predict the measured P(4). The positive value is normally chosen. The reasons for this will become clear when the extraction of the Coulomb deformation parameters \( \beta_2^c \) and \( \beta_4^c \) from the measured moments is discussed. No matrix elements determined above the maximum "safe" bombarding energies discussed in the following section were included in the averages in Table 3. In addition, no measurements of M(E2;0^+\rightarrow 2^+) for \(^{152}\)Sm at incident energies below 10 MeV were included, due to the rather large uncertainties in these determinations stemming from the rapid decrease of P(2) with decreasing incident energy and the accompanying large uncertainties inherent in determining the integrated number of counts in the 2^+ peak.

As is indicated in Table 3, significant discrepancies exist between the excitation probabilities calculated by correcting the semi-classical calculation with the perturbation calculation of Alder, et al., and the results of the full quantal calculation using the code LISA. These will be discussed in more detail in the section on quantal corrections.
2. Coulomb-Nuclear Interference: Determination of "Safe" Bombarding Energy

If the analysis of the data under the assumption that the interaction is pure Coulomb excitation is to be valid, it must be determined that no contributions from scattering by the nuclear potential occur at a level detectable within the accuracy of the experiment. This is equivalent to the classical statement that the interaction takes place below the Coulomb barrier. The onset of Coulomb-nuclear interference effects was studied by measuring the ratio of scattering from samarium to scattering from lead as a function of incident energy through the use of composite $^{152}$Sm-$^{208}$Pb and $^{154}$Sm-$^{208}$Pb targets. The possibility of the effects manifesting themselves differently in inelastic scattering was explored by studying $P(2)$ as a function of incident energy. These two isotopes were chosen because the collective structure of $^{154}$Sm is very similar to that of the heavier nuclei studied, while $^{152}$Sm is somewhat transitional, and conforms less well to the rotational model.

If the Coulomb barrier is calculated, in the classical sense, as the incident energy necessary for the projectile and target to just "touch," overcoming the Coulomb repulsion for a head on collision, that energy is given by:

$$E_b = \frac{Z_1 Z_2 e^2}{\frac{1}{2} r_0 (A_1^{1/3} + A_2^{1/3})}$$
For alpha particles incident on rare earth nuclei, this predicts $E_b = 20\text{MeV}-25\text{MeV}$, which is too high by roughly 10MeV. The prediction can be improved by adding a constant to the denominator, which implies that the range of the nuclear force between projectile and target is greater than the nuclear "radius" by a constant. This has resulted in an empirical rule of the form

$$E_b = \frac{1.44Z_1Z_2}{r_0(A_1^{1/3}+A_2^{1/3})+t(1+A_1/A_2)}$$

where $t$ is typically 5 to 6 fm.\textsuperscript{23) Although the absolute $E_b$ calculated from the classical formula is grossly in error, it is probably not incorrect to assume that the barrier scales according to the classical formula; i.e., as $Z'/Z$. Determination of the "safe" bombarding energy for alpha particles on samarium therefore provides a reasonable determination of the "safe" energy for scattering from heavier nuclei of similar structure.

Below the Coulomb barrier, the scattering cross-section is given by the Rutherford cross-section:

$$d\sigma(\theta) = \left(\frac{Z_1Z_2e^2}{4E}\right)^2 \sin^{-4}(\theta/2)$$

where $E$ is the bombarding energy and $\theta$ is the center of mass scattering angle. For a laboratory scattering angle of $175^\circ$, the center of mass scattering angle, given by:

$$\theta_{cm} = \theta_{lab} + \sin^{-1}\left(\frac{M_1}{M_2}\sin\theta_{lab}\right)$$

(9)
is $174.87^\circ$ and $174.90^\circ$ if the target is $^{152}\text{Sm}$ and $^{208}\text{Pb}$, respectively. The difference in $\sin^{-4}(\theta/2)$ is 0.005%.

The ratio of Rutherford scattering from samarium to Rutherford scattering from lead at the same incident energy is therefore equal to the ratio of the nuclear charges, multiplied by constant factors dependent upon the relative amounts of material in the targets. Since the Coulomb barrier for lead is at least four MeV higher than that for samarium, the lead provides an effective normalization.

The variation of the samarium to lead ratio as a function of incident $^4\text{He}$ energy for both $^{152}\text{Sm}$ and $^{154}\text{Sm}$ are shown in Figure 12. Although the statistical accuracy is 1/2% and 3/4% for $^{152}\text{Sm}$ and $^{154}\text{Sm}$, the consistency in the regions where the curves should be constant indicates that the real uncertainties are closer to 3/4% and 1-1/2%, respectively. Since the beam was very highly collimated, this variation is probably due to changes in the beam profile across the target spot as the beam energy was changed, rather than the beam wandering across the target. The measured ratios are listed in Appendix V. Significant deviations occur for both isotopes at 13MeV to 13.25MeV.

Interference effects in the excitation of the $2^+$ state were studied by examining the behavior of $M(E2;0^+\rightarrow2^+)$, extracted under the assumption of pure Coulomb excitation, as a function of incident energy from 8MeV to 16MeV for $^{152}\text{Sm}$, and over the more limited range from 11MeV for 13MeV for $^{154}\text{Sm}$. The results are shown in Figure 13, and included
Figures IV-12,13.

Figure 12. Ratio of scattering from $^{152}\text{Sm}$ to scattering from $^{208}\text{Pb}$, and $^{154}\text{Sm}$ to $^{208}\text{Pb}$ as a function of incident $^4\text{He}$ projectile energy.

Figure 13. Variation with incident projectile energy of $M(E2; 0^+ \rightarrow 2^+)$ extracted under the assumption of pure Coulomb excitation of $^{152}\text{Sm}$ and $^{154}\text{Sm}$. 
Figure IV-12
Figure IV-13
Deviations on the 1% level occur in the $^{152}\text{Sm}$ data above 12MeV, and in the $^{154}\text{Sm}$ data at the same, or possibly slightly higher, incident energy. This onset of interference effects at a lower energy for inelastic scattering than for elastic scattering has also been observed for $^{152}\text{Sm}$ by Brückner, et al.\textsuperscript{10}; in addition these authors observe interference effects in scattering from the $4^+$ state beginning at about the same incident energy as interference in elastic scattering. Stelson, et al.\textsuperscript{11} have observed interference effects in both the $2^+$ and $4^+$ cross-sections of $^{154}\text{Sm}$, $^{166}\text{Er}$, and $^{182}\text{W}$ at 12MeV, 13.5MeV, and 14MeV, respectively. Assuming 12MeV to be a safe bombarding energy for the samarium isotopes, which is consistent with available data, and scaling according to $Z'/Z$, the following upper limits were adopted:

\begin{align*}
152,154\text{Sm} & \quad 12\text{MeV} \\
158,160\text{Gd} & \quad 12.25\text{MeV} \\
164\text{Dy} & \quad 12.5\text{MeV} \\
166,168\text{Er} & \quad 13\text{MeV} \\
174\text{Yb} & \quad 13\text{MeV}
\end{align*}

A comparison of pure Coulomb and Coulomb plus nuclear potential scattering of 12MeV alpha particles from $^{152}\text{Sm}$ at 165° to 178° performed using the code LISA\textsuperscript{19} and the optical potential parameters determined by Brückner, et al.\textsuperscript{10} demonstrates a nuclear-Coulomb interference which lowers the $2^+$ cross-section by 0.6% and the elastic cross-section by
0.06%. This agrees well with the trends shown in Figures 12 and 13. The 0.4% decrease in the $4^+$ cross-section due to nuclear-Coulomb interference is negligible compared to the accuracy of the measurement.

3. Full Quantal Calculation

The computer code LISA$^{19}$ solves the set of coupled equations of the partial wave expansion of the quantal scattering problem as outlined in Section II-B. The interaction includes both Coulomb and nuclear parts. The long range of the Coulomb force makes it necessary to include a large number of partial waves and integrate over many nuclear radii; in order to insure convergence on the 0.1% level in all calculated cross-sections in the case of $^4$He projectiles scattered from samarium, it was necessary to include 240 partial waves and integrate to 180 fm from the center-of-mass.

The pure Coulomb, Coulomb plus nuclear, and pure nuclear scattering calculations, which included the $0^+$, $2^+$ and $4^+$ states were performed for 12MeV incident projectile scattering from $^{152}$Sm to check nuclear-Coulomb interference near the postulated barrier, and to calculate the difference between the classical and quantal pure Coulomb calculations. The calculated nuclear-Coulomb interference effects in the $0^+$, $2^+$, and $4^+$ cross-sections of 0.06%, 0.6%, and 0.4%, respectively, are in agreement with the experimental results and support both the chosen "safe" bombarding energy and the contention that interference effects become significant
in the $0^+$ and $4^+$ cross-sections only at higher bombarding energies than in the $2^+$ cross-section. The optical parameters used in the calculation are those of Brückner, et al.\textsuperscript{10}) The pure Coulomb calculations were also done for scattering at $12\text{MeV}$ from other nuclei.

The pure Coulomb calculations were compared to a three level deBoer-Winther code calculation. The quantally calculated cross-sections were lower than the classical cross-sections, as is expected; representative differences are:

<table>
<thead>
<tr>
<th></th>
<th>$\Delta\sigma(2^+)$</th>
<th>$\Delta\sigma(4^+)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{152}\text{Sm}$</td>
<td>1.1%</td>
<td>7.7%</td>
</tr>
<tr>
<td>$^{154}\text{Sm}$</td>
<td>1.5%</td>
<td>8.1%</td>
</tr>
<tr>
<td>$^{166}\text{Er}$</td>
<td>1.2%</td>
<td>7.4%</td>
</tr>
</tbody>
</table>

These corrections fluctuate by about 0.25% with center of mass scattering angle in $1^\circ$ steps from $165^\circ$ to $178^\circ$. The correction to the $2^+$ cross-section is 0.5% to 1% less than the correction calculated from perturbation theory treatment of Alder, et al.\textsuperscript{5}) while the correction to $4^+$ cross-section is 1.5% to 2.1% greater than the perturbation theory result. It should be noted that the change in the calculated value of $P(4)$ after $M(E2;0^+\rightarrow2^+)$ has been adjusted to fit the quantally calculated $P(2)$ is about the same in all cases.

A second set of matrix elements $M(E2;0^+\rightarrow2^+)$ and $M(E4;0^+\rightarrow4^+)$ was obtained from each nucleus by applying the quantal corrections discussed above. These are listed in Table 3 as Set II. It is important to note that the
difference in $P(4)$ calculated using the perturbation theory quantal corrections and that calculated using the full quantal calculation after the value of $M(E2;0^+\rightarrow 2^+)$ used in each calculation has been adjusted to make the calculated values of $P(2)$ agree is about 3%. This is a substantial fraction of the entire $E4$ contribution, especially for the nuclei from $^{160}\text{Gd}$ to $^{174}\text{Yb}$ where it causes a change in the extracted $M(E4;0^+\rightarrow 4^+)$ value of about $0.2e\text{b}^2$. $M(E4;0^+\rightarrow 4^+)$ is typically $0.1$ to $0.3e\text{b}^2$ in this region.

4. Deformation Parameters

The measurement of $M(E2;0^+\rightarrow 2^+)$ and $M(E4;0^+\rightarrow 4^+)$ determines two model-independent properties of the nuclei studied. If the assumption is made that the low-lying states of these nuclei are well described by the rotational model—and it has been demonstrated that these states conform to certain predictions of the rotational model on the 1%-2% level—then these matrix elements may be interpreted in terms of the intrinsic quadrupole and hexadecapole moments of the nuclei, as demonstrated in Appendix II. If some further assumptions are made about the shape of the charge distribution (see Section IC) which is described by these moments, the data may be interpreted in terms of the Coulomb deformation parameters $\beta^C_\lambda$, which facilitates comparison with the results of other experiments, but describes the nucleus in terms of parameters that are more model-dependent than either the $E4$ transition matrix elements or the intrinsic moments of the electric field.
The deformation parameters $\beta_2^c$ and $\beta_4^c$ were extracted under the assumption of a Fermi charge distribution with a deformed spheroidal surface, as suggested by the results of electron scattering experiments and mu-mesic atom studies.\(^{12, 13, 14, 15, 16, 18}\)

\[
\rho(r, \theta) = \rho_0 [1 + \exp\left(\frac{r-R(\theta)}{t}\right)]^{-1}
\]

\[
R(\theta) = R_0 [1 + \beta_2 Y_2^0(\theta) + \beta_4 Y_4^0(\theta)]
\]

The measured moments were calculated from the deformation parameters by numerically integrating the equations:

\[
Q_\lambda = M(E\lambda; 0+\lambda) = Ze \frac{\int d\Omega d\Omega' r^2 [1 + e^{\left(\frac{r-R(\theta)}{t}\right)}]^{-1} Y_\lambda^0(\theta, \phi)}{\int d\Omega d\Omega' r^2 [1 + e^{\left(\frac{r-R(\theta)}{t}\right)}]^{-1}}
\]

simultaneously for $\lambda = 2, 4$. The results were plotted as a function of $\beta_2^c$ and $\beta_4^c$ and compared with experiment as shown in Figure 14. The volume and central density, $\rho_0$, are therefore adjusted as the deformation is changed, keeping $r_0$ constant at its experimentally determined value. This conserves total charge. This differs in detail from the treatment employed by some other authors\(^{8, 21}\) in which the central density is kept constant at the value calculated for a spherical nucleus of the same charge and mass, and $r_0$ is allowed to vary as the deformation changes to keep the volume and total charge constant. It has been calculated by Bemis, et al.\(^{8}\) that the difference between these two methods (i.e., fixing $r_0$ and allowing $\rho_0$ to vary,
Figure IV-14. Calculation of intrinsic moments $Q_2' = M(E; 0^+; 2^+)$ and $Q_4' = M(E4; 0^+; 4^+)$ for various values of $\beta_2$ and $\beta_4$. $^{152}\text{Sm}$, $R_0 = 5.763$ fm, $t = 0.6$. Deformed Fermi distribution.
Figure IV-14

$M(E^2;0^+\rightarrow 2^+) \text{ in e\cdot barns}^2$

$M(E^4;0^+\rightarrow 4^+) \text{ in e\cdot barns}^2$

- $\beta_4 = 0.03$
- $\beta_4 = 0.06$
- $\beta_4 = 0.09$
- $\beta_2 = 0.30$
- $\beta_2 = 0.28$
- $\beta_2 = 0.26$
and fixing $\rho_0$ and allowing $r_0$ to vary) changes the calculated values of $\rho_0$, $\beta_2^c$, and $\beta_4^c$ by -1.8%, -0.8%, and -2.0%, respectively, for the highly deformed actinide nucleus $^{234}\text{U}$ ($\beta_2^c = .248$, $\beta_4^c = .142$). Such changes are smaller than experimental uncertainties encountered in this experiment.

The results of recent electron scattering measurements performed on $^{166}\text{Er}$ and $^{176}\text{Yb}$ indicate possible serious deviations from a deformed Fermi distribution, due to regions of density higher than the central density near the poles of the quadrupole shape, which would cause much larger changes in $\rho_0$, $\beta_2^c$, $\beta_4^c$ extracted from measured moments.

The parameters of the radial charge distribution, $r_0 = R_0/A^{1/3}$ and $t$, were obtained from the mu-mesic atom studies of Hitlin, et al.$^{13}$) and the electron scattering experiment of Bertozzi, et al.$^{14}$) The first set of parameters were obtained by Hitlin, et al. by simultaneously extracting $r_0$, $t$, and $\beta_2^c$ from measurements on mesic atoms of $^{150}\text{Nd}$, $^{152}\text{Sm}$, $^{162,164}\text{Dy}$, $^{168,170}\text{Er}$, and several Tungsten isotopes, under the assumption of $\beta_4^c = 0$. Since $r_0$ and $t$ vary slowly throughout the region, interpolated values were used in the analysis of this experiment where the measurements were not available. The electron scattering experiment was performed only for $^{152}\text{Sm}$, and the measured $r_0$ and $t$ were used for all isotopes studied in this experiment. The analysis of the electron scattering experiment included a determination of $\beta_2$ and $\beta_4$. As discussed in Chapter I, the analysis of mu-mesic atom experiments and electron
scattering experiments usually takes the form of the determination of a region in $r_0$-$t$ space. All sets of $r_0$ and $t$ within that region reproduce the data within one standard deviation; the addition of deformation to the charge distribution adds other dimensions, but not degrees of freedom. There are therefore many non-unique sets of $r_0$ and $t$.\(^\text{17}\)

The deformation parameters extracted from given moments are, unfortunately, very sensitive to $r_0$ and $t$ used in the calculation. For example, in the sharp cut-off distribution discussed in Chapter I, $Q_\lambda$ is proportional to $r_0^\lambda \cdot \beta_\lambda$ in lowest order. The results of the calculations using both sets of $r_0$ and $t$ are given in Table 4. The dependence on $r_0$ and $t$ used in the calculation is demonstrated in Table 5 which shows the results of calculations using three different sets of $r_0$ and $t$ to analyze the $^{152}\text{Sm}$ data.

The parameters $r_0$ and $t$ used in the extraction of the deformation parameters have associated with them a value of $\beta_2$ which was obtained simultaneously in the analysis of the mu-mesic atom experiment.\(^\text{13}\) In that analysis $\beta_4$ was assumed to be zero. If a new set of $\beta_2$, $\beta_4$ is to be substituted for that used in the mu-mesic atom experiment, it should not change the calculated values of those quantities measured in the mesic atom experiment, for example the rms radius which is determined uniquely and to a high degree of accuracy. To check the validity of this substitution, the mean square radii of $^{152}\text{Sm}$, $^{164}\text{Dy}$, and $^{168}\text{Er}$ were calculated using both sets of $\beta_2^c$ and $\beta_4^c$ with $r_0$ and $t$ being fixed in
### Table IV-4

Charge deformation parameters extracted from the measured moments under the assumption of a deformed Fermi charge distribution using the radial parameters of A) Ref 13, B) Ref. 14 \( r_0 = 1.08 \). Asterisk implies interpolated values of \( r_0 \) and \( t \) used. I and II have the same meaning as in Table 3.

<table>
<thead>
<tr>
<th>NUCLEUS</th>
<th>( r_0 )</th>
<th>( R_0 )</th>
<th>( t )</th>
<th>Set I</th>
<th>Set II</th>
<th>( R_0 )</th>
<th>( t )</th>
<th>Set I</th>
<th>Set II</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{152}\text{Sm})</td>
<td>1.106</td>
<td>5.90</td>
<td>.538</td>
<td>( B_2 )</td>
<td>( B_4 )</td>
<td>( B_2 )</td>
<td>( B_4 )</td>
<td>5.763</td>
<td>.6</td>
</tr>
<tr>
<td>(^{154}\text{Sm})</td>
<td>1.106*</td>
<td>5.928</td>
<td>.54</td>
<td>( 0.305 \pm 0.006 )</td>
<td>( 0.103 \pm 0.019 )</td>
<td>( 0.299 \pm 0.006 )</td>
<td>( 0.117 \pm 0.019 )</td>
<td>5.788</td>
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<td>(^{158}\text{Gd})</td>
<td>1.111*</td>
<td>6.006</td>
<td>.52</td>
<td>( 0.332 \pm 0.006 )</td>
<td>( 0.025 \pm 0.021 )</td>
<td>( 0.329 \pm 0.006 )</td>
<td>( 0.049 \pm 0.021 )</td>
<td>5.839</td>
<td>.6</td>
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<tr>
<td>(^{160}\text{Gd})</td>
<td>1.111*</td>
<td>6.031</td>
<td>.52</td>
<td>( 0.349 \pm 0.007 )</td>
<td>( -0.02 \pm 0.035 )</td>
<td>( 0.337 \pm 0.007 )</td>
<td>( 0.016 \pm 0.022 )</td>
<td>5.863</td>
<td>.6</td>
</tr>
<tr>
<td>(^{164}\text{Dy})</td>
<td>1.116</td>
<td>6.109</td>
<td>.50</td>
<td>( 0.349 \pm 0.014 )</td>
<td>( 0.04 \pm 0.045 )</td>
<td>( 0.337 \pm 0.014 )</td>
<td>( 0.005 \pm 0.025 )</td>
<td>5.912</td>
<td>.6</td>
</tr>
<tr>
<td>(^{166}\text{Er})</td>
<td>1.118*</td>
<td>6.133</td>
<td>.50</td>
<td>( 0.338 \pm 0.007 )</td>
<td>( 0.017 \pm 0.03 )</td>
<td>( 0.327 \pm 0.007 )</td>
<td>( 0.017 \pm 0.016 )</td>
<td>5.936</td>
<td>.6</td>
</tr>
<tr>
<td>(^{168}\text{Er})</td>
<td>1.118</td>
<td>6.158</td>
<td>.50</td>
<td>( 0.348 \pm 0.010 )</td>
<td>( -0.045 \pm 0.04 )</td>
<td>( 0.335 \pm 0.010 )</td>
<td>( -0.005 \pm 0.028 )</td>
<td>5.959</td>
<td>.6</td>
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<tr>
<td>(^{174}\text{Yb})</td>
<td>1.118*</td>
<td>6.230</td>
<td>.50</td>
<td>( 0.326 \pm 0.010 )</td>
<td>( -0.03 \pm 0.04 )</td>
<td>( 0.317 \pm 0.010 )</td>
<td>( 0.002 \pm 0.023 )</td>
<td>6.029</td>
<td>.6</td>
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</tbody>
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### Table IV-5

Comparison of the deformation parameters extracted from the same moments using three different sets of \( r_0 \) and \( t \). \(^{152}\text{Sm}\)

<table>
<thead>
<tr>
<th>( r_0 )</th>
<th>( t )</th>
<th>Reference</th>
<th>( B_2^C )</th>
<th>( B_4^C )</th>
</tr>
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<tbody>
<tr>
<td>1.106</td>
<td>.538</td>
<td>13</td>
<td>.287</td>
<td>.070</td>
</tr>
<tr>
<td>1.14</td>
<td>.6</td>
<td>12</td>
<td>.268</td>
<td>.058</td>
</tr>
<tr>
<td>1.08</td>
<td>.6</td>
<td>14</td>
<td>.293</td>
<td>.071</td>
</tr>
</tbody>
</table>
each case. The differences in the mean square radii calculated using the deformation parameters of Hitlin, et al.\textsuperscript{13)} and those obtained in this experiment were 0.1\%, 0.2\% and 0.3\% for $^{152}\text{Sm}$, $^{164}\text{Dy}$, and $^{168}\text{Er}$, respectively.

5. The Negative Solution for $\text{M(E}^4;0^{+}\rightarrow4^{+})$

As was seen above, two different values of $\text{M(E}^4;0^{+}\rightarrow4^{+})$, one positive and the other negative, can correctly predict the measured $P(4)$ for each nucleus. An examination of the $\beta_4^c$ deformation parameters obtained for each of the two solutions gives reasonable evidence that the positive solution is indeed the correct one. According to theory, of all the nuclei studied, Yb\textsuperscript{174} should have the most negative $\beta_4^c$, hence the most negative $\text{M(E}^4;0^{+}\rightarrow4^{+})$. This is consistent with the results of the nuclear scattering experiments. The second possible $\text{M(E}^4;0^{+}\rightarrow4^{+})$ for this nucleus is -0.7. This results in a $\beta_4^c$ of approximately -0.2. This is larger in magnitude than any predicted $\beta_4^c$ in the region. In addition, since the magnitude of the negative solution increases as the magnitude of the positive solution increases, it is expected that $\beta_4^c$ obtained from the negative solution will become more negative towards the lighter nuclei studied. This is serious conflict with both theory and nuclear scattering results. In addition, it results in $\beta_4^c$ for $\text{Sm}^{152}$ which is on the order of -0.25 to -0.3 which is in complete disagreement with the value obtained from electron scattering measurements.
REFERENCES

CHAPTER IV


17) C. K. Bockelman, private communication.


CHAPTER V
DISCUSSION

The significance of the data may be considered from three distinct points of view: as a measurement of the reduced electromagnetic transition matrix elements \( M(E2;0^+\rightarrow2^+) \) and \( M(E4;0^+\rightarrow4^+) \); as a measurement of the deformation of the nuclear electric potential, either in terms of the intrinsic quadrupole and hexadecapole moments or as described by the deformation parameters \( \beta^c_2 \) and \( \beta^c_4 \); or as a measurement of the nuclear deformation under the assumption that the charge and mass deformations are identical. The third description may also be considered as a tentative measurement of any differences between the charge and mass distributions.

A. MATRIX ELEMENT MEASUREMENTS

Refinements in nuclear physics instrumentation have made it possible to measure Coulomb excitation probabilities with accuracies of a few percent and better, so that Coulomb excitation has become a precision probe of nuclear properties. This has been accompanied by a greater understanding of the Coulomb excitation process and the properties of the nuclei being studied, specifically the reduced \( EA \) matrix elements. These precise measurements have made it possible
to refine the determination of the transition matrix elements of the dominant E2 mode of excitation, and to measure effects which cause small changes in excitation probabilities, such as the re-orientation effect due to the presence of static quadrupole moments of excited states $1, 2$, and the presence of E4 excitation. As measurements have become more exact, the task of uniquely determining the various contributions has become more complicated, since the presence of small effects can complicate the precise determination of a large effect, and uncertainties in large contributions can completely obscure small ones.

Since the dominant mode of excitation within the ground state band is multiple E2, no measurement of matrix elements within that band can be better than the value of $M(E2; 0^{+} \rightarrow 2^{+})$ used to analyze the data. In this experiment a method has been utilized in which this matrix element can be directly measured to roughly $1/2\%$ without significant interference from other processes. The measurements of $M(E2; 0^{+} \rightarrow 2^{+})$ and $M(E4; 0^{+} \rightarrow 4^{+})$ are compared to other measurements in Table (1). In some cases there is an improvement in accuracy of $M(E2; 0^{+} \rightarrow 2^{+})$ of better than a factor of five over those values accepted a few years ago.

As demonstrated in Chapter II, the influence of E4 excitation on the Coulomb excitation of states in the ground state band of even-even rare earth nuclei can be substantial under a variety of experimental conditions, and can be the dominant mode of excitation in a few limited cases. The
Table V-1

Comparison of measured values of $M(E2;0^+\rightarrow 2^+)$ in $e^+\nu$ and $M(E4,0^+\rightarrow 4^+)$ in $e^+\nu$ with other determinations. Set I indicates matrix elements extracted using the de Heer-Winter calculation corrected for quantum mechanical effects using second order perturbation theory. Set II indicates matrix elements determined using the corrections from the full quantum mechanical calculation.

<table>
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<tr>
<th>Nucleus</th>
<th>$M(12,0^+\rightarrow 2^+)$ Other Measurements</th>
<th>$M(14,0^+\rightarrow 4^+)$ Other Measurements</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{152}\text{Sm}$</td>
<td>-1.864 ± 0.011</td>
<td>-1.864 ± 0.011</td>
</tr>
<tr>
<td></td>
<td>-1.856 ± 0.011</td>
<td>-1.860 ± 0.012</td>
</tr>
<tr>
<td></td>
<td>-1.857 ± 0.075</td>
<td>-1.838 ± 0.007</td>
</tr>
<tr>
<td></td>
<td>-1.866 ± 0.031</td>
<td>-1.841 ± 0.009</td>
</tr>
<tr>
<td></td>
<td>-1.857 ± 0.016</td>
<td>-1.841 ± 0.009</td>
</tr>
<tr>
<td>$^{154}\text{Sm}$</td>
<td>-2.093 ± 0.015</td>
<td>-2.147 ± 0.042</td>
</tr>
<tr>
<td></td>
<td>-2.088 ± 0.015</td>
<td>-2.053 ± 0.026</td>
</tr>
<tr>
<td></td>
<td>-2.076 ± 0.012</td>
<td>-2.074 ± 0.007</td>
</tr>
<tr>
<td></td>
<td>-2.113 ± 0.019</td>
<td>-2.113 ± 0.019</td>
</tr>
<tr>
<td>$^{158}\text{Gd}$</td>
<td>-2.242 ± 0.011</td>
<td>-2.302 ± 0.054</td>
</tr>
<tr>
<td></td>
<td>-2.233 ± 0.011</td>
<td>-2.252 ± 0.014</td>
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<tr>
<td></td>
<td>-2.233 ± 0.014</td>
<td>-2.233 ± 0.031</td>
</tr>
<tr>
<td>$^{160}\text{Gd}$</td>
<td>-2.292 ± 0.014</td>
<td>-2.381 ± 0.052</td>
</tr>
<tr>
<td></td>
<td>-2.283 ± 0.014</td>
<td>-2.292 ± 0.028</td>
</tr>
<tr>
<td></td>
<td>-2.292 ± 0.028</td>
<td>-2.292 ± 0.028</td>
</tr>
<tr>
<td>$^{164}\text{Dy}$</td>
<td>-2.370 ± 0.020</td>
<td>-2.362 ± 0.053</td>
</tr>
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<td>-2.360 ± 0.020</td>
<td>-2.366 ± 0.011</td>
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<td>-2.366 ± 0.011</td>
<td>-2.366 ± 0.011</td>
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<tr>
<td>$^{166}\text{Er}$</td>
<td>-2.414 ± 0.011</td>
<td>-2.404 ± 0.042</td>
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<tr>
<td></td>
<td>-2.404 ± 0.011</td>
<td>-2.393 ± 0.011</td>
</tr>
<tr>
<td></td>
<td>-2.393 ± 0.011</td>
<td>-2.393 ± 0.011</td>
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<tr>
<td>$^{168}\text{Er}$</td>
<td>-2.454 ± 0.021</td>
<td>-2.408 ± 0.042</td>
</tr>
<tr>
<td></td>
<td>-2.444 ± 0.021</td>
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<td></td>
<td>-2.400 ± 0.014</td>
<td>-2.400 ± 0.014</td>
</tr>
<tr>
<td>$^{174}\nu$</td>
<td>-2.448 ± 0.012</td>
<td>-2.387 ± 0.042</td>
</tr>
<tr>
<td></td>
<td>-2.438 ± 0.012</td>
<td>-2.427 ± 0.097</td>
</tr>
<tr>
<td></td>
<td>-2.427 ± 0.097</td>
<td>-2.418 ± 0.023</td>
</tr>
</tbody>
</table>

a) ref. 6  d) ref. 9  j) ref. 12
b) ref. 7  e) ref. 10  k) ref. 13
c) ref. 8  f) ref. 4  l) ref. 5
g) ref. 11  i) ref. 11

*The values presented in this reference are average values for many different experiments.
determination of $M(E^4;0^+\rightarrow 4^+)$ in these nuclei is therefore important to the analysis of many different Coulomb excitation experiments. A knowledge of these contributions is essential if a complete understanding of the excitation process is to be obtained. The $E^4$ contributions near the center of the deformed region are expected to be small, as evidenced by the large uncertainties in the extraction of $M(E^4;0^+\rightarrow 4^+)$ and the small measured values of that matrix element, but are seen to be quite substantial in the region of $A = 150 - 160$.

In addition, other small effects which can have a substantial effect on some Coulomb excitation experiments have been examined within the context of this measurement. Both the assumption of pure Coulomb excitation and the assumption of the validity of the semi-classical calculation have been central to the analysis of Coulomb excitation experiments. It has been observed that Coulomb-nuclear interference effects can become significant at different incident energies for different states, and that these effects can be predicted through the use of a scattering calculation which includes both Coulomb and nuclear potential scattering. The large quantum mechanical corrections necessary to the semi-classically calculated cross-sections—especially the $4^+$ cross-section—and the substantial discrepancies caused in the measured values of $M(E^4;0^+\rightarrow 4^+)$ by the differences in the results of two different calculations of those corrections emphasize the importance of including quantal effects.
in the analysis of Coulomb excitation measurements and the fact that these effects have not been, as yet, completely understood. This is especially important in view of the large quantal corrections predicted for the excitation of higher spin states. 25)

B. DEFORMATION OF THE NUCLEAR CHARGE DISTRIBUTION

The quadrupole moments measured in this experiment, although measured to a higher degree of accuracy, show no significant deviations from the trends exhibited by early measurements of these moments: the quadrupole moment increases by about 25% in the mass region 150-160, and then remains large and relatively constant up to $^{174}$Yb, which is near the center of the deformed region. The small hexadecapole moments, close to being consistent with zero, found in the region of large quadrupole moments from $^{160}$Gd to $^{174}$Yb indicate that the shape of these nuclei is close to being pure quadrupole. The lighter isotopes, however, have a significant hexadecapole component. It is interesting to note that both the quadrupole and hexadecapole moments change most rapidly in the region of mass 152-160, and are both more nearly constant from $^{160}$Gd to $^{174}$Yb.

The Coulomb deformation parameters extracted from both sets of measured moments using the parameters $r_0$ and $t$ from Hitlin, et al 14) are listed in Table (2) along with the results of other experiments. The quoted standard deviations do not include uncertainties in $r_0$ and $t$. The
Table V-7

Measured Coulomb deformation parameters compared with other determinations. The parameters of the radial charge distribution are from Hitlin, et al.\textsuperscript{14}. Interpolated values used where measurements do not exist from Reference 14. I, II have the same meaning as in Table V-1 and.

m) ref. 15

o) ref. 14

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<th>Other Measurements</th>
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<td>.268,\pm,006</td>
<td>.304,\pm,007</td>
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<td></td>
<td>.335,\pm,010</td>
<td></td>
<td>.005,\pm,028</td>
<td></td>
</tr>
<tr>
<td>$^{174}_{\text{Yb}}$</td>
<td>.326,\pm,010</td>
<td>.31,\pm,01</td>
<td></td>
<td>.002,\pm,023</td>
</tr>
</tbody>
</table>
original observation of Stephens, et al\textsuperscript{11}) that $\beta_4^C$ of $^{152}\text{Sm}$ and $^{154}\text{Sm}$ are much larger than those predicted by theory is supported by these measurements. In the case of $^{152}\text{Sm}$, $\beta_4^C$ agrees well with the measurement by electron scattering which is quoted to a high degree of accuracy.\textsuperscript{15}) As has been discussed in Chapter IV, the disagreements with the determinations of $\beta_2^C$ and $\beta_4^C$ of $^{152}\text{Sm}$ by Brückner, et al\textsuperscript{4}) can be accounted for by the different $r_0$ and $t$ used in the analyses, since the values of $M(E2;0^+\rightarrow2^+)$ and $M(E4;0^+\rightarrow4^+)$ measured in the two experiments are nearly identical. The differences with the determinations of Erb, et al\textsuperscript{13}) are more substantial, since the moments measured in the two experiments are different. These latter measurements are analyzed using the quantal corrections of Alder, et al\textsuperscript{3}).

C. COMPARISON OF COULOMB AND NUCLEAR POTENTIAL DEFORMATIONS

The deformed optical potential employed in the analysis of the scattering experiments conducted at incident energies above the Coulomb barrier is parameterized in terms of the products $R_0\beta_\lambda$ (see ref 4, 16 and Chapter I). If a meaningful comparison is to be made between two measurements of the deformation parameters, one must be scaled by the ratio of the radius parameters, $r_0$. The comparison of the nuclear potential deformation parameters, $\beta_\lambda^N$, measured by Hendrie, et al\textsuperscript{16}), Aponick, et al\textsuperscript{17}), and others\textsuperscript{18, 19}) to the Coulomb deformation parameters $\beta_\lambda^C$ measured in this experiment is therefore facilitated by scaling $\beta_\lambda^N$ so that
The charge radius parameter determined from mu-mesic atom experiments is used in all cases and $R_0\beta_\lambda^N$ is kept constant. Recently Hendrie $^{20}$ has considered in greater detail the geometry of the scattering of a projectile such as an alpha particle from a deformed nuclear surface, and has calculated second order corrections to the scaling by $r_0$. Table (3) compares the Coulomb deformation parameters measured in this experiment with various measurements of $\beta_\lambda^N$, and $\beta_\lambda^N$ scaled by the ratio of $r_0$. In addition, the deformation parameters of Hendrie, et al $^{16}$ have been further scaled in accordance with the second order corrections; those values are listed in parenthesis.

The $\beta_2^C$ values are roughly 10% larger than the $\beta_2^N$ values after the latter have been scaled. The second order corrections make the agreement worse. Although the $\beta_4^C$ follow the same general trend as the $\beta_4^N$ (which agree reasonably well with the theoretical $\beta_4$ values shown in Figure (1)) the values of $\beta_4^C$ are larger for the samarium isotopes, and decrease rapidly, remaining approximately constant from $^{160}$Gd to $^{174}$Yb. Theory predicts a smooth decrease with increasing mass.

It should be noted that recent measurements of hexadecapole deformation in the Actinide region by Bemis, et al $^{21}$ have demonstrated a similar trend in $\beta_4^C$ as compared to the predicted values. Theory predicts in this region a rather steady decrease from positive $\beta_4$ to near zero as the neutron number increases. Experiment indicates that $\beta_4^C$ is
Table V-3. Comparison of Coulomb $\beta_4$'s measured in this experiment with nuclear $\beta_4$'s. The nuclear $\beta_4$'s have been scaled (in the column "nuclear scaled") so that the $r_0$ used to extract both Coulomb and nuclear $\beta_4$'s are equal, keeping $r_0\beta_4$ constant. The numbers in parenthesis include the "second order" corrections of Hendrie$^{20}$. I, II have the same meaning as in Table V-1.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$\beta_2$</th>
<th>$\beta_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Coulomb</td>
<td>Nuclear</td>
</tr>
<tr>
<td>$^{152}$Sm</td>
<td>.287±.006 I</td>
<td>.246$^a)$</td>
</tr>
<tr>
<td></td>
<td>.280±.006 II</td>
<td>.25+.02$^c)$</td>
</tr>
<tr>
<td>$^{154}$Sm</td>
<td>.305±.006 I</td>
<td>.270$^a)$</td>
</tr>
<tr>
<td></td>
<td>.299±.006 II</td>
<td>.225+.005$^b)$</td>
</tr>
<tr>
<td>$^{158}$Gd</td>
<td>.332±.006 I</td>
<td>.282$^a)$</td>
</tr>
<tr>
<td></td>
<td>.329±.006 II</td>
<td>.25$^d)$</td>
</tr>
<tr>
<td>$^{166}$Er</td>
<td>.338±.007 I</td>
<td>.276$^a)$</td>
</tr>
<tr>
<td></td>
<td>.327±.007 II</td>
<td>.230+.005$^b)$</td>
</tr>
<tr>
<td>$^{174}$Yb</td>
<td>.326±.010 I</td>
<td>.276$^a)$</td>
</tr>
<tr>
<td></td>
<td>.317±.010 II</td>
<td>.002+.023 II</td>
</tr>
</tbody>
</table>

a) Ref. 16 $r_0 = 1.2$

b) Ref. 17 $r_0 = 1.49$

c) Ref. 18 $r_0 = 1.16$

d) Ref. 19 $r_0 = 1.2$

e) Ref. 4 $r_0 = 1.14$
Figure V-1.

Figure 1a. Two representative contemporary theoretical predictions of $\beta_H$ in the rare earths. The dots are from Reference 22, and the squares from Reference 23.

Figure 1b. Set I.

Figure 1c. Set II.
Figure V-1a
Figure V-1b
Figure V-1c
positive and greater than the prediction in the low mass end of the region, remains fairly constant, and then goes rapidly to equally large negative values as the neutron number increases. The data is again not clearly in conflict with theory, but neither does it agree very well. The measured values for $\beta_2^C$ are again consistently larger than the predicted values.

These numbers are open to several different interpretations. In the $^{152}$Sm measurement of Brückner, et al $^4$), as was discussed in Chapter IV, $r_0$ used to extract the deformation parameters from the measured moments was different from that used in the analysis of this experiment. This resulted in $\beta_2^C$ and $\beta_4^C$ which are in agreement with the nuclear deformation parameters measured in the same experiment, and are more in agreement with the nuclear deformation parameters of Hendrie, et al $^{16}$) than with the Coulomb deformation parameters obtained in this experiment. Following this example, if a value of $r_0$ is generated for each nucleus such that $\beta_2^N = \beta_2^C$, then much better agreement is obtained between the values of $\beta_4^N$ and $\beta_4^C$ which result when $\beta_4^C$ is scaled by $(r_0'/r_0)^4$ and $\beta_4^N$ is scaled by $(r_0'/r_0)$.

Small errors (1% to 2%) in the analysis of the data using the Coulomb excitation calculation could of course cause significant changes in the extracted $\beta_4^C$. This is especially true of the matrix element $M(E2; 2^+ \rightarrow 4^+)$ used in the calculation. Definitive measurements of that matrix element in nuclei other than the samarium isotopes may
reveal significant discrepancies with the rotational model calculation. Calculation of quantum mechanical effects in processes higher in order than the second order processes included in the tables of Alder, et al 3) may reveal effects not presently considered, as indicated by the differences between the two sets of extracted matrix elements.

The interpretation of the deformation parameters as expansion coefficients of a complex deformed optical potential is not as straightforward as the inclusion of those parameters in the deformed Fermi charge distribution. Neither is the equivalence of the two parameterizations of the nuclear shape completely clear, or the relation of either to the nuclear mass distribution 24). The nuclear and Coulomb deformation parameters may in fact be describing different things, and should perhaps not necessarily be equal. It is also possible that there are indeed differences between the nuclear mass distribution and the nuclear charge distribution.24, 16)
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CHAPTER V


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CHAPTER VI
CONCLUSIONS

Through the use of modern high resolution surface barrier detectors and careful attention to beam optics, the direct detection of $^4$He projectiles with incident energies below the Coulomb barrier, elastically and inelastically scattered at back angles has been used to measure the excitation probabilities of the first two excited states of eight even-even rare earth nuclei with 1%-2% and 2%-3% accuracies, respectively. The production of clean spectra with consistent 13-15 keV resolution was aided by the use of a pile-up rejection system. The reduced matrix elements $M(E2;0^+\rightarrow2^+)$ and $M(E4;0^+\rightarrow4^+)$ have been extracted from the measured excitation probabilities.

Careful consideration of both effects included in the Winther-deBoer semi-classical Coulomb excitation calculation - such as the influence of the excitation of other states - and effects not included in that calculation - such as differences between the classical calculation and the quantum mechanical treatment, and Coulomb-nuclear interference - was found to be necessary if the results were to be reliably interpreted to the degree of accuracy with which the excitation probabilities were measured. The determination of
$M(E2;0^+_1\rightarrow2^+_1)$ to about 1/2% accuracy (in some cases this is more accurate by a factor of three to six than previous determinations) is important to the analysis of experiments which measure small effects in multiple excitation of states in the ground state rotational band.

One such small effect is the influence of hexadecapole excitation on the excitation of the $4^+$ state. This experiment has confirmed the observation of $E4$ excitation in $^{152,154}$Sm by Stephens, et al, and extended that measurement to $^{158,160}$Gd, $^{164}$Dy, $^{166}$Er, and $^{174}$Yb. These eight nuclei span the region from the beginning of the good rotors to the middle of the rare earths. It has been calculated that $E4$ excitation can contribute significantly to multiple Coulomb excitation processes initiated by many different incident projectiles within ground state rotational bands in this region.

The interpretation of these matrix elements, within the context of the rotational model, in terms of the intrinsic quadrupole and hexadecapole moments of these nuclei provides a measurement of the deformation of the nuclear electric potential in this region. Previous descriptions have been in terms of measured quadrupole moments only. The hexadecapole moment is largest in the beginning of the region, at $^{152}$Sm and $^{154}$Sm, and decreases towards the center of the region where it is close to zero.

The calculation of these moments from a nuclear charge distribution having a Fermi form with a deformed surface,
and the subsequent extraction of the deformation parameters $\beta_2^c$ and $\beta_4^c$, involves an important uncertainty due to the strong dependence of the calculated moments on the radius parameter used in the calculation. This makes it difficult to believe the extracted deformation parameters as absolute numbers within the uncertainties dictated solely by the uncertainties in the measured moments.

Using one set of experimentally determined $r_0$ and $\ell$, deformation parameters were obtained which, although they are in agreement with other measurements of the Coulomb deformation parameters, disagree in detail with the deformation parameters used to describe the nuclear potential, while following the general trends displayed by those parameters. Those nuclear deformation parameters agree reasonably well with recent theoretical calculations of equilibrium ground state $\beta_2$ and $\beta_4$. Adjustments on the order of 10% in $r_0$ can greatly improve the agreement of nuclear and Coulomb deformation parameters. This adjustment is greater than quoted experimental uncertainties in $r_0$ by more than an order of magnitude, and can only be considered on an ad hoc basis.

Although the determination of the primary Coulomb excitation data is straightforward, the interpretation remains somewhat ambiguous until certain difficulties are resolved. It would be advantageous, for example, if the exact shape of the nuclear charge distribution in the rare earth region were better determined. While available data is fit with a
deformed Fermi distribution, there is some evidence that this distribution is a simplification of the true distribution which may include, for example, regions of high density near the poles at the ends of the symmetry axis of the basic quadrupole shape. Other formulations have also been proposed, but sufficient data has not existed in the past to determine a distribution described by many more parameters than is the Fermi distribution. Since measurements of the radial form of the charge distribution and the deformation are interrelated, it would be useful to analyze the data from several sources - such as Coulomb excitation, electron scattering and mu mesic atoms - simultaneously in order to obtain a consistent determination of the charge distribution. This body of primary data is currently expanding. By fitting many types of data simultaneously it may prove possible to determine a charge distribution described by a larger number of parameters. If the Fermi distribution is found to provide a good description of the true charge distribution in light of new data and analysis techniques, it should be possible to determine a consistent, and possibly unique, parameterization in terms of $r_0$, $t$, $\beta^C_2$, and $\beta^C_4$.

Finally, a unique correspondence between the deformations described by $\beta^C_\lambda$ and $\beta^N_\lambda$ is necessary, as is a method of relating the charge distribution and the nuclear potential distribution to the nuclear mass distribution. Until the relationship is firmly established between Coulomb deformation as parameterized by a deformed Fermi charge
distribution, and the nuclear potential deformation described by the parameters of a deformed optical potential, these two descriptions of nuclear deformation should be viewed as complementary information and not necessarily measurements of the same thing.
APPENDIX I

CALCULATION OF INTRINSIC MOMENTS FOR THE UNIFORM, SHARP CUT-OFF CHARGE DISTRIBUTION
APPENDIX I

CALCULATION OF INTRINSIC MOMENTS FOR THE
UNIFORM, SHARP CUT-OFF CHARGE DISTRIBUTION

The intrinsic moments are defined by:

\[ q_\lambda = \int d\tau r^\lambda Y^0_\lambda(\Omega) \rho(\vec{r}) \]

where the integral is over the nuclear volume. The uniform, sharp cut-off charge distribution:

\[ \rho(\vec{r}) = \rho(r) = \rho_0 \quad r \leq R(\theta) \]
\[ = 0 \quad r > R(\theta) \]

\[ \rho_0 = \frac{Z e}{V} \approx \frac{Z e}{\frac{4}{3} \pi R_0^3} \]

\[ R(\theta) = R_0 \left[ 1 + \beta_2 Y^0_2(\theta) + \beta_4 Y^0_4(\theta) + \beta_6 Y^0_6(\theta) \right] = R_0 f(\theta) \]

\[ q_\lambda = \rho_0 \int d\omega Y^0_\lambda(\Omega) \int_0^R \theta r^{\lambda+2} dr \]

\[ = \rho_0 \frac{R^\lambda_0}{\lambda+3} \int d\omega Y^0_\lambda(\Omega) f(\theta) \]

Let \( a_\lambda = \beta_\lambda Y^0_\lambda(\Omega) \)

so \( f(\theta) = [1 + a_2 + a_4 + a_6] \)

using the binomial expansion:

\[ (a+x)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \]
Where \(^{n}\!\!\!\!\!\!\!\!\!_{k}\) are the usual binomial coefficients

Then: \[ f^{L}(\theta) = \sum_{i=0}^{L} \binom{L}{i} \sum_{k=0}^{1} \binom{1}{k} a_{2}^{1-k} \sum_{j=0}^{k} \binom{k}{j} a_{4}^{j} a_{6}^{k-j} \]

The evaluations will involve integrals over products of spherical harmonics

\[ \int d\Omega Y_{l}^{0}(\Omega)Y_{l'}^{0}(\Omega) = \delta_{ll'} \] from definition of normalization of \( Y_{l}^{m}(\Omega) \). From Rose, p. 61:

\[ Y_{l_{1}}^{0}(\Omega)Y_{l_{2}}^{0}(\Omega) = \sum_{l} \left[ \frac{(2l_{1}+1)(2l_{2}+1)}{4\pi(2l+1)} \right]^{1/2} C^{2}(l_{1}l_{2}l;00)Y_{l}^{0}(\Omega) \]

\( C(l_{1}l_{2}l;00) = \) Clebsh-Gordon coefficient

\[ \int d\Omega Y_{l_{1}}^{0}(\Omega)Y_{l_{2}}^{0}(\Omega)Y_{l_{3}}^{0}(\Omega) = \sum_{l} \left[ \frac{(2l_{1}+1)(2l_{2}+1)}{4\pi(2l+1)} \right]^{1/2} x C^{2}(l_{1}l_{2}l_{3};00) \]

from substituting expression for \( Y_{l_{1}}Y_{l_{2}} \) and doing the integral.

Applying the technique to higher products yields:

\[ \int d\Omega Y_{l_{4}}^{0}(\Omega)Y_{l_{3}}^{0}(\Omega)Y_{l_{2}}^{0}(\Omega)Y_{l_{1}}^{0}(\Omega) = \]

\[ \sum_{l} \left[ \frac{(2l_{1}+1)(2l_{2}+1)}{4\pi(2l+1)} \right]^{1/2} \left[ \frac{(2l_{3}+1)(2l_{4}+1)}{4\pi(2l+1)} \right]^{1/2} C^{2}(l_{1}l_{2}l_{3};00) \]

\[ x C^{2}(l_{3}l_{4}l;00) \]
\[ \int d\Omega Y_{l_6}^0(\Omega) \ldots Y_{l_1}^0(\Omega) = \sum_{\ell, \ell'} \left[ \frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell_1+1)} \right]^{1/2} \left[ \frac{(2\ell_3+1)(2\ell_4+1)}{4\pi(2\ell_5+1)} \right]^{1/2} \]
\[
\times \left[ \frac{(2\ell_1+1)(2\ell_6+1)}{4\pi(2\ell_5+1)} \right]^{1/2} C^2(\ell_1, \ell_2; 00) C^2(\ell_3, \ell_4; 00) C^2(\ell_5, \ell_6; 00)
\]

\[ \int d\Omega Y_{l_6}^0(\Omega) \ldots Y_{l_1}^0(\Omega) = \sum_{\ell, \ell', \ell''} \left[ \frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell_1+1)} \right]^{1/2} \left[ \frac{(2\ell_3+1)(2\ell_4+1)}{4\pi(2\ell_5+1)} \right]^{1/2} \left[ \frac{(2\ell_5+1)(2\ell_6+1)}{4\pi(2\ell_5+1)} \right]^{1/2} \]
\[
\times C^2(\ell_1, \ell_2; 00) C^2(\ell_3, \ell_4; 00) C^2(\ell_5, \ell_6; 00)
\]

All orders of \( \beta \), up to \( \beta^{\lambda+3} \) (i.e., \( \beta_1 \beta_2 \beta_3 \beta_4 \ldots \beta_{\lambda+2} \)) will contribute to \( f^{\lambda+3}(\theta) \). In general only first order terms \( \beta_1 \beta_2 \ldots \) will go to zero when the integral is done. Since \( \beta_\lambda \leq 1 \) always, and decreases with \( \lambda \) (\( \beta_2 > \beta_4 > \beta_6 \)) only terms up to second in \( \beta_{\lambda} \) (i.e., \( \beta_1 \beta_2 \beta_3 \)) will be kept:

\[ q_2 = \rho_0 R^5 I_2 \]

\[ I_2 = \int d\Omega Y_2^0(\Omega) \left[ \sum_{i=0}^{5} (i) \sum_{k=0}^{i} (i)_k a_2^{-1-k} \sum_{j=0}^{k} (k)_j a_4^j a_6^{k-j} \right] \]

keeping only up to second order:
\[ I_2 = (5)(1/0)\beta_2 + (5)(2/0)\beta_2 \left( \frac{5x5}{5x4\pi} \right)^{1/2} C^2(222;00) \]

\[ + (5)(1)(1)\beta_2 \beta_6 \left[ \frac{5x13}{5x4\pi} \right]^{1/2} C^2(226;00) \]

\[ + (5)(2)(1)\beta_2 \beta_4 \left[ \frac{5x5}{9x4\pi} \right]^{1/2} C^2(244;00) \]

\[ + (5)(2)(2)\beta_6 \left[ \frac{5x13}{13x4\pi} \right]^{1/2} C^2(626;00) \]

\[ + (5)(2)(2)\beta_4 \beta_6 \left[ \frac{9x5}{4\pi x13} \right]^{1/2} C^2(424;00) \]

\[ + (5)(2)(2)\beta_4 \beta_6 \left[ \frac{9x5}{4\pi x9} \right]^{1/2} C^2(424;00) \]

Using:  \( C(222;00) = -0.5345 \)

\( C(226;00) = 0 \)

\( C(224;00) = 0.7171 \)

\( C(626;00) = -0.5045 \)

\( C(426;00) = 0.6742 \)

\( C(424;00) = -0.5096 \)

\[ I_2 = 5\beta_2 + 1.8021\beta_2 \beta_4 + 4.8351\beta_2 \beta_4 + 1.6053\beta_6 \]

\[ + 4.7708\beta_4 \beta_6 + 1.6381\beta_4 \]

\[ q_2 = Ze(r_0 A^{1/3})^2 \left[ 0.2387\beta_2 + 0.0861\beta_2 \beta_4 + 0.2309\beta_4 \beta_4 + 0.0782\beta_4 \beta_6 \right. \]

\[ + 0.0766\beta_6 \beta_6 + 0.2278\beta_4 \beta_6 \]
To demonstrate the relative sizes of the terms:

\[
q_2 = 0.2387Z e(r_0 A^{1/3})^2 \left[ \beta_2 + 0.3607 \beta_2^2 + 0.9673 \beta_2 \beta_4 + 0.3276 \beta_4^2 + 0.3209 \beta_6^2 + 0.9543 \beta_4 \beta_6 \right]
\]

As an example: \( \text{Sm}^{152} \)  
\( \beta_2 = 0.29 \)  \( \beta_4 = 0.07 \)  \( \beta_6 = -0.01 \)  
\( r_0 = 1.15 \)  
\( q_2 = 23.3536 e^{0.06922 + 0.00724 + 0.00469 + 0.00038 + 0.00008 - 0.00016} = 1.900 \text{ e\cdot barns} \)

The next term in the integral \( O(\beta^3) \) is given by:

\[
\begin{align*}
(3) \left\{ (3)_3 a^3_3 + (3)_2 a^2_2 \left[ (1)_0 a_6 + (1)_1 a_4 \right] + (3)_2 a_2 \left[ (2)_0 a_6^2 + (2)_1 a_4 a_6 \right] \\
+ (2)_1 a_1 \right\} + (3)_3 \left[ (3)_3 a^3_3 + (3)_2 a_2 a_6 + (3)_1 a_1 a_6 \\
+ (3)_2 a_1 a_6 \right] + (3)_3 \left[ (3)_3 a^3_3 + (3)_2 a_2 a_6 + (3)_1 a_1 a_6 \\
+ (3)_2 a_1 a_6 \right]
\end{align*}
\]

Again the terms are in general mixed (different \( \beta_\lambda \)'s).

Similarly,

\[
q_4 = \frac{2Z e}{\pi} R_0 q^{4} I_4
\]

\[
I_4 = \int d\Omega Y_{4}^{0}(\Omega) \left\{ \sum_{i=1}^{7} \sum_{k=0}^{1} (1) a_{2}^{i-k} \sum_{j=0}^{k} (1) a_{4} a_{6}^{k-j} \right\}
\]
\[ I_4 = \left( \frac{7}{1} \right)^{\left( \frac{1}{1} \right)} B_4 + \left( \frac{7}{2} \right)^{\left( \frac{2}{1} \right)} B_2^2 \left[ \frac{5 \times 5}{9 \times 4 \pi} \right]^{1/2} C^2(224;00) \]
\[ + \left( \frac{7}{1} \right)^{\left( \frac{1}{0} \right)} B_2 B_6 \left[ \frac{9 \times 5}{13 \times 4 \pi} \right]^{1/2} C^2(426;00) \]
\[ + \left( \frac{7}{2} \right)^{\left( \frac{2}{1} \right)} B_2^2 \left[ \frac{9 \times 5}{9 \times 4 \pi} \right]^{1/2} C^2(424;00) \]
\[ + \left( \frac{7}{2} \right)^{\left( \frac{2}{0} \right)} B_6 \left[ \frac{13 \times 9}{13 \times 4 \pi} \right]^{1/2} C^2(646;00) \]
\[ + \left( \frac{7}{2} \right)^{\left( \frac{2}{1} \right)} B_4 B_6 \left[ \frac{9 \times 9}{4 \pi \times 13} \right]^{1/2} C^2(446;00) \]
\[ + \left( \frac{7}{2} \right)^{\left( \frac{2}{2} \right)} B_4^2 \left[ \frac{9 \times 9}{4 \pi \times 9} \right]^{1/2} C^2(444;00) \]

\[ C(224;00) = .7171 \]
\[ C(426;00) = .6742 \]
\[ C(424;00) = -.5096 \]
\[ C(646;00) = .3870 \]
\[ C(446;00) = -.4495 \]
\[ C(444;00) = .4023 \]

\[ q_4 = Ze r_0 A^{1/3} \left[ .2387 B_4 + .1732 B_2^2 + .3417 B_2 B_6 + .2346 B_2 B_4 \right. \]
\[ + .09078 B_6^2 + .2038 B_4 B_6 + .0981 B_4^2 \]
Looking at Sm\textsuperscript{152} again:

\[ q_4 = 2.0997 \left[ 0.07 + 0.061 - 0.004 + 0.02 + 0 - 0.001 + 0.002 \right] = 0.71 \]

Note in the case of \( q_4 \):

1) \( \beta_2^0 \) contributes about as much as \( \beta_4^0 \). Thus even for \( \beta_4^0 = 0 \) a substantial \( q_4 \) can exist (in this case about half its value).

2) \( \beta_2^0 \beta_4^0 \) contributes \( \sim 1/3 \) of each of the major terms. Thus the fact that \( q_4 \) (\( \lambda = 4 \)) does not involve only \( \beta_4^0 (\lambda = 4) \) is very important.

REFERENCE

APPENDIX II

STATIC AND TRANSITION MOMENTS OF
ROTATIONAL NUCLEI
A. STATIC MOMENTS IN THE COLLECTIVE MODEL

As has been noted in Chapter I, one of the basic assumptions of the collective model is the separability of collective and intrinsic (single-particle) degrees of freedom. For low-lying states - which are rotational in nature - the vibrational modes may be included with the intrinsic modes, and the wave function written in terms of the intrinsic wave functions, $x_K$, and the rotational wave functions which are given by the $D_{MK}^J$ functions:

$$\Psi_{JMK} = \left[ \frac{2J+1}{16\pi^2(1+\delta_{K0})} \right]^{1/2} \left(D_{MK}^J x_K + (-)^J D_{M,K}^J x_{-K} \right)$$

The moment of multipolarity $\lambda$ is defined by:

$$Q_{\lambda} = \langle q_{\lambda}^O \rangle_{M=J}$$

where $q_{\lambda}^O = \int r^{''} \lambda y_{\lambda}^O(\theta', \phi') \rho(r^{''}) d\tau^{''}$

$$= \int r^{''} \lambda y_{\lambda}^O(\theta', \phi') \rho(\vec{r}^{''}) d\tau^{''}$$

This can be rewritten in terms of the operator in the body-fixed system:

$$q_{\lambda}^\mu = \sum_v D^{\lambda}_{\mu, v} q_{\lambda}^v$$

$$q_{\lambda}^O = \sum_v D^{\lambda}_{0, v} q_{\lambda}^v$$
Inserting this into the expression for the expectation value yields:

\[
\langle q^0_\lambda \rangle_{M=J} = \int d\tau \psi^*_{JJ\lambda} q^0_{\lambda} \psi_{JJ\lambda}
\]

\[
= \int d\tau \psi^*_{JJ\lambda} \sum_{\nu} D^\lambda_{0\nu} q^\nu_{\lambda} \psi_{JJ\lambda}
\]

\[
= \left( \frac{2J+1}{16\pi^2(1+\delta_{K0})} \right) \int d\tau \left( D^J_{JK} x_K^* + (-)^J D^J_{JK} x_K^* \right)
\]

\[
\times \sum_{\nu} D^\lambda_{0\nu} q^\nu_{\lambda} \left( D^J_{JK} x_K^* + (-)^J D^J_{JK} x_K^* \right)
\]

and expanding results in four terms:

\[
\langle q^0_\lambda \rangle_{M=J} = \left( \frac{2J+1}{16\pi^2(1+\delta_{K0})} \right) \sum_{\nu} \left\{ \langle D^J_{JK} x_K^* | D^\lambda_{0\nu} | D^J_{JK} \rangle <x_K^*| q^\nu_{\lambda} | x_K \rangle 
\right. 
\]

\[
+ (-)^J \langle D^J_{JK} x_K^* | D^\lambda_{0\nu} | D^J_{JK} \rangle <x_K^*| q^\nu_{\lambda} | x_K \rangle 
\]

\[
+ (-)^J \langle D^J_{JK} x_K^* | D^\lambda_{0\nu} | D^J_{JK} \rangle <x_K^*| q^\nu_{\lambda} | x_K \rangle 
\]

\[
+ (-)^{J-J} \langle D^J_{JK} x_K^* | D^\lambda_{0\nu} | D^J_{JK} \rangle <x_K^*| q^\nu_{\lambda} | x_K \rangle 
\]
The matrix elements of the D functions are given by:

\[ <D_{MK}^J|D_{\mu\nu}^J|D_{MK}'^J> = \frac{8\pi^2}{2J + 1} <JkM\mu|J'M'|<Jk\nu|J'\nu'> \]

where the following must hold:

1) \( M + \mu = M' \)
2) \( K + \nu = K' \)

\( <j_1j_2m_1m_2|JM> \) are Clebsch-Gordan coefficients.

Rewriting the moment:

\[ <q_\lambda^O M=J> = \frac{8\pi^2}{2J+1} \frac{2J+1}{16\pi^2 (1+\delta_{KO})} <J\lambda J0|JJ> \]

\[ (\langle J\lambda KO|JK \rangle \langle x_K'|q_\lambda^O|x_K \rangle \]

\[ + (-)^J \langle J\lambda-K 2K|JK \rangle \langle x_K'|q_\lambda^{2K}|x-K \rangle \]

\[ + \langle J\lambda-KO|J-K \rangle \langle x_K'|q_\lambda^O|x-K \rangle \]

\[ + (-)^J \langle J\lambda-K-2K|J-K \rangle \langle x_K'|q_\lambda^{-2K}|x_K \rangle \}

applying a Clebsch-Gordan identity:

\[ <j_{J'} m'|JM> = (-)^{J'+J} <JJ'-m-m'|J-M> \]
and a property of these tensor operators:

\[ \langle -K|q^\mu|_K' \rangle = (-)^\lambda \langle K|q^\mu|_K' \rangle \]

Then:

\[ \langle J\lambda K|_J K \rangle = (-)^{\lambda+2J+2K} \langle J\lambda K|_J K \rangle \]

\[ (-)^{-J} \langle J\lambda K-2K|_J K \rangle = (-)^{J+\lambda+2K} \langle J\lambda K-2K|_J K \rangle \]

\[ \langle -K|q^\nu|_K \rangle = \langle K|q^\nu|_K \rangle \]

so that the moment is now given by:

\[ \langle q^0\rangle_{M=J} = \frac{1}{2(1+\delta_{K0})} \langle J\lambda K|_J K \rangle \]

\[ \times \{ \langle J\lambda K|_J K \rangle \langle x^*_K|q^\nu|x_K \rangle (1+(-)^{2J+2\lambda+2K}) \]

\[ + \langle J\lambda K-2K|_K J \rangle \langle x^*_K|q^\nu|2K|x_{-K} \rangle ((-)^J + (-)^{J+2\lambda+2K}) \} \]

Examining the factors which multiply each of the terms, it is seen that since \( \lambda \) is always an integer:

\[ a = (1+(-)^{2J+2\lambda+2K}) = (1+(-)^{2J+2K}) \]

\[ b = (-)^J (1+(-)^{2\lambda+2K}) = (-)^J (1+(-)^{2\lambda+2K}) \]
The factor a is always 2, since K must be an integer if J is, and a half integer if J is half integer. The factor b is equal to 2 if K is an integer, and zero if K is half integer. Therefore, there are three cases:

1) \( K = 0 \)
\[
<q^0_\lambda | M=J >= <J_{\lambda J}O|JJ> <J_{\lambda K}O|JK> <q^0_\lambda >
\]

2) \( K = \) half integer = \( n/2 \)
\[
<q^0_\lambda | M=J >= <J_{\lambda J}O|JJ> <J_{\lambda K}O|JK> <q^0_\lambda >
\]

3) \( K = n, \) an integer \( \leq \lambda/2 \)
\[
<q^0_\lambda > = <J_{\lambda J}O|JJ> \{ <J_{\lambda K}O|JK> <q^0_\lambda > \\
+ (-)^J <J_{\lambda-K 2K}|JK> <q^{2K}_\lambda > \}
\]

For cases 1 and 2 the moment in the laboratory frame of reference is proportional to the intrinsic moment and can be written as:
\[
Q_\lambda = <J_{\lambda J}O|JJ> <J_{\lambda K}O|JK> Q_\lambda ^{
}
\]

In the third case, \( 2K \) must be less than or equal to \( \lambda \) to satisfy triangulation requirements. For \( K \) equal to an integer less than or equal to \( \lambda/2 \) the second term must be taken into account. Some examples are:

1) the quadrupole moment for \( K=1 \)
2) the hexadecapole moment for \( K=1,2 \)
3) the \( \lambda=6 \) moment for \( K=1,2,3 \)
\[ K=2 \text{ is the intrinsic projection of gamma vibrational bands of even-even nuclei. Experiments cannot as yet determine hexadecapole moments of excited states such as gamma vibrational states, but this may become possible in the future.} \]

The relationship between the intrinsic quadrupole moment and the spectroscopic quadrupole moment has been derived from several different sources (see, for example, Preston\(^1\), p. 67) and is given by:

\[ Q_2 = \frac{3K^2 - J(J+1)}{(2J+3)(J+1)} Q_2' \]

The general expression derived above should reduce to this for \( \lambda = 2, \ K \neq 1 \).

\[ Q_2 = <J2J0|JJ> <J2K0|JK> Q_2' \]

Using two explicit expressions for Clebsch-Gordan coefficients:

\[ <J2J0|JJ> = (+1) \left\{ \frac{(2J+1)12!}{(2J+3)!(2J-2)!2!} \right\} \left\{ \frac{(2J)!2!12!}{012!} \right\} = \sqrt{\frac{2J(2J-1)}{(2J+3)(2J+2)}} \]

(Messiah\(^2\), p. 1059)

\[ <J2K0|JK> = (-)^{-2} (2J+1)^{1/2} \left\{ \frac{(2J-2)!(J+K)!}{(2J+3)!(J+K)!} 2[3K^2 - J(J+1)] \right\} \]

(Preston\(^1\), p. 606)

The product reduces to:
\[ 2 [3K^2 - J(J+1)] \sqrt{2J+1} \frac{\sqrt{(2J+1)!(2J)!(2J-2)!}}{(2J+3)!(2J+3)!(2J-2)!} \]

\[ = 2 [3K^2 - J(J+1)] \times \sqrt{\frac{1}{(2J+3)^2(2J+2)^2}} \]

\[ Q_2 = \frac{3K^2 - J(J+1)}{(2J+3)(J+1)} Q_2' \]

which is exactly the usual form. It is important to note that the spectroscopic quadrupole moment is often defined as \( \sqrt{\frac{16\pi}{5}} Q_2' \). This is due to the fact that the first derivations of the quadrupole moment were in terms of the \( Q_{33} \) member of the quadrupole tensor which is \((3z^2 - r^2)\) and differs from \( P_2(\cos\theta) \) by a factor of \( \sqrt{\frac{5}{16\pi}} \). On the other hand, the quadrupole operator used in theories of multipole radiation (and Coulomb excitation theory) is given by: \( r^2Y_2^0 (\theta \phi) \). (See Preston\(^1\), or Ford and Hill\(^3\).) This latter form will be assumed throughout this analysis.

**B. RELATION OF THE TRANSITION MATRIX ELEMENTS TO THE STATIC MOMENTS OF THE NUCLEAR ELECTRIC FIELD**

The operators used to obtain the transition matrix elements, \( M(\varepsilon\lambda, \mu) \) are identical in form to the operators which yield the moments of the nuclear field:
The integral is over the nuclear volume. Since the low-lying states are rotational states, having the same charge distribution, it would seem intuitively clear that there is a relationship between the spectroscopic moments and the transition moments. The reduced transition rate $B(E\lambda; J_1 \rightarrow J_F)$ is defined by:

$$B(E\lambda; J_1 \rightarrow J_F) = \sum_{M_\lambda M_F \mu} |<f|M(E\lambda, \mu)|J>|^2$$

$$= (2J_1+1)^{-1}|<f||M(E\lambda)||1>|^2$$

where the sum is over all possible magnetic substates. Transforming into the body-fixed frame and proceeding as in the calculation of the moments in Section A:

$$B(E\lambda; J_1 \rightarrow J_F) = \sum_{M_\lambda M_F \mu} |\int \psi_f^* \psi_{J_1 M_1 K_1} \sum_v D_{\mu \nu}^\lambda q_{\nu}^\lambda \psi_{J_1 M_1 K_1} |^2$$

Using the collective wave functions and considering a transition from ground state $|J=K,M_1,K>$ to state $|J=K+\lambda,M_\lambda,K>$:

$$B(E\lambda; J_1 \rightarrow J_F) = \left(\frac{2K+1}{16\pi^2(1+\delta_{K0})}\right)^2 \sum_{M_\lambda M_F \mu} \sum_v f(D_{J_1-\lambda}^{K+\lambda} x_{K+\lambda}^{(-)} x_{-K-\lambda}^{(-)}$$

$$D_{M_\lambda}^{K+\lambda} x_{-K-K}^{(-)} x_{K-K}^{(-)} D_{M_\lambda}^{K+\lambda} x_{-K-K}^{(-)} x_{K-K}^{(-)} |^2$$
and writing all terms explicitly:

\[ \left( \frac{2K+1}{16\pi^2(1+\delta_{K0})} \right)^2 \sum_{M_1M_f} \sum_{\nu} \left\{ \langle D_{M_fK}^{K+\lambda^*} | D_{\mu\nu}^{\lambda} | D_{M_1K}^{K} \rangle \langle x_{K}^{*} | q_{\lambda}^\nu | x_{K}^{*} \rangle 
\right. \\
+ (-)^K \langle D_{M_fK}^{K+\lambda^*} | D_{\mu\nu}^{\lambda} | D_{M_1K}^{K} \rangle \langle x_{K}^{*} | q_{\lambda}^\nu | x_{-K} \rangle \\
+ (-)^{-K-\lambda} \langle D_{M_fK}^{K+\lambda^*} | D_{\mu\nu}^{\lambda} | D_{M_1K}^{K} \rangle \langle x_{-K} | q_{\lambda}^\nu | x_{K} \rangle \\
+ (-)^{-\lambda} \langle D_{M_fK}^{K+\lambda^*} | D_{\mu\nu}^{\lambda} | D_{M_1K}^{K} \rangle \langle x_{-K} | q_{\lambda}^\nu | x_{-K} \rangle \right\} \right|^2 

Using the same relationship for the matrix elements of the D functions used in Section A, the sums collapse:

1) \( M_1 + \mu = M_f \) so \( \sum_{M_f} \) collapses

2) \( \mp K + \nu = \mp K \) so \( \sum_{\nu} \) collapses leaving \( \nu = 2, \mp 2K \)

the expression becomes:

\[ B(E\lambda; J, K\rightarrow J+\lambda, K) = \left( \frac{\delta_{K0}}{2K+1} \right)^2 \frac{2K+1}{16\pi^2(1+\delta_{K0})} \left( \sum_{M_\mu} \left\{ \langle K\lambda M_\mu | K+\lambda \ u+M > < K\lambda KO | K+\lambda K > < K^* | q_{\lambda}^0 | K > 
\right. \\
+ (-)^K \langle K\lambda M_\mu | K+\lambda \ M+\mu > < K\lambda K-2K | K+\lambda , -K > < K^* | q_{\lambda}^{-2K} | -K > \\
+ (-)^{-K-\lambda} \langle K\lambda M_\mu | K+\lambda M+\mu > < K\lambda -K 2K | K+\lambda K > < -K^* | q_{\lambda}^{2K} | K > \\
+ (-)^{-\lambda} \langle K\lambda M_\mu | K+\lambda M+\mu > < K\lambda KO | K+\lambda , -K > < -K^* | q_{\lambda}^0 | -K > \right\} \right|^2 
\]
Finally, applying the expressions for the Clebsch-Gordan coefficients and the relationships between the matrix elements used in Chapter II:

\[
B(E\lambda) = \left(\frac{1}{2(1+\delta_{KO})}\right)^2 \sum_{M\mu} |<K\lambda M\mu|K+\lambda M+\mu> \times \left\{ (1+(-)^{\lambda})<K\lambda KO|K+\lambda K><K^*|q^0|K> \\
+ ((-)^{K+(-)^{\lambda}K})<K\lambda K-2K|K+\lambda,-K><K^*|q^0|2K|K> \right\} |^2
\]

The coefficients of the two terms become:

\( (1+(-)^{\lambda}) = 2 \) for even \( \lambda \)

\( (-)^{K+(-)^{\lambda}K} = \begin{cases} 0 & \text{for } K=n/2, \text{n odd} \\ \neq 0 & \text{for } K = \text{integer} \end{cases} \)

so \( B(E\lambda;J,K\rightarrow J+\lambda, K) \) has only one term except in the case where \( K \) is an integer less than or equal to \( \lambda/2 \). For \( K=0 \) or half integer it has the form:

\[
B(E\lambda;J=K,K\rightarrow J=K+\lambda K) = \left(\frac{2(1+\delta_{KO})}{2(1+\delta_{KO})}\right)^2 \sum_{M\mu} |<K\lambda M\mu|K+\lambda M+\mu><K\lambda KO|K+\lambda K> \\
\times |<K^*|q^0|K>|^2
\]

which becomes:
\[
= \left( \frac{2(1+\delta_{K0})}{2(1+\delta_{K0})} \right)^2 \left| <K\lambda KO|K+\lambda K> \right|^2 Q_\lambda^2 \sum_{M,\mu} \left| <K\lambda M\mu|K+\lambda M+\mu> \right|^2
\]

Making use of the relationship (Messiah, p. 1051):

\[
\sum_{m_1} \sum_{m_2} <J_1 J_2 m_1 m_2 |JM> <J_1 J_2 m_1 m_2 |J'M'> = \delta_{JJ'} \delta_{MM'}
\]

the expression finally reduces to:

\[
B(E\lambda;J=K,K+J=K+\lambda,K) = <K\lambda KO|K+\lambda K>^2 (Q_\lambda')^2
\]

In the special case where \(K=0\) (the case for the ground state band of even-even nuclei), this becomes:

\[
B(E\lambda;0,0,0,\lambda,0) = (Q_\lambda')^2
\]

Comparing both forms of the reduced transition rate:

\[
B(E\lambda;0,0,0,\lambda,0) = \left| <\lambda|M(E\lambda)||0> \right|^2 = (Q_\lambda')^2
\]

\[
\left| <\lambda|M(E\lambda)||0> \right| = \left| M(E\lambda;0,\lambda) \right| = |Q_\lambda'| \quad [\text{See Section C}]
\]

The reduced matrix elements from the ground state to the state \(J=\lambda\) in the rotational band have special importance, because they yield the intrinsic moments (and hence the spectroscopic moments directly. It is therefore advantageous to obtain the quadrupole moment by studying the excitation of the \(2^+\) state, and the hexadecapole moment by studying the \(4^+\)
state. It is also in these low-lying states that the rotational model is known to be most valid.

C. ANOTHER DERIVATION

The relationship between the reduced matrix elements and the intrinsic moments can be deduced from the individual matrix elements as follows:

For $K=0$, calculate:

$$<J=0, K=0, M=0 | M(E\lambda, \mu) | J=\lambda, K=0, \mu> = <00 | q^\mu_\lambda | \lambda \mu>$$

$$\psi_{J, N, 0} = 2 \left( \frac{2J+1}{16\pi^2(2)} \right)^{1/2} D^J_{\text{MO}X_0}$$

$$= \left( \frac{(2J+1)}{8\pi^2} \right)^{1/2} D^J_{\text{MO}X_0}$$

$$<00 | q^\mu_\lambda | \lambda \mu> = \frac{1}{8\pi^2} \sqrt{2\lambda+1} \sum_v <D^\lambda_{\text{MO}} | D^\lambda_{\nu} | D^0_{00}> <x_0^* | q^\nu_\lambda | x_0>$$

$$= \frac{8\pi^2}{2\lambda+1} \frac{\sqrt{2\lambda+1}}{8\pi^2} <0\lambda 0\mu | \lambda \mu> <0\lambda 00 | \lambda 0> <x_0^* | q^0_\lambda | x_0>$$

from the same relationships used in Sections A and B.

$$<0\lambda 00 | \lambda 0> = 1$$

So,
\[ <00|q_{\lambda}^\mu|\lambda\mu> = \frac{1}{\sqrt{2\lambda+1}} <0\lambda0\mu|\lambda\mu>Q_{\lambda}^{0'} \]

Definition of reduced matrix element:

\[ <00|M(E\lambda,\mu)|\lambda\mu> \equiv \langle 0\lambda0\mu|0\rangle\langle 0||M(E\lambda)||\lambda \rangle \]

\[ \langle 0\lambda0\mu|\lambda\mu> = \frac{1}{\sqrt{2\lambda+1}} <0\lambda0\mu|\lambda\mu> \]

\[ \frac{1}{\sqrt{2\lambda+1}} <0\lambda0\mu|\lambda\mu><0||M(E\lambda)||\lambda \rangle = \frac{1}{\sqrt{2\lambda+1}} <0\lambda0\mu|\lambda\mu>Q_{\lambda}^{0'} \]

\[ <0||M(E\lambda)||\lambda \rangle = Q_{\lambda}^{0'} \]

which is the same expression obtained from calculating \( B(E\lambda) \), but does not have the sign ambiguity.

Thus, for an even-even rotational nucleus:

A) \( Q_{\lambda} = <J\lambda00|JJ><J\lambda00|JK>Q_{\lambda}^{0'} \)

B) \( B(E\lambda;J=0,K=0\rightarrow J=\lambda,K=0) = |<\lambda||M(E\lambda)||0\rangle|^2 = (Q_{\lambda}^{0'})^2 \)

C) \( <\lambda||M(E\lambda)||0\rangle = Q_{\lambda}^{0'} \)
References


APPENDIX III

ENERGIES AT WHICH EXCITATION PROBABILITIES WERE MEASURED
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Numbers in parentheses indicate more than one measurement.
APPENDIX IV
MEASURED EXCITATION PROBABILITIES
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APPENDIX V

RATIO OF SCATTERING FROM Sm
TO SCATTERING FROM Pb
\( \theta_{\text{lab}} = 174.50 \)

\( R = \text{ratio to average for 8-12 MeV} \)

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**Averages:**

- 8-12 MeV: .3526, .4646
- 8-12.5: .3527, .4658
- 8-13: .3525, .4649

**Statistics:**
- 0.5% for $^{152\text{Sm}}\text{Pb}$, 0.8% for $^{154\text{Sm}}\text{Pb}$

**Maximum Scatter:**
- 1.5% for $^{152\text{Sm}}\text{Pb}$, 3% for $^{154\text{Sm}}\text{Pb}$
APPENDIX VI

INFORMATION INPUT INTO DeBOER-WINTHER CALCULATIONS
APPENDIX VI

INFORMATION INPUT INTO DEBOER-WINTHER CALCULATIONS

A) All matrix elements not specifically mentioned, except 
M(E2; 0⁺→2⁺) and M(E4; 0⁺→4⁺), are calculated from the 
rotational model.

B) All excitation energies are from reference 1 unless 
otherwise stated.

C) ACCUR = 0.0000010

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$M(E2)$ e·bars

0+ 2+γ  -0.3  Ref. 9
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\[ M(E2) \quad \text{e\cdot barns} \]

0+\rightarrow 2+\gamma \quad -0.3 \quad \text{Ref. 10}
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$M(E2)$ e·barns

$0^+ \rightarrow 2+\gamma$  -0.35  Ref. 6
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**M(E2) e·barns**

| 0+→2+γ | -0.4 | Ref. 7 |

166 Er
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$M(E2)$ e·barns.

$0^+ \rightarrow 2^+\gamma$ -0.4 Ref. 7
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Ref. 8
References Appendix VI


2) J.S.Greenberg, R.G.Stokstad, I.A.Fraser, private communication.


10) B.Elbek, B.Herskind, M.C.Olesen, Y.Toshizawa, Third Conference on Reactions Between Complex Nuclei p. 84 (4/63) (quoted in Nuclear Data sheets 5-6-132, April 1964).
APPENDIX VII

TARGET FABRICATION
APPENDIX VII

TARGET FABRICATION

The equipment used for the evaporation is shown in Figure A-VII. The method usually employed in the preparation of rare earth targets is to heat the rare earth oxide with a quantity of Lanthanum in a Tantalum crucible. This proved unsatisfactory because both Lanthanum and Tantalum evaporate, and their presence in the target produces elastic scatter lines close in energy to the lines of interest. Using Zirconium as a reducing agent in a Carbon boat eliminated this problem. Zirconium evaporates at a higher temperature than does Lanthanum, hence is present in smaller concentrations in the final target than Lanthanum would be. In those few targets in which any trace of Zirconium was discernable in the final spectrum, the elastic scatter lines from the isotopes of Zirconium were lower in energy than the region of interest, and sufficiently separate to cause no problems. The carbon crucible was designed as indicated in the figure to localize heating and impede heat flow to the copper electrodes. In addition, the electrodes were water-cooled, to further reduce the possibility of heating copper (which evaporates at about 1100°C) to the 1300-1800°C temperatures in the crucible. Evaporation times were 5 seconds to one minute, depending on the material used and the thickness desired.
Figure A-VII. Equipment used to fabricate targets. The pump is behind the trap.
Figure A-VII
Typically, 5 to 10 milligrams of the oxide were used to simultaneously produce deposition on four slides; the thickness of the deposition varied among the slides. In this way a variety of thicknesses could be produced in single evaporation. Since it proved difficult to reliably predict the areal density of these thin targets, it was necessary to produce a large variety in order to insure having the proper target in the experiment.

Several attempts were made to reduce Gadolinium oxide, but it proved extremely difficult to reliably obtain the necessary temperature using resistance heating. In addition, that temperature appeared near the point at which large quantities of Zirconium evaporated. Due to this difficulty, samples of isotopically enriched Gadolinium metal were obtained and evaporated. Although some Gadolinium targets were produced by resistance heating in a Carbon boat, the cleanest method for evaporating the samples at the high temperatures was evaporation in an electron bombardment gun. Several configurations were tried, the most reliable being a small Carbon boat, containing the sample, which was supported on a thin tungsten rod attached to the water-cooled base of the gun.